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A note on simple bifurcation of equilibrium forms of an elastic rod on a deformable foundation

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To Professor José M.R. Sanjurjo, our excellent friend who has contributed so much to topology and its applications, on the occasion of his 70th birthday.

Abstract

We study bifurcation of equilibrium states of an elastic rod on a two-parameter Winkler foundation. In [5] the existence of simple bifurcation points was proved by the use of the Crandall-Rabinowitz theorem. In this paper we want to present an alternative proof of this fact based on the Krasnosielski theorem, which seems to be simpler than that of [5].

key words: Bifurcation, buckling, Brouwer degree, Winkler foundation
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running head: A note on simple bifurcation of an elastic rod

1 Introduction

In this work we will be concerned with the study of simple bifurcation of equilibrium forms of an isotropic elastic beam on a two-parameter Winkler foundation, compressed by forces at the ends. We have been working under the assumption that the beam is free at the left end, and simply supported at the right end (see Fig. 1).

In [5] the authors and N. Waterstraat and A. Zgorzelska derived the equations describing the behaviour of the rod by using Euler's method. Namely they showed that equilibrium forms of the rod satisfy the boundary value problem

$$\begin{cases} x^{(4)} + \alpha x'' + \beta x - \gamma x^3 - 3x''^3 - 12x'x''x''' + \\ -3x'^2\left(x^{(4)} - \frac{\alpha}{2}x''\right) = 0, & \text{in } [-r, r], \\ x'(-r) = x'''(-r) = 0, \\ x(r) = x''(r) = 0, \end{cases} \tag{1}$$

where $\alpha > 0$ is a parameter of the compressive force, $\beta > 0$ and $\gamma > 0$ are parameters of the elastic foundation, $l = 2r$ denotes the length of the rod. Moreover, they found and corrected a mistake in the earlier model proposed by A. Borisovich and J. Dymkowska in [2]. Similar models for buckling of an elastic rod were also investigated for example in [1, 3, 4].

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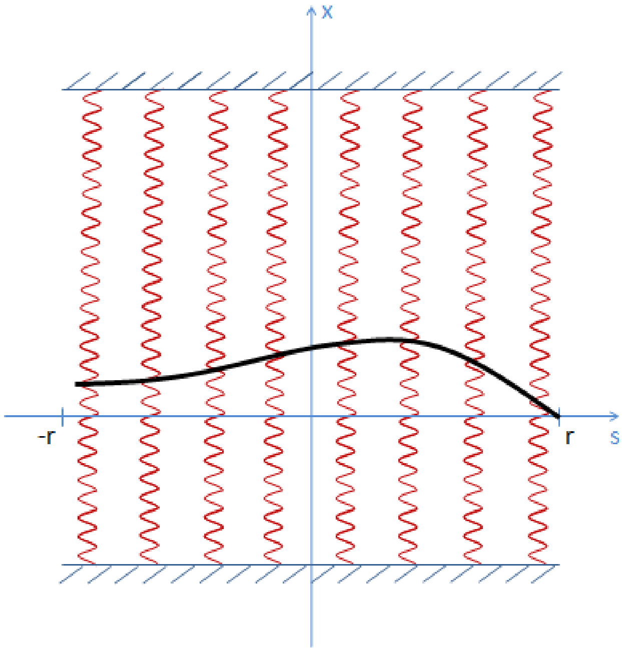

Figure 1: A buckling of an elastic beam

Let us denote by $N(\alpha,\beta)$ the space of solutions of the linear boundary value problem

$$\begin{cases} x^{(4)} + \alpha x'' + \beta x = 0, & \text{in } [-r,r], \\ x'(-r) = x'''(-r) = 0, \\ x(r) = x''(r) = 0. \end{cases} \tag{2}$$

Set

$$Z = \{(\alpha,\beta) \in \mathbb{R}^2_+ : 4\beta \leq \alpha^2\}.$$

For each $m \in \mathbb{N}$ we define

$$l_m = \{(\alpha,\beta) \in Z \colon \beta = -c_m\alpha - c_m^2\},$$

where

$$c_m = -\left(\frac{\pi}{r}\right)^2 \left(\frac{2m-1}{4}\right)^2. \tag{3}$$

Theorem 1.1 (see [2, 7]) *For each $(\alpha,\beta) \in \mathbb{R}^2_+$ one of the following three cases hold:*

(i) If the point (α,β) does not belong to any ray l_m, then

$$\dim N(\alpha,\beta) = 0$$

and the linear boundary value problem (2) possesses only the trivial solution.

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(ii) If the point (α, β) belongs to one and only one ray l_m, then

$$\dim N(\alpha, \beta) = 1$$

and $N(\alpha, \beta)$ is generated by

$$e_m(s) = \sqrt{2} \cos\left(\sqrt{-c_m}(s + r)\right).$$

(iii) If the point (α, β) belongs to the intersection of two rays l_{m_1} and l_{m_2}, then

$$\dim N(\alpha, \beta) = 2$$

and the two linearly independent functions

$$e_{m_1}(s) = \sqrt{2} \cos\left(\sqrt{-c_{m_1}}(s + r)\right)$$

and

$$e_{m_2}(s) = \sqrt{2} \cos\left(\sqrt{-c_{m_2}}(s + r)\right)$$

are a basis of $N(\alpha, \beta)$.

In [5] we proved that a necessary and sufficient condition for bifurcation of the beam is that $\dim N(\alpha, \beta) \neq 0$, which shows that the parameter γ has no influence on the occurrence of bifurcation. Our proof was divided into three steps. It follows from the implicit function theorem that there is no bifurcation at points $(0, \alpha, \beta, \gamma)$ if $\dim N(\alpha, \beta) = 0$. The existence of simple bifurcation points ($\dim N(\alpha, \beta) = 1$) was studied by applying the Crandall-Rabinowitz theorem, and the existence of multiple bifurcation points ($\dim N(\alpha, \beta) = 2$) by the use of topological degree. In this paper we want to present an alternative proof for the case $\dim N(\alpha, \beta) = 1$ based on the Krasnosielski theorem, which seems to be simpler than that of [5]. In Section 2 we set up notation and terminology. Section 3 contains a new proof of the existence of simple bifurcation points of (1).

2 Preliminaries

Let us denote by X the Banach space

$$X = \{x \in C^4[-r, r] \colon x'(-r) = x'''(-r) = 0, \quad x(r) = x''(r) = 0\}$$

with the standard norm

$$\|x\|_X = \sum_{k=0}^{4} \max_{s \in [-r, r]} |x^{(k)}(s)|,$$

and we will denote by Y the space $C[-r, r]$ with the maximum norm

$$\|y\|_Y = \max_{s \in [-r, r]} |y(s)|.$$

Let a functional $E \colon X \times \mathbb{R}^3_+ \to \mathbb{R}$ be given by

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$$
\begin{aligned}
E(x,\alpha,\beta,\gamma) =& \frac{1}{4r}\int_{-r}^{r}\left(x''(s)^2 - 3x'(s)^2x''(s)^2\right) ds \\
&- \frac{1}{4r}\int_{-r}^{r}\left(\alpha x'(s)^2 + \frac{\alpha}{4}x'(s)^4\right) ds \\
&+ \frac{1}{4r}\int_{-r}^{r}\left(\beta x(s)^2 - \frac{\gamma}{2}x(s)^4\right) ds.
\end{aligned} \tag{4}
$$

We call E *the energy functional.* E is the approximating functional of the total potential energy of the system composed of the rod and the foundation. Roughly speaking, to derive the formula (4), we write down the total potential energy as a function of x and we omit the terms of higher order than 4. (For the exact proof we refer the reader to [5].) An easy computation shows that

$$
\begin{aligned}
E'_x(x,\alpha,\beta,\gamma)h =& \frac{1}{2r}\int_{-r}^{r}\left(\beta x(s) - \gamma x(s)^3\right) h(s)ds \\
&- \frac{1}{2r}\int_{-r}^{r}\left(\alpha x'(s) + \frac{\alpha}{2}x'(s)^3 + 3x'(s)x''(s)^2\right) h'(s)ds \\
&+ \frac{1}{2r}\int_{-r}^{r}\left(x''(s) - 3x'(s)^2x''(s)\right) h''(s)ds
\end{aligned} \tag{5}
$$

for all $x, h \in X$ and $\alpha, \beta, \gamma \in \mathbb{R}_+$. Moreover, integrating by parts in (5), we get

$$
\begin{aligned}
E'_x(x,\alpha,\beta,\gamma)h =& \ \frac{1}{2r}\int_{-r}^{r}\left(x^{(4)}(s) + \alpha x''(s) + \beta x(s)\right) h(s)ds \\
&- \frac{1}{2r}\int_{-r}^{r}\left(\gamma x(s)^3 + 3x''(s)^3 + 12x'(s)x''(s)x'''(s)\right) h(s)ds \\
&- \frac{1}{2r}\int_{-r}^{r} 3x'(s)^2\left(x^{(4)}(s) - \frac{\alpha}{2}x''(s)\right) h(s)ds.
\end{aligned} \tag{6}
$$

Let a map $F\colon X \times \mathbb{R}^3_+ \to Y$ be defined by

$$
\begin{aligned}
F(x,\alpha,\beta,\gamma) =& \, x^{(4)} + \alpha x'' + \beta x - \gamma x^3 - 3x''^3 - 12x'x''x''' \\
&- 3x'^2\left(x^{(4)} - \frac{\alpha}{2}x''\right).
\end{aligned} \tag{7}
$$

Consider the equation

$$
F(x,\alpha,\beta,\gamma) = 0. \tag{8}
$$

The operator equation (8) is equivalent to the boundary value problem (1). Remark that the trivial function $x_0(s) = 0$, $s \in [-r, r]$, satisfies the equation (8) for all values of parameters α, β and γ. Set

$$
\Gamma = \{(0,\alpha,\beta,\gamma) \in X \times \mathbb{R}^3_+ \colon \alpha,\beta,\gamma \in \mathbb{R}_+\}.
$$

We call Γ *the trivial family* of solutions of the equation (8), which corresponds to the straight rod in our model. A solution of (8) is said to be *nontrivial* if it does not belong to Γ.

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Definition 2.1 $(0, \alpha_0, \beta_0, \gamma_0) \in \Gamma$ *is called a bifurcation point of the equation* (8) *if in every neighbourhood of this point in* $X \times \mathbb{R}^3_+$ *there is a nontrivial solution of* (8).

From now on, we denote by $\langle \cdot, \cdot \rangle$ the standard inner product in $L^2(-r, r)$, i.e.

$$\langle g, h \rangle = \frac{1}{2r} \int_{-r}^{r} g(s)h(s)ds, \quad g, h \in L^2(-r, r).$$

Note that

$$E'_x(x, \alpha, \beta, \gamma)h = \langle F(x, \alpha, \beta, \gamma), h \rangle \tag{9}$$

for all $x, h \in X$ and $\alpha, \beta, \gamma \in \mathbb{R}_+$. This follows by (6) and (7). We call the map F *the variational gradient* of the potential E. The formula (9) implies that solutions of the equation (8) are critical points of the energy functional (4).

Differentiating the map F with respect to the space variable x at $x_0 \equiv 0$ we obtain

$$F'_x(0, \alpha, \beta, \gamma)h = h^{(4)} + \alpha h'' + \beta h \tag{10}$$

for every $h \in X$ and $\alpha, \beta, \gamma \in \mathbb{R}_+$, which yields

$$N(\alpha, \beta) = \ker F'_x(0, \alpha, \beta, \gamma).$$

Proposition 2.1 (see [5]) *For all values of parameters* $\alpha, \beta, \gamma \in \mathbb{R}_+$ *the linear operator* $F'_x(0, \alpha, \beta, \gamma)\colon X \to Y$ *is Fredholm of index zero.*

The proof is based on the observation that $F'_x(0, \alpha, \beta, \gamma)$ is a compact perturbation of $A\colon X \to Y$, $Ah = h^{(4)}$, a Fredholm map of index 0.

Proposition 2.2 *For all* $\alpha, \beta, \gamma \in \mathbb{R}_+$ *the map* $F'_x(0, \alpha, \beta, \gamma)\colon X \to Y$ *is symmetric with respect to the inner product* $\langle \cdot, \cdot \rangle$*, i.e.*

$$\langle F'_x(0, \alpha, \beta, \gamma)h, g \rangle = \langle h, F'_x(0, \alpha, \beta, \gamma)g \rangle$$

for all $h, g \in X$.

This follows by the equality (9).

3 Simple bifurcation points

We now turn to simple bifurcation points.

Theorem 3.1 *Let* $(0, \alpha_0, \beta_0, \gamma_0) \in \Gamma$. *Assume that* $\dim N(\alpha_0, \beta_0) = 1$. *Then* $(0, \alpha_0, \beta_0, \gamma_0)$ *is a bifurcation point of the equation* (8).

We first make a finite-dimensional reduction of Lyapunov-Schmidt type. By Theorem 1.1, there is exactly one $m \in \mathbb{N}$ such that $(\alpha_0, \beta_0) \in l_m$ and the linear space $N(\alpha_0, \beta_0)$ is spanned by e_m. Let a map $G\colon X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \to Y$ be given by

$$G(x, \xi, \alpha, \beta) = F(x, \alpha, \beta, \gamma_0) + (\xi - \langle x, e_m \rangle)\, e_m,$$

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where $x \in X$, $\xi \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}_+$. We see at once that

$$G'_x(0,0,\alpha_0,\beta_0)h = F'_x(0,\alpha_0,\beta_0,\gamma_0)h - \langle h, e_m \rangle\, e_m,$$

where $h \in X$. As $F'_x(0,\alpha_0,\beta_0,\gamma_0)\colon X \to Y$ is Fredholm of index 0 we conclude by Proposition 2.2 that $G'_x(0,0,\alpha_0,\beta_0)$ is an isomorphism of X onto Y. From the implicit function theorem it follows that there exist open subsets $U \subset X$ and $S \subset \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ such that $0 \in U$, $(0,\alpha_0,\beta_0) \in S$, and the set

$$\{(x,\xi,\alpha,\beta) \in U \times S \colon G(x,\xi,\alpha,\beta) = 0\}$$

is the graph of a smooth function $\tilde{x}\colon S \to U$ satisfying $\tilde{x}(0,\alpha_0,\beta_0) = 0$. Moreover, since $G(0,0,\alpha,\beta) = 0$ for all $\alpha, \beta \in \mathbb{R}_+$, we have $\tilde{x}(0,\alpha,\beta) = 0$ for all $(0,\alpha,\beta) \in S$.

A function $\varphi\colon S \to \mathbb{R}$ is defined by

$$\varphi(\xi,\alpha,\beta) = \xi - \langle \tilde{x}(\xi,\alpha,\beta), e_m \rangle\,. \tag{11}$$

It is easily seen to be smooth and $\varphi(0,\alpha,\beta) = 0$ for all $(0,\alpha,\beta) \in S$.

Theorem 3.2 (see [6]) *The point $(0,\alpha_0,\beta_0,\gamma_0) \in X \times \mathbb{R}^3_+$ is a bifurcation point of (8) if and only if the point $(0,\alpha_0,\beta_0) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ is a bifurcation point of the equation*

$$\varphi(\xi,\alpha,\beta) = 0. \tag{12}$$

The rest of the proof of Theorem 3.1 is based on the Krasnosielski theorem, which we recall for the convenience of the reader.

Theorem 3.3 (see [6]) *If $(0,\alpha_0,\beta_0) \in S$ is not a bifurcation point of the equation (12) then there exist open sets $V_1 \subset \mathbb{R}$ and $V_2 \subset \mathbb{R}_+ \times \mathbb{R}_+$ satisfying:*

- *(i)* $(0,\alpha_0,\beta_0) \in V_1 \times V_2 \subset S$.
- *(ii)* *For each open subset $V \subset V_1$ such that $0 \in V$ and for all $(\alpha,\beta), (\tilde{\alpha},\tilde{\beta}) \in V_2$ the mappings $\varphi(\cdot,\alpha,\beta)$ and $\varphi(\cdot,\tilde{\alpha},\tilde{\beta})$ have no zeros on the boundary of V and*

$$\deg(\varphi(\cdot,\alpha,\beta), V, 0) = \deg(\varphi(\cdot,\tilde{\alpha},\tilde{\beta}), V, 0), \tag{13}$$

where $\deg(\varphi(\cdot,\alpha,\beta), V, 0)$ denotes the Brouwer degree of the map $\varphi(\cdot,\alpha,\beta)$ on the set V with respect to 0.

By differentiating

$$G(\tilde{x}(\xi,\alpha,\beta),\xi,\alpha,\beta) = 0$$

with respect to ξ we get

$$F'_x(\tilde{x}(\xi,\alpha,\beta),\alpha,\beta,\gamma_0)\frac{\partial \tilde{x}}{\partial \xi}(\xi,\alpha,\beta) + e_m - \left\langle \frac{\partial \tilde{x}}{\partial \xi}(\xi,\alpha,\beta), e_m \right\rangle e_m = 0,$$

and hence

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$$F_x'(0,\alpha,\beta,\gamma_0)\frac{\partial\tilde{x}}{\partial\xi}(0,\alpha,\beta)+e_m-\left\langle\frac{\partial\tilde{x}}{\partial\xi}(0,\alpha,\beta),e_m\right\rangle e_m=0.$$

As $\langle e_m,e_m\rangle=1$ we have

$$\left\langle F_x'(0,\alpha,\beta,\gamma_0)\frac{\partial\tilde{x}}{\partial\xi}(0,\alpha,\beta),e_m\right\rangle+1-\left\langle\frac{\partial\tilde{x}}{\partial\xi}(0,\alpha,\beta),e_m\right\rangle=0.$$

Applying Proposition 2.2 we see that

$$\left\langle\frac{\partial\tilde{x}}{\partial\xi}(0,\alpha,\beta),e_m^{(4)}+\alpha e_m''+\beta e_m-e_m\right\rangle=-1.$$

Since $e_m''=c_me_m$ and $e_m^{(4)}=c_m^2e_m$, we obtain

$$\left\langle\frac{\partial\tilde{x}}{\partial\xi}(0,\alpha,\beta),e_m\right\rangle=-\frac{1}{c_m^2+\alpha c_m+\beta-1}.$$

Now (11) yields

$$\frac{\partial\varphi}{\partial\xi}(0,\alpha,\beta)=\frac{c_m^2+\alpha c_m+\beta}{c_m^2+\alpha c_m+\beta-1}.$$

Furthermore, it follows from Theorem 1.1 that

$$\beta_0=-c_m\alpha_0-c_m^2,$$

and so

$$c_m^2=-c_m\alpha_0-\beta_0.$$

Hence

$$\frac{\partial\varphi}{\partial\xi}(0,\alpha,\beta)=\frac{(\alpha-\alpha_0)c_m+\beta-\beta_0}{(\alpha-\alpha_0)c_m+\beta-\beta_0-1}.$$

Let us now suppose, contrary to our claim, that $(0,\alpha_0,\beta_0)$ is not a bifurcation point of the equation (12). Let $V_1\subset\mathbb{R}$ and $V_2\subset\mathbb{R}_+\times\mathbb{R}_+$ be the open sets as in Theorem 3.3. Take $0<\varepsilon<1$ such that $(\alpha_0,\beta_0-\varepsilon)$ and $(\alpha_0,\beta_0+\varepsilon)$ belong to V_2 and take a neighbourhood $V\subset V_1$ of 0 such that the Brouwer degrees of $\varphi(\cdot,\alpha_0,\beta_0-\varepsilon)$ and $\varphi(\cdot,\alpha_0,\beta_0+\varepsilon)$ on V with respect to 0 are the same as the signs of $\frac{\partial\varphi}{\partial\xi}(0,\alpha_0,\beta_0-\varepsilon)$ and $\frac{\partial\varphi}{\partial\xi}(0,\alpha_0,\beta_0+\varepsilon)$ respectively.
We get

$$\deg(\varphi(\cdot,\alpha_0,\beta_0-\varepsilon),V,0)=\operatorname{sgn}\frac{\partial\varphi}{\partial\xi}(0,\alpha_0,\beta_0-\varepsilon)=\operatorname{sgn}\frac{\varepsilon}{\varepsilon+1}=1$$

and

$$\deg(\varphi(\cdot,\alpha_0,\beta_0+\varepsilon),V,0)=\operatorname{sgn}\frac{\partial\varphi}{\partial\xi}(0,\alpha_0,\beta_0+\varepsilon)=\operatorname{sgn}\frac{\varepsilon}{\varepsilon-1}=-1,$$

which contradicts the equality (13). Hence $(0,\alpha_0,\beta_0)$ is a bifurcation point of the equation (12), and, in consequence, $(0,\alpha_0,\beta_0,\gamma_0)$ is a bifurcation point of the equation (8).

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Remark 3.4 *Suppose that* $\dim N(\alpha_0, \beta_0) = 2$. *It is easily seen that the point* $(0, \alpha_0, \beta_0, \gamma_0) \in \Gamma$ *is a bifurcation point of* (8), *which is clear from*

$$(\alpha_0, \beta_0) \in \overline{\{(\alpha, \beta) \in \mathbb{R}^2_+ : \dim N(\alpha, \beta) = 1\}}.$$

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