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A Noether theorem for stochastic operators on Schatten classes

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ABSTRACT

We show that a stochastic (Markov) operator S acting on a Schatten class $\mathcal{C}_1$ satisfies the Noether condition (i.e. $S'(A) = A$ and $S'(A^2) = A^2$, where $A \in \mathcal{C}_\infty$ is a Hermitian and bounded operator on a fixed separable and complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$), if and only if $S(E^A(G)XE^A(G)) = E^A(G)S(X)E^A(G)$ for any state $X \in \mathcal{C}_1$ and all Borel sets $G \subseteq \mathbb{R}$, where $E^A(G)$ denotes the orthogonal projection coming from the spectral resolution $A = \int_{\sigma(A)} zE^A(dz)$. Similar results are obtained for stochastic one-parameter continuous semigroups.

1. Introduction

Noether's famous theorem linking symmetries of a physical system to its conserved quantities has been attracting the interest of theoretical physicists for a long time. Discussions on it extend from scientific journals to Internet forums (see e.g. [6,11,13]). In a classical setting it is defined in terms of Lagrangian (differential) dynamical systems. Contemporary discussions initiated by this theorem sometimes concern deep philosophical aspects of modern quantum physics (see e.g. [6,8,15]). Baez and Fong have recently proposed in [8] to replace the classical Poisson brackets with commutators. Such an idea turned out to be very fruitful and finally resulted in Markovian models. Namely, it has been proved in [8] (see also [7]) that given a σ-finite measure space $(X, \mathcal{F}, \mu)$ and a stochastic operator $S : L^1(\mu) \mapsto L^1(\mu)$, a function $h \in L^\infty(\mu)$ will satisfy $S(fh) = S(f)h$ for all $f \in L^1(\mu)$ if and only if $S'(h) = h$ and $S'(h^2) = h^2$ (i.e. h and h^2 are both S' invariant, where $S' : L^\infty(\mu) \mapsto L^\infty(\mu)$ denotes the dual operator to S). The second part of the condition, $S'(h^2) = h^2$, unlike in Noether's original theoretical mechanics theorem case, appears to be necessary. Baez and Fong [8] discuss at length a related commutator's equation:

$$S(\cdot h) - S(\cdot)h = [S, M_h] = 0, \quad \text{(BF0)}$$

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where M_h stands for a multiplication operator by a function h, i.e. $M_h(f) = f \cdot h$. An analogous equation $[S_t, H] = 0$ for one parameter continuous Markovian semigroups is worked through in [8]. In a very recent paper [15] results from [8] and [7] have been extended to a fully quantum setting of quantum Markov dynamics with a strictly positive stationary density matrix. However in [15] the authors frequently exploit the complete positivity assumption.

In this paper we go in the same direction but we look on this problem from a purely mathematical perspective. In particular, we do not discuss physical paradigms at full length (see [3,7,8,15,18]). We focus only on the mathematical model of quantum measurement.

However, we drop the complete positivity and strict stationarity assumptions. Let us stress that there do exist positive but not completely positive linear operators; even finite dimensional (see [1], page 154, Example 8.4). Therefore, we can formulate our results for general Markovian (stochastic) dynamics. The main question we address is: when does a given stochastic (Markov) operator S on a Schatten class $\mathcal{C}_1$ commute with quantum measurement operations $M_{A,\Delta}(X) = E^A(\Delta)XE^A(\Delta)$? In the case of $\Delta \subseteq \mathbb{R}$ being Borel, we obtain corresponding results to those of [8] and [7]. As in the commutative case, we characterize Hermitian bounded operators $A \in \mathcal{C}_\infty$ satisfying $S'(A) = A$ and $S'(A^2) = A^2$. In the main result of the paper, Theorem 5.9, we provide several equivalent conditions which may be expressed in a compact form as $[S, M_{A,\Delta}] = 0$ for all Borel sets Δ. Finally we arrive to continuous time models and obtain a characterization of Noether strongly continuous Markovian semigroups.

2. Basics on Markov operators

In this section we remain in a commutative case but switch to Markov operators on $C(K)$. This leads to new simplified proofs.

Let us start with the notion of abstract Markov operators.

Definition 2.1. Let K be a topological compact Hausdorff space and $C(K)$ denote the Banach lattice of real valued continuous functions on K endowed with the sup norm $\|f\|_{\sup} = \sup\{|f(\kappa)| : \kappa \in K\}$. Let $\mathbf{1}$ denote the constant function equal to 1 on the whole space K. A linear operator $T : C(K) \to C(K)$ is called Markov (Markovian) if

$$T\mathbf{1} = \mathbf{1}, \tag{M1}$$

$$Tf \geq 0 \quad \text{if} \quad f \geq 0. \tag{M2}$$

Following standard notation of the Banach lattices theory we may say that Markov operators are positive (we simply write $T \geq 0$) and preserve the order unit $\mathbf{1}$. It follows that the operator norm $\|T\| = 1$. It is well known that any Markov operator defined on $C(K)$ may be canonically extended to the Banach lattice of Borel bounded functions. Indeed, given a Borel function g we set

$$Tg(\kappa) = \int g(u)dT'\delta_\kappa(u) \ , \tag{M3}$$

where $T' : \mathcal{M}(K) \mapsto \mathcal{M}(K)$ is the dual operator acting on the Banach lattice $\mathcal{M}(K)$ of all Radon bounded (signed) measures with the variation norm, where δ_κ stands for the Dirac measure at κ. In particular, if $\Delta \subseteq K$ is a Borel set, then for its indicator function $\mathbf{1}_\Delta$ the function $T\mathbf{1}_\Delta(\kappa) = T'\delta_\kappa(\Delta)$ is well defined and Borel.

It can be easily proved, that the family of probability measures $\{T'\delta_\kappa : \kappa \in K\}$ forms the so-called transition probability function. Namely, $K \ni \kappa \mapsto T'\delta_\kappa(D) \in [0, 1]$ is Borel measurable for every fixed Borel set $D \subseteq K$. Moreover, this transition probability function is Feller as $\langle g, T'\delta_\kappa\rangle = Tg(\kappa)$. Hence $T'\delta_{\kappa_\alpha} \to T'\delta_\kappa$

in the weak* topology, whenever $\kappa_\alpha \to \kappa$ in K. In the theory of Markov operators $\{T'\delta_\kappa : \kappa \in K\}$ is traditionally denoted as $\{P(\kappa, D) : \kappa \in K, D \text{ Borel}\}$. In order to define a Markov operator (acting on $C(K)$) it is sufficient to describe its transition function, which is Feller. In other words, T is well defined if defined are $T'\delta_\kappa$ for all $\kappa \in K$. In fact,

$$T'\mu = \int T'\delta_\kappa d\mu(\kappa) \tag{M4}$$

for all Radon (finite) measures μ on K.

Functional analysis language is an alternative to the classical probabilistic approach and importantly Markov operators completely describe the category of stochastic Markov processes. To keep the paper compact we do not further develop this topic, and refer the reader to a recent monograph [28] if necessary. Let us only add that it would be difficult to introduce Markov processes to theoretical quantum physics if we solely used probabilistic notions. An interesting point of view on Markovian dynamics, invertibility and determinism in the context of open systems is presented in [5].

3. Preparatory results

We remark, that from the lattice theory point of view $L^\infty(\mu)$ may be identified with $C(K)$, for some compact, Hausdorff space K (see [24], page 106, Example 4(5) or [2], page 201, Theorem 4.29). Given a stochastic operator $S : L^1(\mu) \mapsto L^1(\mu)$ (i.e. $Sf \geq 0$ and $\int Sf d\mu = \int f d\mu$ for all nonnegative $f \in L^1(\mu)$) and assuming that the base measure space is σ-finite, then the dual operator $T = S' : L^\infty(\mu) = C(K) \to L^\infty(\mu) = C(K)$ satisfies $S' \geq 0$ and $S'\mathbf{1} = \mathbf{1}$. Hence, we can interpret every stochastic operator S as a predual operator to some Markov operator $T : C(K) \mapsto C(K)$, for some compact Hausdorff space K. The relevant compact topology and identification of the space K are commonly known as Stonian (see [24], page 106). Our tactic to switch from stochastic operators to more general Markovian models will bring a two-folded improvement. We will generalize and simplify some proofs originally coming from [8]. Then, we use this approach with appropriate modifications in a noncommutative situation.

Proposition 3.1. *Let $C(K)$ be the (real) Banach lattice of real continuous functions on a Hausdorff compact space K, endowed with the supremum norm $\|\cdot\|_{sup}$. If $T : C(K) \mapsto C(K)$ is a Markov operator and $a \in C(K)$ satisfies $Ta = a$ and $Ta^2 = a^2$, then for every $r \in \mathbb{R}$*

$$T\mathbf{1}_{\Delta_r} = \mathbf{1}_{\Delta_r}, \ \textit{where } \Delta_r = a^{-1}(\{r\}) \textit{ is the level set.} \tag{M5}$$

Moreover, for each Borel bounded function $g : K \mapsto \mathbb{R}$ we have

$$T(\mathbf{1}_{\Delta_r} g) = \mathbf{1}_{\Delta_r} T(g) \quad \textit{and} \quad T(ag) = aT(g). \tag{M6}$$

Proof. Let us fix $\kappa \in K$. We have

$$T(a - a(\kappa))^2 = T(a^2 - 2a(\kappa)a + a(\kappa)^2) = a^2 - 2a(\kappa)a + a(\kappa)^2.$$

Evaluating it at κ we get

$$\begin{aligned}\int_K (a(u) - a(\kappa))^2 dT'\delta_\kappa(u) &= \langle T((a - a(\kappa))^2), \delta_\kappa\rangle \\ &= \langle (a - a(\kappa))^2, T'\delta_\kappa\rangle = T[(a - a(\kappa))^2](\kappa) = 0.\end{aligned}$$

It follows that $a(u) - a(\kappa) = 0$ for $T'\delta_\kappa$ almost all $u \in K$, or equivalently we may write

$$\mathrm{supp}(T'\delta_\kappa) \subseteq \{v \in K : a(v) = a(\kappa)\} = \Delta_{a(\kappa)}. \tag{Δ7}$$

Hence, $T'\delta_\kappa(\Delta_r) = 0$ or 1 for every $r \in \mathbb{R}$. From the above we easily get

$$T\mathbf{1}_{a^{-1}(G)}(\kappa) = \mathbf{1}_{a^{-1}(G)}(\kappa)$$

for every Borel set $G \in \mathcal{B}_\mathbb{R}$ and all $\kappa \in K$. Indeed, if $T\mathbf{1}_{a^{-1}(G)}(\kappa) = 1$ then the measure $T'\delta_\kappa$ must be concentrated on the set $a^{-1}(G)$. It has been already proved in (Δ7) that $T'\delta_\kappa$ is concentrated on the fiber $\{k \in K : a(k) = a(\kappa)\} = a^{-1}(a(\kappa))$. Hence $\kappa \in a^{-1}(a(\kappa)) \subseteq a^{-1}(G)$ as $a^{-1}(G)$ is a union of full level sets. For the reverse suppose that $T\mathbf{1}_{a^{-1}(G)}(\kappa) = 0$. Then $T\mathbf{1}_{a^{-1}(\mathbb{R}\setminus G)}(\kappa) = 1$. It follows that $\kappa \in a^{-1}(\mathbb{R} \setminus G) = K \setminus a^{-1}(G)$.

Now let g be any bounded Borel function on K. We get

$$T(\mathbf{1}_{\Delta_r} g)(\kappa) = \langle T(\mathbf{1}_{\Delta_r} g), \delta_\kappa\rangle = \langle \mathbf{1}_{\Delta_r} g, T'\delta_\kappa\rangle = 0 = \mathbf{1}_{\Delta_r}(\kappa) Tg(\kappa)$$

if $a(\kappa) \neq r$ and

$$T(\mathbf{1}_{\Delta_r} g)(\kappa) = \langle \mathbf{1}_{\Delta_r} g, T'\delta_\kappa\rangle = \langle g, T'\delta_\kappa\rangle = Tg(\kappa) = \mathbf{1}_{\Delta_r}(\kappa) Tg(\kappa)$$

if $a(\kappa) = r$. Now the statement $T(ag) = aT(g)$ is obvious. Indeed, $a = \lim_{n\to\infty} a_n$ (the convergence is uniform on K), where

$$a_n = \sum_{j=-J_n}^{J_n} \tfrac{j}{n} \mathbf{1}_{\Delta_j^n} \ ,$$

for $\Delta_j^n = a^{-1}([\frac{j}{n}, \frac{j+1}{n}))$ and $J_n = n\|a\|_{\sup}$. Now

$$T(ag) = T(\lim_{n\to\infty} a_n g) = \lim_{n\to\infty} T(a_n g) = \lim_{n\to\infty} T((\sum_{j=-J_n}^{J_n} \tfrac{j}{n}\mathbf{1}_{\Delta_j^n})g)$$
$$= \lim_{n\to\infty} \sum_{j=-J_n}^{J_n} \tfrac{j}{n} T((\mathbf{1}_{\Delta_j^n} g) = \lim_{n\to\infty} \sum_{j=-J_n}^{J_n} \tfrac{j}{n} \mathbf{1}_{\Delta_j^n} T(g) = aT(g). \quad \square$$

Due to positivity and linearity, Markov operators act "independently" on real and imaginary parts of complex functions. Therefore, results like the above can be extended "for free" to complex Banach lattices (see [24], page 137 for the so-called complexification of Banach lattices). Modifying the last proposition, without essential changes in its proof, we have:

Proposition 3.2. *Let $C(K)$ be the complex Banach lattice of complex continuous functions on a Hausdorff compact space K, endowed with the supremum norm $\|\cdot\|_{sup}$. If $T : C(K) \mapsto C(K)$ is a Markov operator and a real valued function $a \in C(K)$ satisfies $Ta = a$ and $Ta^2 = a^2$, then for each complex valued Borel bounded function $g : K \mapsto \mathbb{C}$ we have $T(ag) = aT(g)$.*

Now, let us introduce

Definition 3.3. A Markov operator $T : C(K) \to C(K)$ is called Noether with respect to a continuous function $a \in C(K)$ if

$$T(a) = a \quad \text{and} \quad T(a^2) = a^2. \tag{N8}$$

The following theorem fully characterizes Noether's Markov operators acting on $C(K)$. It refines and generalizes some parts of [7,8,15].

Theorem 3.4. *Let T be a Markov operator on (real or complex Banach space) $C(K)$, where K is compact Hausdorff space and $a \in C(K)$ a fixed real valued continuous function. Then the following conditions are equivalent:*

(1) $T(a) = a$ and $T(a^2) = a^2$,
(2) $T(a^n) = a^n$ for all $n \geq 0$,
(3) $T(ag) = aT(g)$ for all continuous $g \in C(K)$,
(4) $T(af) = aT(f)$ for all Borel bounded functions on K,
(5) $T'(a\mu) = aT'(\mu)$ for all $\mu \in \mathcal{M}(K)$.

Proof. Applying our Proposition 3.1 we may easily deduce the equivalences (1) ⇔ (2) ⇔ (3) ⇔ (4). Next (3) ⇔ (5) follows from the Riesz–Markov characterization $C(K)' = \mathcal{M}(K)$. Indeed, if $T(ag) = aT(g)$ then for all $g \in C(K)$ and $\mu \in \mathcal{M}(K)$ we have

$$\langle g, T'(a\mu)\rangle = \langle aT(g), \mu\rangle = \langle T(ag), \mu\rangle = \langle ag, T'(\mu)\rangle = \langle g, aT'(\mu)\rangle.$$

It follows that $T'(a\mu) = aT'(\mu)$. Hence (3) ⇒ (5). The reverse implication may be proved similarly. In fact, for all $g \in C(K)$ and $\mu \in C'(K)$ we have

$$\langle T(ag), \mu\rangle = \langle ag, T'(\mu)\rangle = \int_K gadT'(\mu) = \int_K gd(aT'(\mu))$$
$$= \langle g, T'(a\mu)\rangle = \langle T(g), a\mu\rangle = \langle aT(g), \mu\rangle \ . \quad \square$$

4. Stochastic operators on $\mathcal{C}_1$

We begin this section by introducing the notion of Markov (stochastic) operators on the von Neumann algebra of all bounded operators on a separable complex Hilbert space. Our notation and approach is similar to [9,10,17]. For the convenience of the reader and the completeness of the paper we repeat necessary definitions and notation.

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a separable (finite or infinite dimensional) complex Hilbert space. As usual the norm is denoted by $\|\cdot\|$ and the Banach algebra of linear and bounded operators on $(\mathcal{H}, \|\cdot\|)$ is denoted by $\mathcal{L}(\mathcal{H}) = \mathcal{C}_\infty$ (without confusion the operator norm in $\mathcal{L}(\mathcal{H})$ will be denoted by $\|\cdot\|$ too). The paper is concerned with positive linear operators acting on an ordered Banach space of trace-class operators on $\mathcal{H}$. For necessary theoretical background the reader is referred to any standard book on operators on Hilbert spaces (e.g. [12,21–23,25–27,29]). The (Hilbert) adjoint operator to A is denoted by A^*. An operator $A \in \mathcal{L}(\mathcal{H})$ is called Hermitian if $A = A^*$ i.e. $\langle Ax, y\rangle = \langle x, Ay\rangle$ holds for all $x, y \in \mathcal{H}$. Equivalently, an operator A is Hermitian if $\langle Ax, x\rangle \in \mathbb{R}$ for any $x \in \mathcal{H}$ (see [29], page 57). Moreover, if A is a Hermitian operator and $\langle Ax, x\rangle \in [0, \infty)$ holds for all $x \in \mathcal{H}$, then we say that A is positive. Clearly, positive operators on $\mathcal{H}$ form a cone in $\mathcal{L}(\mathcal{H})$, denoted by $\mathcal{L}(\mathcal{H})_+$. Each Hermitian operator A may be uniquely decomposed as $A = A_+ - A_-$ (with $A_+A_- = A_-A_+ = 0$), where A_+ and A_- are called respectively the positive and negative part of A. By $|A|$ we mean $A_+ + A_-$. Obviously $|A| \in \mathcal{L}(\mathcal{H})_+$ and it is called the modulus of A. The modulus may be equivalently introduced as $|A| = \sqrt{A^*A}$ (see [12], page 42 or [29], page 63). Having the cone we introduce in $\mathcal{L}(\mathcal{H})$ the following partial order: $A \leq B$ if and only if $B - A \in \mathcal{L}(\mathcal{H})_+$. It is well known that $\mathcal{L}(\mathcal{H})$ endowed with this order is not a (vector) lattice and it does not satisfy the so-called Riesz decomposition property (see [27], page 29 or [25], page 1). A general bounded operator A may be written as $A = B + iC = (B_+ - B_-) + i(C_+ - C_-)$, where B_+, B_-, C_+, C_- are positive. Compact operators play an important role in Hilbert space theory. Let us recall that A is compact if $A(x_n)$ has a (norm) convergent subsequence for each bounded sequence $x_n \in \mathcal{H}$ (or equivalently (see [20], page 207 or [21], page 222) when A is a limit of finite dimensional operators). They form a (closed) ideal in $\mathcal{L}(\mathcal{H})$ which is denoted in our paper by $\mathcal{C}_0$. We say that an operator $X \in \mathcal{L}(\mathcal{H})$ is trace-class if for each (some; see [21], page 230, [23], page 82 or [25], pages 37, 44 for all details) orthonormal base $e_1, e_2, \dots \in \mathcal{H}$ the series $\sum_{j=1} \langle |X| e_j, e_j \rangle < \infty$. The trace is defined as $\sum_{j=1} \langle X e_j, e_j \rangle$ and it is denoted by $\mathrm{tr}(X)$. Then the functional

$$X \to \mathrm{tr}(|X|) = \|X\|_1 \tag{T9}$$

defines (see [21], page 233, [22], page 97, [23], page 93 or [25], page 37) a complete norm (stronger than the operator norm). The trace-class operators form a two sided ideal in $\mathcal{L}(\mathcal{H})$, which is called the Schatten class 1 (see [21], page 231 or [23], page 83) and it is denoted by $\mathcal{C}_1$. It may be easily verified that whenever $\mathcal{H}$ is not finite dimensional then $\mathcal{C}_1$ is not closed in the operator norm in $\mathcal{L}(\mathcal{H})$. It is well known (see [21], page 236 or [23], page 99) that through the dual operation $\langle B, X \rangle = \mathrm{tr}(XB)$, where $B \in \mathcal{C}_0$ and $X \in \mathcal{C}_1$, the dual space to $(\mathcal{C}_0, \|\cdot\|)$ may be identified with $(\mathcal{C}_1, \|\cdot\|_1)$. Further, the dual space to $(\mathcal{C}_1, \|\cdot\|_1)$ is $(\mathcal{L}(\mathcal{H}), \|\cdot\|)$ with dual operation $\langle X, B \rangle = \mathrm{tr}(BX)$, where $B \in \mathcal{C}_\infty$ and $X \in \mathcal{C}_1$. In order to be consistent with generally accepted notation we shall denote $\mathcal{L}(\mathcal{H})$ as $\mathcal{C}_\infty$. In particular, $\mathcal{C}_1$ is not reflexive (unless $\mathcal{H}$ is finite dimensional). The space $\mathcal{C}_1$ is commonly recognized as the noncommutative counterpart of the ℓ^1 space. Since the operators of finite rank are norm dense in $\mathcal{C}_1$ (see [21], page 233 or [23], page 93), and the Hilbert space $\mathcal{H}$ is separable (by our assumption), thus $(\mathcal{C}_1, \|\cdot\|_1)$ is separable too. The following additivity property (sometimes called (AL) condition when we deal with Banach lattices) of the norm $\|\cdot\|_1$ is preserved:

$$\forall_{X_1, X_2 \in \mathcal{C}_1} \ (\ X_1, \ X_2 \geq 0 \quad \Rightarrow \|X_1 + X_2\|_1 = \|X_1\|_1 + \|X_2\|_1). \tag{T10}$$

The cone of all positive trace operators is denoted by $\mathcal{C}_{1+}$ Therefore $(\mathcal{C}, \|\cdot\|_1, \mathcal{C}_{1+})$ becomes an ordered Banach space, even though, it is not a Riesz space (see [25], page 1).

Definition 4.1. A positive operator X from $\mathcal{C}_1$ is called a state if $\mathrm{tr}(X) = 1$. The set of all states is denoted by $\mathcal{D}$ (analog of probability distributions in statistical physics).

It is easy to verify that $\mathcal{D}$ is a convex and closed subset of $\mathcal{C}_1$, for the weak topology (hence for both operator and trace norms). By a direct inspection it can be shown that it is not closed for the weak* topology (if $\dim \mathcal{H} = \infty$).

By $\mathrm{Proj}(\mathcal{H})$ we denote the family of all orthogonal (self-adjoint) projections in $\mathcal{H}$. An operator valued mapping

$$E : \mathcal{B}_\mathbb{R} \mapsto \mathrm{Proj}(\mathcal{H})$$

defined on the σ-algebra $\mathcal{B}_\mathbb{R}$ (in general spectral measures are defined on Borel subsets of the complex plain $\mathbb{C}$) of all Borel subsets of $\mathbb{R}$ is called a spectral measure if

- $E(\emptyset) = 0$,
- $E(\mathbb{R}) = \mathbf{Id}$,
- $E(\bigcup_{j=1}^\infty F_j) =$ s.o.t. $\sum_{j=1}^\infty E(F_j)$ for any sequence of Borel and pairwise disjoint sets $F_j \in \mathcal{B}_\mathbb{R}$, where s.o.t. means the convergence in the strong operator topology,
- $E(F \cap G) = E(F)E(G) = E(G)E(F)$

(see [20], page 273 or [29], page 118). When we fix vectors $g, h \in \mathcal{H}$ then $\mathcal{B}_{\mathbb{R}} \ni F \mapsto \langle E(F)g, h\rangle = E_{g,h}(F)$ becomes a complex valued σ-additive measure. It is well known (Spectral Theorem, see [21], pages 247, 250 or [29], pages 60, 120) that each Hermitian operator $A \in \mathcal{L}(\mathcal{H})_H$ is uniquely determined by its spectral measure E^A. Actually we have

$$A = \int_{-\infty}^{+\infty} tE^A(dt) \ , \text{ or equivalently } \ \forall_{g,h\in\mathcal{H}} \ \langle Ag, h\rangle = \int_{-\infty}^{+\infty} tE^A_{g,h}(dt) \ . \tag{S11}$$

Moreover, we may confine the integration to the spectrum, i.e. the set $\sigma(A) = \{\lambda \in \mathbb{C} : (\lambda I - A)$ is not invertible$\}$, which is included in $\mathbb{R}$ whenever A is Hermitian. The reader is referred to the already cited monographs for other facts on the spectral theory of Hermitian (normal) operators. Here let us only mention (see [29], page 62) that $A^2 = \int_{-\infty}^{+\infty} t^2 E^A(dt)$.

Definition 4.2. A bounded linear operator $S : \mathcal{C}_1 \to \mathcal{C}_1$ is said to be positive if $S(\mathcal{C}_{1+}) \subseteq \mathcal{C}_{1+}$. A positive operator S is called stochastic if for every $X \in \mathcal{C}_{1+}$ we have $\|S(X)\|_1 = \|X\|_1$ (equivalently we may say that $S(\mathcal{D}) \subseteq \mathcal{D}$). The adjoint operator S' acting on $\mathcal{C}_\infty$ will be called Markov (Markovian). The set of all stochastic operators on $\mathcal{C}_1$ is denoted by $\mathcal{S}$.

The notion of quantum stochastic operators is at the core of many monographs (e.g. [1,4,12,26]). The reader is advised to consult them if necessary. Most physical papers concerning noncommutative stochastic dynamics assume a stronger condition on positivity, i.e. complete positivity (see [1,26] for a definition if necessary). We on the other hand do not consider any specific quantum model and hence may consider positivity in a weaker and therefore more general form. The reader is referred to [3,7,11,13,16,18] for the physical background of the problem. A Markov operator S' preserves the (ordered) Banach space $\mathcal{L}(\mathcal{H})_H$ of Hermitian operators, which generate the whole space $\mathcal{L}(\mathcal{H})$. Thus instead of studying the evolution $\{S'^n : n \geq 0\}$ on the whole domain it suffices to focus on the restriction to the Hermitian part. In other words, we shall study $S'^n : \mathcal{L}(\mathcal{H})_H \to \mathcal{L}(\mathcal{H})_H$ for $n \geq 0$.

We omit the obvious proof of the following result.

Lemma 4.3. *The set $\mathcal{S}$ of all stochastic operators on $\mathcal{C}_1$ is convex and a w.o.t. closed subsemigroup of $\mathcal{L}(\mathcal{C}_1, \mathcal{C}_1)$. However it is not closed for the w*.o.t. in the infinite dimensional case.*

Now we give a few examples of stochastic operators.

Example 4.4. Let U be a unitary operator on $\mathcal{H}$. Clearly the operator $S(X) = U^*XU$ is stochastic. Moreover, it is an invertible isometry of $\mathcal{C}_1$.

Example 4.5. Let V be a linear operator on $\mathcal{H}$ such that V^* is isometric. Similarly as above we define $R(X) = V^*XV$. It is easy to check that R is a stochastic (non-invertible in general) operator on $\mathcal{C}_1$.

5. Noether stochastic operators on $\mathcal{C}_1$

Definition 5.1. Let A be a bounded Hermitian operator on $\mathcal{H}$. A stochastic operator $S : \mathcal{C}_1 \mapsto \mathcal{C}_1$ is called Noether with respect to an operator $A \in \mathcal{C}_\infty$ if

$$S'(A) = A \quad \text{and} \quad S'(A^2) = A^2. \tag{N12}$$

The dual operator S' of a Noether stochastic operator S will be called Noether Markov operator (with respect to A).

Given a stochastic operator $S : \mathcal{C}_1 \to \mathcal{C}_1$ its second dual $S'' : \mathcal{C}'_\infty \to \mathcal{C}'_\infty$ preserves the positive cone $\mathcal{C}'_{\infty,+}$ which consists of all continuous linear functionals ξ on $\mathcal{C}_\infty$ satisfying $\xi(A) = \langle A, \xi\rangle \geq 0$ for all positive $A \in \mathcal{C}_\infty$. Clearly the cone $\mathcal{C}'_{\infty,+}$ is weak* (hence) norm closed and nontrivial. Moreover $\mathcal{C}'_{\infty,+} = \overline{\{\lambda X : \lambda \geq 0, X \in \mathcal{S}\}}^{w*}$ (by Goldstein's theorem, see [2], page 158).

We begin the main part of the paper with lemma below.

Lemma 5.2. *Let $S : \mathcal{C}_1 \to \mathcal{C}_1$ be a Noether stochastic operator with respect to a Hermitian bounded operator $A \in \mathcal{C}_\infty$. Then for every continuous function $\Psi : \mathbb{R} \to \mathbb{C}$ we have $S'\Psi(A) = \Psi(A)$.*

Proof. Denote by $C^*(A)$ the smallest C^*-subalgebra of $\mathcal{C}_\infty$, generated by $\{\mathbf{1}, A\}$, where $\mathbf{1} = \mathbf{Id}$ stands for the identity operator. Clearly $C^*(A)$ is commutative with $\{\sum_{j=0}^{n} t_j A^j : t_j \in \mathbb{C}, n \in \mathbb{N}_0\}$ being a dense subalgebra. By the Gelfand–Naimark theorem $C^*(A) = C(K_A)$ for some compact (Hausdorff) topological space K_A (with a little abuse of the notation we will denote the function corresponding to an element $D \in C^*(A)$ by the same letter D; clearly this function is the Gelfand transform $\Gamma(D)$). Let us recall (see [29], pages 27, 32, 35, 56) that the topology on K_A is the weak* topology coming from its predual space $C^*(A)$. If $\xi \in \mathcal{C}'_{\infty,+}$ satisfies $\xi(\mathbf{1}) = \langle \mathbf{1}, \xi\rangle = 1$ then its restriction $\xi|_{C^*(A)}$ may be viewed as a Radon probability measure μ_ξ on K_A. For general $\xi \in \mathcal{C}'_\infty$ the measure μ_ξ is complex valued. We identify $\xi|_{C^*(A)} = \mu_\xi$.

A local linear functional ξ_μ defined on $C^*(A)$, considered over real scalars, and such that $\langle \mathbf{1}, \xi_\mu\rangle = 1$ with the norm $\|\xi_\mu\| = 1$ (on $C^*(A)$) is represented (the Riesz–Markov theorem) by a unique probability (Radon) measure μ on K_A such that $\langle D, \xi_\mu\rangle = \int_{K_A} D(\kappa) d\mu(\kappa)$ and by the Hahn–Banach theorem ξ_μ may be extended on the whole $\mathcal{C}_\infty$ to some norm 1 linear functional ξ (i.e. $\xi|_{C^*(A)} = \xi_\mu$).

Given a norm 1 linear functional ξ, which restricted to $C^*(A)$ represents a probability measure on K_A, we want to look at its image $S''(\xi)$. Because we do not control ξ outside $C^*(A)$, the behavior of $S''(\xi)$ (or its positivity) is not clear at this stage. Let us consider the real part $\mathrm{Re}S''(\xi)$ instead. We notice

$$\langle \mathbf{1}, \mathrm{Re}S''(\xi)\rangle = \mathrm{Re}\langle \mathbf{1}, S''(\xi)\rangle = \mathrm{Re}\langle S'(\mathbf{1}), \xi\rangle = \mathrm{Re}\langle \mathbf{1}, \xi\rangle = 1 \ ,$$

and

$$\|\mathrm{Re}S''(\xi)|_{C^*(A)}\| \leq \|S''(\xi)\| \leq \|\xi\| = 1.$$

Hence $\mathrm{Re}S''(\xi)|_{C^*(A)}$ is represented by a unique probability (Radon) measure on K_A.

If $\xi|_{C^*(A)}$ is multiplicative on $C^*(A)$ then $\mu_\xi = \delta_{\kappa_\xi}$ for some $\kappa_\xi \in K_A$ (see [29], page 35), where δ_{κ_ξ} stands for the Dirac delta measure at κ_ξ. We denote $P(\kappa_\xi, \cdot)$ to be the probability measure on K_A defined by $\mathrm{Re}S''(\xi)|_{C^*(A)}$.

We shall show that $P(\kappa_\xi, \cdot), \kappa_\xi \in K_A$ are well defined. For this let us suppose that $\xi|_{C^*(A)} = \zeta|_{C^*(A)}$ are multiplicative on $C^*(A)$, where $\xi, \zeta \in \mathcal{C}'_\infty$ are norm 1 (then $\kappa_\xi = \kappa_\zeta$). We have to prove $\mathrm{Re}S''(\xi)|_{C^*(A)} = \mathrm{Re}S''(\zeta)|_{C^*(A)}$ (it is well known that if real parts of linear functionals coincide then functionals coincide; hence we will prove that $S''(\xi)|_{C^*(A)} = S''(\zeta)|_{C^*(A)}$). Let us note $S'((A - \xi(A)\mathbf{1})^2 = (A - \xi(A)\mathbf{1})^2$ and

$$\langle (A - \xi(A)\mathbf{1})^2, \xi\rangle = \langle A^2 - 2\xi(A)A + \xi(A)^2, \xi\rangle = 0 = \xi(A^2) - 2\xi(A)\xi(A) + \xi(A)^2 \ .$$

Thus

$$\int_{K_A} (A - \xi(A)\mathbf{1})^2(\kappa) d\mu_{\mathrm{Re}S''(\xi)}(\kappa) = \mathrm{Re}\langle (A - \xi(A)\mathbf{1})^2, S''(\xi)\rangle$$
$$= \mathrm{Re}\langle S'((A - \xi(A)\mathbf{1})^2), \xi\rangle = \int_{K_A} (A - \xi(A)\mathbf{1})^2(\kappa) d\mu_\xi(\kappa) = 0 \ .$$

The measure $\mathrm{Re}S''(\xi)|_{C^*(A)}$ is concentrated on the zero level set

$$Z_\xi = \{\kappa \in K_A : (A - \xi(A)\mathbf{1})^2(\kappa) = 0\} = \{\kappa \in K_A : A(\kappa) = \xi(A)\} \; .$$

We get

$$\int_{K_A} A^n(\tau) d\mu_{\mathrm{Re}S''(\xi)|_{C^*(A)}}(\tau) = \int_{K_A} \xi(A)^n d\mu_{\mathrm{Re}S''(\xi)|_{C^*(A)}}(\tau) = \xi(A)^n$$

for all $n \in \mathbb{N}_0$, as ξ is multiplicative on $C^*(A)$. Since ζ and ξ coincide on $C^*(A)$ thus

$$\int_{K_A} A^n(\tau) d\mu_{\mathrm{Re}S''(\xi)|_{C^*(A)}}(\tau) = \xi(A)^n = \zeta(A)^n = \int_{K_A} A^n(\tau) d\mu_{\mathrm{Re}S''(\zeta)|_{C^*(A)}}(\tau) \; .$$

The functions $A^n, n \geq 0$ are linearly dense in $C(K_A)$, hence $\mathrm{Re}S''(\xi) = \mathrm{Re}S''(\zeta)$ on $C^*(A)$ and as already mentioned $S''(\xi) = S''(\zeta)$ on $C^*(A)$. It follows that $P(\kappa_\xi, \cdot) = P(\kappa_\zeta, \cdot)$.

The weak*–weak* continuity of $\mathcal{C}'_\infty \ni \xi \to S''(\xi) \in \mathcal{C}'_\infty$ is obvious. So is the general remark that $\kappa_{\xi_\alpha} \to \kappa_\xi$ in the topological space K_A if and only if $D(\xi_\alpha) \to D(\kappa_\xi)$ for all $D \in C^*(A) = C(K_A)$. Now, let $\kappa_{\xi_\alpha} \to \kappa_\xi$ in K_A, where $\xi_\alpha, \xi \in \mathcal{C}_\infty$ are multiplicative on $C^*(A)$. Choosing a subnet (if necessary) we may guarantee that $\xi_\alpha \to \widetilde{\xi}$ in $\mathcal{C}'_\infty$ in the weak* topology. Clearly

$$\langle D, \widetilde{\xi} \rangle = \lim_\alpha \langle D, \xi_\alpha \rangle = \lim_\alpha D(\kappa_{\xi_\alpha}) = D(\kappa_\xi) = \langle D, \xi \rangle$$

for all $D \in C^*(A)$. Hence $\kappa_{\widetilde{\xi}} = \kappa_\xi$. In other words we may assume that $\kappa_{\xi_\alpha} \to \kappa_\xi$ in K_A and $\xi_\alpha \to \xi$ in $\mathcal{C}'_\infty$ in the weak* topology. Now if $D \in C(K_A)$ is real valued ($D \in C^*(A)$ is Hermitian) then

$$\begin{aligned} \langle D, P(\kappa_{\xi_\alpha}, \cdot) \rangle &= \mathrm{Re}\langle D, S''(\xi_\alpha) \rangle = \mathrm{Re}\langle S'(D), \xi_\alpha \rangle \to \mathrm{Re}\langle S'(D), \xi \rangle \\ &= \mathrm{Re}\langle D, S''(\xi)) \rangle = \langle D, P(\kappa_\xi, \cdot) \rangle \; . \end{aligned}$$

Thus the Feller transition probability function $\{P(\kappa, \cdot) : \kappa \in K_A\}$ properly defines a Markov operator T on $C(K_A) = C^*(A)$. Clearly $T'\delta_\kappa = P(\kappa, \cdot) = \mathrm{Re}S''(\xi_\kappa)$ for some $\xi_\kappa \in \mathcal{C}'_\infty$. It follows from our construction that $T(A) = A$ and $T(A^2) = A^2$. Indeed, for every Hermitian $D \in C^*(A) = C(K_A)$ and all $\kappa \in K_A$ we have

$$\begin{aligned} T(D)(\kappa) &= \int_{K_A} D(\tau) P(\kappa, d\tau) = \int_{K_A} D(\tau) d\mu_{\mathrm{Re}S''(\xi_\kappa)}(\tau) = \mathrm{Re}\langle D, S''(\xi_\kappa) \rangle \\ &= \mathrm{Re}\langle S'(D), \xi_\kappa \rangle. \end{aligned}$$

Substituting $D = A$ and then $D = A^2$, for all $\kappa \in K_A$ we get

$$T(A)(\kappa) = \mathrm{Re}\langle S'(A), \xi_\kappa \rangle = \mathrm{Re}\langle A, \xi_\kappa \rangle = A(\kappa),$$

and

$$T(A^2)(\kappa) = \mathrm{Re}\langle S'(A^2), \xi_\kappa \rangle = \mathrm{Re}\langle A^2, \xi_\kappa \rangle = A^2(\kappa).$$

By Theorem 3.4 we obtain $T = \mathbf{Id}$ on $C^*(A)$. It follows that $T'\delta_\kappa = \delta_\kappa$ for all $\kappa \in K_A$ or equivalently $S''(\xi)|_{C^*(A)} = \xi|_{C^*(A)}$ as long as $\xi \in \mathcal{C}'_\infty$ is multiplicative on $C^*(A)$. Thus $D(\kappa) = \langle D, \xi_\kappa \rangle = \langle S'(D), \xi_\kappa \rangle$ for all $D \in C^*(A)$.

It remains to prove that $S' : C^*(A) \to C^*(A)$ and $S' = \mathbf{Id}$ on $C^*(A)$. Let us introduce a relation between $\xi, \eta \in \mathcal{C}'_\infty$ writing $\xi \sim_A \eta$ if and only if $\langle D, \xi\rangle = \langle D, \eta\rangle$ for all $D \in C^*(A)$. We have already proved that if $\xi \in \mathcal{C}'_\infty$ is multiplicative on $C^*(A)$ then $\xi \sim_A S''(\xi)$. Define $\mathrm{Fix}_A = \{\xi \in \mathcal{C}'_\infty : S''(\xi) \sim_A \xi\}$. Clearly Fix_A is a weak* closed linear subspace of $\mathcal{C}'_\infty$ as

$$\mathrm{Fix}_A = \bigcap_{D\in C^*(A)} \{\xi \in \mathcal{C}'_\infty : \langle D, \xi\rangle = \langle S'(D), \xi\rangle\},$$

and $K_A \subseteq \mathrm{Fix}_A$. Let $\xi \in \mathrm{Fix}_A$ and μ_ξ be a (complex) σ-additive (Radon) measure on K_A representing ξ on $C^*(A) = C(K_A)$. Namely, $\int_{K_A} D(\kappa)d\mu_\xi(\kappa) = \langle D, \xi\rangle$. By the Hahn–Banach theorem every σ-additive (complex) finite measure μ on K_A is of the form μ_ξ for some $\xi \in \mathcal{C}'_\infty$. We have

$$\int_{K_A} D(\kappa)d\mu_\xi(\kappa) = \langle D, \xi\rangle = \langle S'(D), \xi\rangle = \langle D, S''(\xi)\rangle$$

$$= \int_{K_A} D(\kappa)d\mu_{S''(\xi)}$$

for all $D \in C^*(A)$. We can state this property as $\xi \in \mathrm{Fix}_A$ if and only if $\mu_\xi = \mu_{S''(\xi)}$. Since δ_κ, $\kappa \in K_A$ are linearly dense (in the weak* topology) in the set of finite σ-additive measures, thus $\mu_\xi = \mu_{S''(\xi)}$ for all $\xi \in \mathcal{C}'_\infty$. In particular if $\xi \in \mathcal{C}'_\infty$ satisfies $\xi|_{C^*(A)} = 0$ then $S''(\xi)|_{C^*(A)} = 0$. Hence $S'(C^*(A)) \subseteq C^*(A)$.

We find $S' : C(K_A) = C^*(A) \to C^*(A) = C(K_A)$ being a Markov operator satisfying $S'(A) = A$ and $S'(A^2) = A^2$. Applying Theorem 3.4 we get $S'(D) = D$ for all $D \in C^*(A)$.

By the functional calculus theorem (see [29], page 62) we obtain $S'(\Psi(A)) = T(\Psi(A)) = \Psi(A)$ for all continuous functions $\Psi : \mathbb{R} \to \mathbb{R}$. Applying the same arguments as we used in Proposition 3.2 we have $T(\Psi(A)) = \Psi(A)$ for all continuous complex valued functions $\Psi : \mathbb{R} \mapsto \mathbb{C}$. □

The proof of the next lemma could be shortened if we directly apply the Borel functional calculus theorem (see [21], page 288 or [29], page 119). For the sake of the completeness of the paper we have decided for its full presentation.

Lemma 5.3. *Let $S : \mathcal{C}_1 \to \mathcal{C}_1$ be a Noether stochastic operator with respect to a Hermitian operator $A \in \mathcal{C}_\infty$. Then $S'(E^A(\alpha, \beta]) = E^A(\alpha, \beta]$ for all $\alpha < \beta$, where $A = \int_{\sigma(A)} tE^A(dt)$ is the spectral resolution representation. Moreover, $S'(A_+) = A_+$, $S'(A_-) = A_-$, and $S'(|A|) = |A|$.*

Proof. Clearly $A_+ = \Psi_+(A)$, $A_- = \Psi_-(A)$ and $|A| = \Psi_|(A)$, where $\Psi_+(t) = \max\{0, t\}$, $\Psi_-(t) = -\min\{0, t\}$ and $\Psi_|(t) = |t|$. The S'-invariance of A_+, A_- and $|A|$ follow from Lemma 5.2.

Now define

$$\Psi_{\alpha,\beta,n}(t) = \begin{cases} 0 & : \quad t \le \alpha + 1/n \\ n(t - \alpha - 1/n) & : \quad \alpha + 1/n < t \le \alpha + 2/n \\ 1 & : \quad \alpha + 2/n \le t < \beta \\ n(\beta + 1/n - t) & : \quad \beta \le t < \beta + 1/n \\ 0 & : \quad t \ge \beta + 1/n, \end{cases}$$

where $n \in \mathbb{N}$. Clearly $\Psi_{\alpha,\beta,n}$ are continuous and converge pointwise to $\mathbf{1}_{(\alpha,\beta]}$. By Lemma 5.2 we have $S'(\Psi_{\alpha,\beta,n}(A)) = \Psi_{\alpha,\beta,n}(A)$. Hence for any trace operator $X \in \mathcal{C}_1$ we have

$$\langle S(X), \Psi_{\alpha,\beta,n}(A)\rangle = \langle X, S'\Psi_{\alpha,\beta,n}(A)\rangle = \langle X, \Psi_{\alpha,\beta,n}(A)\rangle .$$

Applying the spectral theorem for any orthonormal base $e_1, e_2, \ldots$ in $\mathcal{H}$ we have

$$\begin{aligned}\langle S(X), \Psi_{\alpha,\beta,n}(A)\rangle &= \sum_{j=1} \langle \Psi_{\alpha,\beta,n}(A)S(X)e_j, e_j\rangle \\ &= \sum_{j=1} \int_{\sigma(A)} \Psi_{\alpha,\beta,n}(t)\langle E^A(dt)S(X)e_j, e_j\rangle \\ &\to \sum_{j=1} \int_{\sigma(A)} \mathbf{1}_{(\alpha,\beta]}(t)\langle E^A(dt)S(X)e_j, e_j\rangle \\ &= \sum_{j=1} \langle \mathbf{1}_{(\alpha,\beta]}(A)S(X)e_j, e_j\rangle = \langle S(X), \mathbf{1}_{(\alpha,\beta]}(A)\rangle = \langle X, S'(\mathbf{1}_{(\alpha,\beta]}(A))\rangle \ .\end{aligned}$$

Similarly we obtain $\langle X, \Psi_{\alpha,\beta,n}(A)\rangle \to \langle X, \mathbf{1}_{(\alpha,\beta]}(A)\rangle$. Hence $\langle X, S'(\mathbf{1}_{(\alpha,\beta]}(A))\rangle = \langle X, \mathbf{1}_{(\alpha,\beta]}(A)\rangle$ for all $X \in \mathcal{C}_1$. It follows that $S'(\mathbf{1}_{(\alpha,\beta]}(A)) = \mathbf{1}_{(\alpha,\beta]}(A)\rangle$. Clearly $\mathbf{1}_{(\alpha,\beta]}(A) = \int_{\sigma(A)} \mathbf{1}_{(\alpha,\beta]}(t)E^A(dt) = E^A((\alpha,\beta])$. Finally we get $S'(E^A((\alpha,\beta])) = E^A((\alpha,\beta])$. □

Before formulating further properties of Noether stochastic operators we first show the below.

Remark 5.4. If

$$\|EXE\|_1 = \|X\|_1, \tag{T13}$$

where E is an orthogonal projection and $X \in \mathcal{C}_1$ is positive, then $EXE = X$. In fact, let $X(\cdot) = \sum_{j=1} \lambda_j \langle \cdot, e_j\rangle e_j \geq 0$, for $\lambda_j \geq 0$ and $e_1, e_2, \ldots$ being eigenvectors of X. Then $EXE(\cdot) = \sum_{j=1} \lambda_j \langle \cdot, Ee_j\rangle Ee_j$ and for any orthonormal basis $\xi_1, \xi_2, \ldots$ in $\mathcal{H}$ we have

$$\begin{aligned}\|X\|_1 = \|EXE\|_1 &= \sum_{k=1} \langle \sum_{j=1} \lambda_j \langle \xi_k, Ee_j\rangle Ee_j, \xi_k\rangle \\ &= \sum_{j=1} \lambda_j \sum_{k=1} |\langle \xi_k, Ee_j\rangle|^2 \leq \sum_{j=1} \lambda_j = \|X\|_1 \ .\end{aligned}$$

Hence $\sum_{k=1} |\langle \xi_k, Ee_j\rangle|^2 = 1$ for every $j = 1, 2, \ldots$. It follows that $\|Ee_j\| = 1$ if $\lambda_j > 0$, and finally $EXE = X$.

In particular, if $S : \mathcal{C}_1 \mapsto \mathcal{C}_1$ is a stochastic operator and

$$\|ES(EXE)E\|_1 = \|S(EXE)\|_1 \tag{T14}$$

holds for some state X and an orthogonal projection E, then $ES(EXE)E = S(EXE)$. Moreover, denoting $E^\perp = \mathbf{Id} - E$ we also have $E^\perp S(EXE)E^\perp = 0$.

We notice also that (T14) holds whenever $S'(E) = E$. Indeed, for every $X \in \mathcal{C}_{1,+}$ then we have

$$\begin{aligned}\|ES(EXE)E\|_1 &= \mathrm{tr}(ES(EXE)E) = \mathrm{tr}(ES(EXE)) = \langle S(EXE), E\rangle = \langle EXE, S'(E)\rangle \\ &= \langle EXE, E\rangle = \mathrm{tr}(EXE) = \|EXE\|_1 \ .\end{aligned}$$

We are in a position to present other preparatory facts.

Corollary 5.5. *If for a fixed orthogonal projection E a stochastic operator S on $\mathcal{C}_1$ has the property that $\|ES(EXE)E\|_1 = \|S(EXE)\|_1$ holds for all states X, then*

$$\text{range } S(EYE) \subseteq E\mathcal{H} \tag{R15}$$

and

$$\ker\, S(EYE) \supseteq (E\mathcal{H})^{\perp}, \tag{K16}$$

for all $Y \in \mathcal{C}_1$.

Proof. By the last remark, the inclusion (R15) is obvious for Y being positive. In general (R15) holds by linearity and positivity of S, and a representation $Y = (X_1 - X_2) + i(X_3 - X_4)$, where $X_1, X_2, X_3, X_4 \in \mathcal{C}_{1,+}$.

In order to prove (K16) let us assume that Y is positive. Then $S(EYE)$ is positive too and therefore it is Hermitian. It follows from (R15) that

$$\ker S(EYE) = \ker S(EYE)^* = (\text{range} S(EYE))^{\perp} \supseteq (E\mathcal{H})^{\perp} = E^{\perp}\mathcal{H}.$$

For a general trace class operator Y we once again use the linearity of S and the previously mentioned representation for Y. □

In the same spirit we obtain

Corollary 5.6. *If E is an orthogonal projection, such that $S'(E) = E$, then for every $B \in \mathcal{C}_\infty$ we have*

$$\text{range}\ S'(EBE) \subseteq E\mathcal{H} \tag{R17}$$

and

$$\ker S'(EBE) \supseteq (E\mathcal{H})^{\perp}. \tag{K18}$$

Proof. Assume first that $B \in \mathcal{C}_{\infty,+}$. Clearly there exists and a positive scalar $r > 0$ such that $0 \le B \le r\mathbf{Id}$, so $0 \le EBE \le rE$. It follows that $S'(EBE) \le rE$. If $h \in (E\mathcal{H})^{\perp}$ then

$$0 \le \langle S'(EBE)h, h\rangle \le \langle rEh, h\rangle = 0\ .$$

By positivity of $S'(EBE)$ we obtain $S'(EBE)h = 0$ and $\ker S'(EBE) \supseteq (E\mathcal{H})^{\perp}$ follows (the property (K18) is proved). As $S'(EBE)$ is positive (so Hermitian) it follows that $\text{range} S'(EBE) \subseteq E\mathcal{H}$. In order to get (R17) and (K18) for general $B \in \mathcal{C}_\infty$ we proceed as before, representing $B = (B_1, -B_2) + i(B_3 - B_4)$, where $B_1, B_2, B_3, B_4 \in \mathcal{C}_{\infty,+}$. □

Now we are in a position to prove the main lemma. For the sake of the completeness of the paper and the convenience of the reader we include a full proof. Alternatively the reader is directed to Lemma 2 in [19] for its origins.

Lemma 5.7. *Let $S : \mathcal{C}_1 \to \mathcal{C}_1$ be a stochastic operator such that $S'(E) = E$ for some orthogonal projection E. Then*

$$S(EXE) = ES(X)E \tag{N19}$$

and

$$S'(EBE) = ES'(B)E \tag{N20}$$

for all $X \in \mathcal{C}_1$ and $B \in \mathcal{C}_\infty$.

Proof. Without loss of generality we may assume that both X, B are positive. The operator $S(EXE) - ES(X)E$ is Hermitian. In order to prove (N19) we notice

$$\begin{aligned}&\|(ES(X)E - S(EXE))^2\|_1 = \operatorname{tr}(ES(X)E - S(EXE))^2 = \\ &= \operatorname{tr}(ES(X)E)^2 - \operatorname{tr}ES(X)ES(EXE) + \operatorname{tr}(S(EXE))^2 - \\ &\operatorname{tr}S(EXE)ES(X)E = 0\end{aligned}$$

as

$$\begin{aligned}&\operatorname{tr}ES(X)ES(EXE) = \langle S(EXE), ES(X)E\rangle = \langle EXE, S'(ES(X)E)\rangle \\ &= \langle X, S'(ES(X)E)\rangle = \langle S(X), ES(X)E\rangle = \operatorname{tr}ES(X)ES(X)E = \operatorname{tr}(ES(X)E)^2 ,\end{aligned}$$

and

$$\begin{aligned}&\operatorname{tr}S(EXE)ES(X)E = \langle S(X), ES(EXE)E\rangle = \langle X, S'(ES(EXE)E)\rangle \\ &= \langle EXE, S'(ES(EXE)E)\rangle = \langle S(EXE), ES(EXE)E\rangle \\ &= \langle S(EXE), S(EXE)\rangle = \operatorname{tr}S(EXE)^2 .\end{aligned}$$

Now identity (N20), $S'(EBE) = ES'(B)E$, easily follows from the dual operation. In fact, for any $X \in \mathcal{C}_1$ we have

$$\langle X, S'(EBE)\rangle = \langle ES(X)E, B\rangle = \langle S(EXE), B\rangle = \langle X, ES'(B)E\rangle. \quad \square$$

Remark 5.8. Let S be a stochastic operator on $\mathcal{C}_1$ and E_1, E_2 be orthogonal projections on $\mathcal{H}$ such that $E_1E_2 = E_2E_1 = 0$. If $S'(E_1) = E_1$ and $S'(E_2) = E_2$ then for every $X \in \mathcal{C}_1$ and $B \in \mathcal{C}_\infty$ we have

$$S(E_1XE_2 + E_2XE_1) = E_1S(X)E_2 + E_2S(X)E_1$$

and

$$S'(E_1BE_2 + E_2BE_1) = E_1S'(B)E_2 + E_2S'(B)E_1 .$$

In fact, $E = E_1 + E_2$ is an S' invariant projection. By the previous lemma $S(EXE) = ES(X)E$. It follows from linearity that

$$\begin{aligned}&S(E_1XE_1) + S(E_2XE_2) + S(E_1XE_2 + E_2XE_1) \\ &= E_1S(X)E_1 + E_2S(X)E_2 + E_1S(X)E_2 + E_2S(X)E_1 .\end{aligned}$$

Then $S(E_1XE_2 + E_2XE_1) = E_1S(X)E_2 + E_2S(X)E_1$. The second identity is an easy consequence of the dual argument.

Now we formulate the main result of the paper.

Theorem 5.9. *Let S be a stochastic operator on $\mathcal{C}_1$ and $A \in \mathcal{C}_\infty$ be a positive bounded operator (observable). Then the following conditions are equivalent:*

(1) $S(A^{1/2}XA^{1/2}) = A^{1/2}S(X)A^{1/2}$ for all $X \in \mathcal{C}_1$,
(2) $S'(A^{1/2}BA^{1/2}) = A^{1/2}S'(B)A^{1/2}$ for all operators $B \in \mathcal{C}_\infty$,

(3) $S'(A^m) = A^m$ *for all positive integers* m*,*
(4) $S'(A) = A$ *and* $S'(A^2) = A^2$*,*
(5) $S(E^A(G)XE^A(G)) = E^A(G)S(X)E^A(G)$ *for all* $X \in \mathcal{C}_1$ *and any Borel* $G \subseteq \mathbb{R}$*,*
(6) $S'(E^A(G)BE^A(G)) = E^A(G)S'(B)E^A(G)$ *for all* $B \in \mathcal{C}_\infty$ *and any Borel set* $G \subseteq \mathbb{R}$*,*
(7) $S(\Psi(A)X\Psi(A)) = \Psi(A)S(X)\Psi(A)$ *for all* $X \in \mathcal{C}_1$ *and any continuous function* $\Psi : \mathbb{R} \to \mathbb{R}$*.*

Proof. (1) ⇔ (2) Easily follows from the dual operation.

(2) ⇒ (3) Let $B = \mathbf{Id}$. Then

$$S'(A) = S'(A^{1/2}\mathbf{Id}A^{1/2}) = A^{1/2}S'(\mathbf{Id})A^{1/2} = A\ .$$

By induction, for $m \geq 1$, we have

$$S'(A^m) = S'(A^{1/2}A^{m-1}A^{1/2}) = A^{1/2}S'(A^{m-1})A^{1/2} = A^{1/2}A^{m-1}A^{1/2} = A^m\ .$$

(3) ⇒ (4) is obvious.

(4) ⇒ (5) Let

$$\mathcal{G} = \{G \in \mathcal{B}_\mathbb{R} : S'(E^A(G)) = E^A(G)\}.$$

For every pair of real numbers $\alpha < \beta$, by our Lemma 5.3, the interval $(\alpha, \beta] \in \mathcal{G}$. Similarly as in the proof of Lemma 5.3 we infer that $\mathcal{G}$ is a monotone class. Obviously, by the additivity of the spectral measure, the finite unions (forming an algebra of sets) $\bigcup_{m=1}^{M}(\alpha_m, \beta_m] \in \mathcal{G}$. It follows from basic measure theory that $\mathcal{G} = \mathcal{B}_\mathbb{R}$. Now (5) follows from Lemma 5.7 (N19).

(5) ⇔ (6) directly follows from properties of the dual operation.

(5) ⇒ (7) Let $X \in \mathcal{D}$ be a fixed state. Let us choose a sequence of partitions $\alpha_1^{[n]} < \alpha_2^{[n]} < \cdots < \alpha_{m_n}^{[n]}$ satisfying $\alpha_1^{[n]} \leq \inf\sigma(A)$ and $\alpha_{m_n}^{[n]} \geq \sup\sigma(A)$ and

$$\lim_{n\to\infty} \sup_{1\leq k<m_n} |\alpha_{k+1}^{[n]} - \alpha_k^{[n]}| = 0\ .$$

By linearity we have

$$\begin{aligned}
&S((\sum_{j=1}^{m_n-1} \Psi(\alpha_j^{[n]})E^A((\alpha_j^{[n]}, \alpha_{j+1}^{[n]}]))X(\sum_{k=1}^{m_n-1} \Psi(\alpha_k^{[n]})E^A((\alpha_k^{[n]}, \alpha_{k+1}^{[n]}])))\\
&= S(\sum_{j,k=1}^{m_n-1} \Psi(\alpha_j^{[n]})\Psi(\alpha_k^{[n]})E^A((\alpha_j^{[n]}, \alpha_{j+1}^{[n]}])XE^A((\alpha_k^{[n]}, \alpha_{k+1}^{[n]}]))\\
&= \sum_{j=1}^{m_n-1} S(\Psi^2(\alpha_j^{[n]})E^A((\alpha_j^{[n]}, \alpha_{j+1}^{[n]}])XE^A((\alpha_j^{[n]}, \alpha_{j+1}^{[n]}]))\\
&+ S(\sum_{1\leq j<k\leq m_n-1} \Psi(\alpha_j^{[n]})\Psi(\alpha_k^{[n]})(E^A((\alpha_j^{[n]}, \alpha_{j+1}^{[n]}])XE^A((\alpha_k^{[n]}, \alpha_{k+1}^{[n]}]))\\
&+ E^A((\alpha_k^{[n]}, \alpha_{k+1}^{[n]}])XE^A((\alpha_j^{[n]}, \alpha_{j+1}^{[n]}]))\\
&= \sum_{j=1}^{m_n-1} \Psi^2(\alpha_j^{[n]})E^A((\alpha_j^{[n]}, \alpha_{j+1}^{[n]}])S(X)E^A((\alpha_j^{[n]}, \alpha_{j+1}^{[n]}])\\
&+ \sum_{1\leq j<k\leq m_n-1} \Psi(\alpha_j^{[n]})\Psi(\alpha_k^{[n]})(E^A((\alpha_j^{[n]}, \alpha_{j+1}^{[n]}])S(X)E^A((\alpha_k^{[n]}, \alpha_{k+1}^{[n]}])
\end{aligned}$$

$$+ E^A((\alpha_k^{[n]}, \alpha_{k+1}^{[n]}])S(X)E^A((\alpha_j^{[n]}, \alpha_{j+1}^{[n]}]))$$
$$= (\sum_{j=1}^{m_n-1} \Psi(\alpha_j^{[n]})E^A((\alpha_j^{[n]}, \alpha_{j+1}^{[n]}]))S(X)(\sum_{k=1}^{m_n-1} \Psi(\alpha_k^{[n]})E^A((\alpha_k^{[n]}, \alpha_{k+1}^{[n]}])).$$

By the Spectral Theorem the Riemann sums converge

$$\lim_{n\to\infty} \sum_{j=1}^{m_n-1} \Psi(\alpha_j^{[n]})E^A((\alpha_j^{[n]}, \alpha_{j+1}^{[n]}]) = \Psi(A)$$

and the convergence is in the sup norm. By the Borel functional calculus theorem

$$B(\sigma(A)) \ni f \mapsto \int f dE^A \in \mathcal{C}_\infty$$

is a C^*-homomorphism (see [21], page 288 or [29], page 119).

Let us abbreviate $\sum_{j=1}^{m_n-1} \Psi(\alpha_j^{[n]})E^A((\alpha_j^{[n]}, \alpha_{j+1}^{[n]}]) = B_n$. Applying the basic inequality $\|D_1 Y D_2\|_1 \le \|D_1\|\|Y\|_1\|D_2\|$, where $Y \in \mathcal{C}_1$ and $D_1, D_2 \in \mathcal{C}_\infty$ (see [23], page 98) we have

$$\|B_n Y B_n - \Psi(A)Y\Psi(A)\|_1 \le \|B_n X B_n - \Psi(A)Y B_n\|_1 + \|\Psi(A)Y B_n - \Psi(A)Y\Psi(A)\|_1$$
$$\le \|B_n - \Psi(A)\|\|Y\|_1\|B_n\| + \|\Psi(A)\|\|Y\|_1\|B_n - \Psi(A)\| \to 0$$

as $n \to \infty$. The operator S is continuous for the trace norm $\|\cdot\|_1$. Thus

$$S(\Psi(A)X\Psi(A)) = \lim_{n\to\infty} S(B_n X B_n) = \lim_{n\to\infty} B_n S(X) B_n = \Psi(A)S(X)\Psi(A).$$

(7) ⇒ (1) Substitute $\Psi(t) = \sqrt{t}$ for nonnegative t. □

Remark 5.10. If a positive operator A is identified with a measurement operator $\mathcal{C}_1 \ni X \mapsto AXA = M_A(X) \in \mathcal{C}_1$, then the condition (1) in our Theorem 5.9 may be reformulated as

$$[S, M_{A^{1/2}}] = 0,$$

which directly corresponds to its commutative counterpart [BF0]. Let us finally add that directly from trace properties we have $\mathrm{tr} M_{A^{1/2}}(X) = \mathrm{tr}(A^{1/2}XA^{1/2}) = \mathrm{tr}(XA)$. Therefore our restriction to a positive observable A does not essentially harm generality. In fact, if A is Hermitian and bounded then there exists a positive α such that $\alpha\mathbf{Id} + A \ge 0$. Clearly $S'(\alpha\mathbf{Id} + A) = \alpha\mathbf{Id} + A$ and $S'(\alpha\mathbf{Id} + A)^2 = (\alpha\mathbf{Id} + A)^2$ if and only if S is Noether with respect to A. Moreover $\mathrm{tr} XA = \mathrm{tr}(X(\alpha\mathbf{Id} + A)) - \alpha$ for all states X. Hence positive observables suffice.

6. Noether one-parameter stochastic semigroups on $\mathcal{C}_1$

Definition 6.1. A stochastic one-parameter semigroup $\mathfrak{S}$ on $\mathcal{C}_1$ is a family of stochastic operators $\mathfrak{S} = \{S_t : t \ge 0\} \subseteq \mathcal{S}$ satisfying

(i) $S_t S_s = S_{t+s}$ for all $t, s \ge 0$,
(ii) $S_0 = I$,
(iii) $[0, \infty) \ni t \mapsto S_t(X) \in \mathcal{C}_1$ is continuous for each $X \in \mathcal{C}_1$.

The (infinitesimal) generator $\mathfrak{s}$ of $\mathfrak{S}$ is defined by the formula

$$\mathfrak{s}(X) = \lim_{t\to 0^+} \frac{S_t(X) - X}{t} = \frac{d}{dt} S_t(X)|_{t=0},$$

where the domain of $\mathfrak{s}$, $D(\mathfrak{s})$, is the set of all $X \in \mathcal{C}_1$ for which the limit defined above exists (in the $\|\cdot\|_1$ norm). Clearly S_t restricted to the Hermitian part of $\mathcal{C}_1$ are positive linear contractions. It follows that on the whole Schatten class $\mathcal{C}_1$ the semigroup is norm bounded. Hence the Hille–Yoshida Theorem (see [14], pages 14–18) is applicable and $\mathfrak{s}$ is densely defined, and closed (i.e. if $(X_n, \mathfrak{s}(X_n)) \to (X, Y)$ in the product topology, where $X_n \in D(\mathfrak{s})$, then $X \in D(\mathfrak{s})$ and $\mathfrak{s}(X) = Y$).

Definition 6.2. Let A be a bounded Hermitian operator on $\mathcal{H}$. A stochastic one-parameter semigroup $\mathfrak{S} = \{S_t : t \geq 0\}$, on a Schatten class $\mathcal{C}_1$, is called Noether with respect to an operator $A \in \mathcal{C}_\infty$ if

$$S_t'(A) = A \quad \text{and} \quad S_t'(A^2) = A^2 \quad \text{for all} \quad t \geq 0. \tag{N21}$$

The final theorem of our paper characterizes Noether stochastic semigroups.

Theorem 6.3. *Let $\{S_t : t \geq 0\}$ be a strongly continuous semigroup of stochastic operators on a Schatten class $\mathcal{C}_1$ and $\mathfrak{s}$ be its (infinitesimal) generator with domain $D(\mathfrak{s})$. If $A \in \mathcal{C}_\infty$ is a positive operator then the following conditions are equivalent:*

(1) $\{S_t : t \geq 0\}$ is a Noether semigroup with respect to an operator $A \in \mathcal{C}_\infty$,
(2) $\frac{d}{dt}\langle S_t(X), A\rangle = \frac{d}{dt}\langle S_t(X), A^2\rangle = 0$ for every $X \in \mathcal{D}$,
(3) $\frac{d}{dt}\langle S_t(X), f(A)\rangle = 0$ for every $X \in \mathcal{D}$ and all polynomials $f : \mathbb{R} \mapsto \mathbb{R}$,
(4) $S_t(A^{1/2}XA^{1/2}) = A^{1/2}S_t(X)A^{1/2}$ for every $X \in \mathcal{C}_1$ and all $t \geq 0$,
(5) $S_t'(A^{1/2}BA^{1/2}) = A^{1/2}S_t'(B)A^{1/2}$ for every operator $B \in \mathcal{C}_\infty$ and all $t \geq 0$,
(6) for every $G \in \mathcal{B}_\mathbb{R}$ and all $X \in D(\mathfrak{s})$ we have $E^A(G)XE^A(G) \in D(\mathfrak{s})$ and $\mathfrak{s}(E^A(G)XE^A(G)) = E^A(G)\mathfrak{s}(X)E^A(G)$, i.e. $[\mathfrak{s}, M_{E^A(G)}] = 0$,
(7) for all $X \in D(\mathfrak{s})$ we have $A^{1/2}XA^{1/2} \in D(\mathfrak{s})$ and $\mathfrak{s}(A^{1/2}XA^{1/2}) = A^{1/2}\mathfrak{s}(X)A^{1/2}$, i.e. $[\mathfrak{s}, M_{A^{1/2}}] = 0$.

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (1) follow directly from Theorem 5.9 and general properties of the dual operation.

(1) ⇒ (6) It follows from Theorem 5.9 that

$$S_t(E^A(G)XE^A(G)) = E^A(G)S_t(X)E^A(G)$$

holds true for all Borel $G \in \mathcal{B}_\mathbb{R}$, all $X \in \mathcal{C}_1$, and all $t \geq 0$. Let $X \in D(\mathfrak{s})$. We have

$$\begin{aligned} &\lim_{t\to 0^+} \frac{S_t(E^A(G)XE^A(G)) - E^A(G)XE^A(G)}{t} \\ &= \lim_{t\to 0^+} \frac{E^A(G)S_t(X)E^A(G) - E^A(G)XE^A(G)}{t} \\ &= \lim_{t\to 0^+} E^A(G)\left(\frac{S_t(X) - X}{t}\right)E^A(G) = E^A(G)\mathfrak{s}(X)E^A(G). \end{aligned}$$

It follows that $E^A(G)XE^A(G) \in D(\mathfrak{s})$ and $\mathfrak{s}(E^A(G)XE^A(G)) = E^A(G)\mathfrak{s}(X)E^A(G)$, or simply

$$[\mathfrak{s}, M_{E^A(G)}] = 0.$$

Let us notice that, by the same arguments as before (see Remark 5.8), for disjoint Borel sets $G_1, G_2 \subseteq \mathbb{R}$, we can prove

$$\begin{aligned}&\mathfrak{s}(E^A(G_1)XE^A(G_2) + E^A(G_2)XE^A(G_1))\\&= E^A(G_1)\mathfrak{s}(X)E^A(G_2) + E^A(G_2)\mathfrak{s}(X)E^A(G_1)\ .\end{aligned}$$

(6) ⇒ (7) Let $X \in D(\mathfrak{s})$ be fixed. Repeating the methods from the proof of Theorem 5.9 let us choose a relevant sequence of partitions $\alpha_1^{[n]} < \alpha_2^{[n]} < \cdots < \alpha_{m_n}^{[n]}$ satisfying

$$\lim_{n\to\infty} \sup_{1\le k<m_n} |\alpha_{k+1}^{[n]} - \alpha_k^{[n]}| = 0\ .$$

By linearity we have

$$\begin{aligned}&\mathfrak{s}((\sum_{j=1}^{m_n-1} \sqrt{\alpha_j^{[n]}}E^A((\alpha_j^{[n]}, \alpha_{j+1}^{[n]}]))X(\sum_{k=1}^{m_n-1} \sqrt{\alpha_k^{[n]}}E^A((\alpha_k^{[n]}, \alpha_{k+1}^{[n]}])))\\&= \sum_{j,k=1}^{m_n-1} \sqrt{\alpha_j^{[n]}}\sqrt{\alpha_k^{[n]}}\mathfrak{s}(E^A((\alpha_j^{[n]}, \alpha_{j+1}^{[n]}])XE^A((\alpha_k^{[n]}, \alpha_{k+1}^{[n]}]))\\&= (\sum_{j=1}^{m_n-1} \sqrt{\alpha_j^{[n]}}E^A((\alpha_j^{[n]}, \alpha_{j+1}^{[n]}]))\mathfrak{s}(X)(\sum_{k=1}^{m_n-1} \sqrt{\alpha_k^{[n]}}E^A((\alpha_k^{[n]}, \alpha_{k+1}^{[n]}]))\\&\to A^{1/2}\mathfrak{s}(X)A^{1/2}.\end{aligned}$$

It follows from closedness (see [14], pages 13–18) of $\mathfrak{s}$ that $A^{1/2}XA^{1/2} \in D(\mathfrak{s})$ and $\mathfrak{s}(A^{1/2}XA^{1/2}) = A^{1/2}\mathfrak{s}(X)A^{1/2}$.

(7) ⇒ (4) Applying the Hille–Yoshida theorem (see [14], page 18) for each $t \ge 0$ and every $X \in \mathcal{C}_1$ we have

$$\begin{aligned}S_t(X) &= \lim_{\lambda\to\infty} \exp(t(\lambda^2(\lambda - \mathfrak{s})^{-1} - \lambda\mathbf{Id}))(X)\\&= \lim_{\lambda\to\infty} \exp(-\lambda t)\sum_{n=0}^{\infty} \frac{(t\lambda^2(\lambda - \mathfrak{s})^{-1})^n(X)}{n!}.\end{aligned}$$

Now let us suppose that $Y \in D(\mathfrak{s})$. Then by (7) we get $(\lambda - \mathfrak{s})(A^{1/2}YA^{1/2}) = A^{1/2}((\lambda - \mathfrak{s})(Y))A^{1/2}$. Substituting $Y = (\lambda - \mathfrak{s})^{-1}(X)$ we obtain

$$(\lambda - \mathfrak{s})(A^{1/2}(\lambda - \mathfrak{s})^{-1}(X)A^{1/2}) = A^{1/2}((\lambda - \mathfrak{s})(\lambda - \mathfrak{s})^{-1}(X))A^{1/2} = A^{1/2}XA^{1/2}.$$

Hence

$$A^{1/2}((\lambda - \mathfrak{s})^{-1}(X))A^{1/2} = (\lambda - \mathfrak{s})^{-1}(A^{1/2}XA^{1/2}).$$

Iterating the above property for every natural n we have

$$A^{1/2}[t\lambda^2(\lambda - \mathfrak{s})^{-1}]^n(X)A^{1/2} = [t\lambda^2(\lambda - \mathfrak{s})^{-1}]^n(A^{1/2}XA^{1/2}).$$

It follows that,

$$\begin{aligned} S_t(A^{1/2}XA^{1/2}) &= \lim_{\lambda\to\infty} \exp(t(\lambda^2(\lambda-\mathfrak{s})^{-1} - \lambda\mathbf{Id}))(A^{1/2}XA^{1/2}) \\ &= \lim_{\lambda\to\infty} \exp(-\lambda t)\sum_{k=0}^{\infty} \frac{(t\lambda^2(\lambda-\mathfrak{s})^{-1})^n(A^{1/2}XA^{1/2})}{n!} \\ &= A^{1/2}\lim_{\lambda\to\infty} \exp(-\lambda t)\sum_{k=0}^{\infty} \frac{(t\lambda^2(\lambda-\mathfrak{s})^{-1})^n(X)}{n!}A^{1/2} \\ &= A^{1/2}S_t(X)A^{1/2}. \quad \square \end{aligned}$$

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References

[1] R. Alicki, M. Fannes, Quantum Dynamical Systems, Oxford University Press, 2001.
[2] C.D. Aliprantis, O. Burkinshaw, Positive Operators, Springer, 2006.
[3] A. Arias, A. Gheondea, S. Gudder, Fixed points of quantum operations, J. Math. Phys. 43 (12) (2002) 5872–5881.
[4] W. Arveson, Noncommutative Dynamics and E-Semigroups, Springer-Verlag, 2003.
[5] S. Attal, Markov chains and dynamical systems: the open system point of view, Commun. Stoch. Anal. 4 (4) (2010) 523–540.
[6] J. Baez, Network theory, http://math.ucr.edu/home/baez/networks/.
[7] J.C. Baez, J.D. Biamonte, A course on quantum techniques for stochastic mechanics, arXiv:1209.3632, 17 September 2012.
[8] J.C. Baez, B. Fong, A Noether theorem for Markov processes, J. Math. Phys. 54 (1) (2013) 013301.
[9] W. Bartoszek, B. Kuna, On residualities in the set of Markov operators on $\mathcal{C}_1$, Proc. Amer. Math. Soc. 133 (2005) 2119–2129.
[10] W. Bartoszek, B. Kuna, Strong mixing Markov semigroups on $\mathcal{C}_1$ are meager, Colloq. Math. 105 (2) (2006) 311–317.
[11] J. Biamonte, Quantum vs stochastic al la (Baez–Fong) Noether's theorem, http://forum.azimuthproject.org/discussion/1221/quantum-vs-stoch.
[12] O. Bratteli, D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics 1, Springer Verlag, 1979.
[13] N. Byers, E. Noether's discovery of the deep connection between symmetries and conservation laws, arXiv:physics/9807044.
[14] J.A. Goldstein, Semigroups of Linear Operators and Applications, Oxford University Press, 1985.
[15] J.E. Gough, T.S. Ratiu, O.G. Smolyanov, Noether's theorem for dissipativequantum dynamical semi-groups, J. Math. Phys. 56 (2015) 022108.
[16] S. Gudder, Quantum Markov chains, J. Math. Phys. 49 (2008) 072105.
[17] B. Kuna, On residualities in the set of Markov continuous semigroups on $\mathcal{C}_1$, Demonstratio Math. 39 (2006) 439–453.
[18] C. Liu, N. Petulante, On limiting distributions of quantum Markov chains, Int. J. Math. Math. Sci. 2011 (2011) 740816, http://dx.doi.org/10.1155/2011/740816.
[19] A. Łuczak, Ergodic projection for quantum dynamical semigroups, Internat. J. Theoret. Phys. 34 (8) (1995) 1533–1540.
[20] W. Mlak, Wstep do teorii przestrzeni Hilberta, Biblioteka Matematyczna, vol. 35, PWN, 1970.
[21] M. Reed, B. Simon, Methods of Modern Mathematical Physics, vol. 1, Functional Analysis, Academic Press, 1972.
[22] J.R. Retherford, Hilbert Space: Compact Operators and the Trace Theorem, London Mathematical Society Student Texts, vol. 27, 1993.
[23] J.R. Ringrose, Compact Non-Selfadjoint Operators, Van Nostrand, 1971.
[24] H.H. Schaefer, Banach Lattices and Positive Operators, Springer, 1974.
[25] B. Simon, Trace Ideals and Their Applications, Cambridge Univ. Press, 1979.
[26] E. Størmer, Positive Linear Maps of Operator Algebras, Springer, 2013.
[27] M. Takesaki, Theory of Operator Algebras I, Springer, 1979.
[28] R. Zaharopol, Invariant Probabilities of Transition Functions, Springer, 2014.
[29] K. Zhu, An Introduction to Operator Algebras, CRC Press, 1993.