DOI: 10.2478/v.10169-011-0014-4

A SOLUTION OF NON-LINEAR DIFFERENTIAL PROBLEM WITH APPLICATION TO SELECTED GEOTECHNICAL PROBLEMS

A. MILEWSKA[1]

A certain non-linear differential equation containing a power of unknown function being the solution is considered with application to selected geotechnical problems. The equation can be derived to a linear differential equation by a proper substitution and properties of operations G and S.

Key words: non-linear problem, geotechnical application, linear operator theory.

1. Introduction

Contact issues [3], where one can consider many interaction problems, e.g. in particular the "stick-slip" phenomena, lead mainly to non-linear problems with differential operators.

Similarly, as a specific geotechnical problem, a model of an interaction between geosynthetic and the subsoil in a given situation (Fig. 1) – [12] can be considered as a linear problem or non-linear one.

Such problem of the equilibrium can be presented in form of a differential equation of the second order (elliptic or hyperbolic), furthermore, a large set of formulations is discussed in e.g. [13].

Another example is the problem of vibration of a foundation loaded with dynamic forces with using an isolation layer (Fig. 2) – [9]; this problem can be again analyzed as a linear or non-linear one.

In the analysis of the process of pile driving into the soil (Fig. 3), or in the process of driving an open pile with a soil plug creation, or models showing the interaction between pile and soil using the basic rheological elements e. g. spring and slide with a parallel connected piston [6], these problems are non-linear by nature and could be understood as relevant geotechnical ones.

[1] Faculty of Applied Physics and Mathematics, Gdansk University of Technology, 11/12 Narutowicza St., 80-233 Gdansk, Poland, e-mail: amilewska@mif.pg.gda.pl

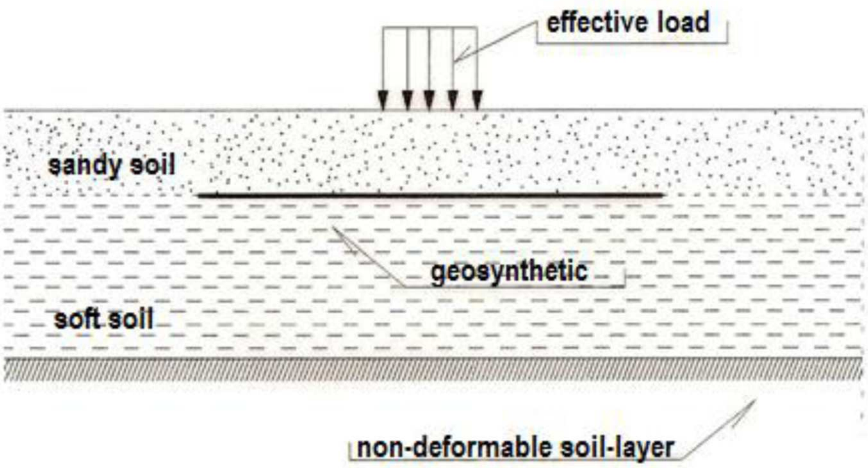

Fig. 1. Model of an interaction between geosynthetic and subsoil.
Rys. 1. Model współpracy geowłókniny z podłożem

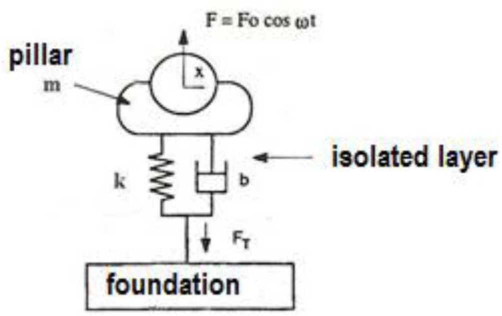

Fig. 2. A foundation loaded with dynamic forces with using an isolation layer.
Rys. 2. Fundament obciążony dynamicznie z warstwą izolacyjną

Settling of banks, diffusion in the porous medium, identifying dynamic geotechnical systems, etc., are only some of the problems that can be modelled as a linear or non-linear one.

Linear models are only a rough approximation of the phenomena and processes occurring in the soil. Usually they lead to differential equations, linear equations, or systems of equations.

However, their advantage is often the fact that it is possible to find an analytic solution to these models, which is crucial from the practical point of view.

In the case of the non-linear models, which are closer to reality, it is quite impossible to find any analytic solution for the non-linear differential equations.

The aim of this paper is to find an analytic solution of the non – linear differential equation system, and therefore an introduction of methods of linearization of the non-linear systems is needed. Such linearization is understood as an approximate analogue of the non-linear model with the appropriate linear one. Such linearization procedure is 'justified' for 'soft' non-linearities occurring in the system in question. In other cases numerical methods, e.g. genetic algorithms, are used but a disadvantage of these methods is a difficulty to estimate the error of obtained approximation.

In this paper, a method for solving the above mentioned non-linear problems without any resort to the approximate methods is described. This is an important advantage of the method presented below. An additional advantage of the method is that the results obtained for the non-linear cases can be used, or they are even useful in the analysis of linear problems.

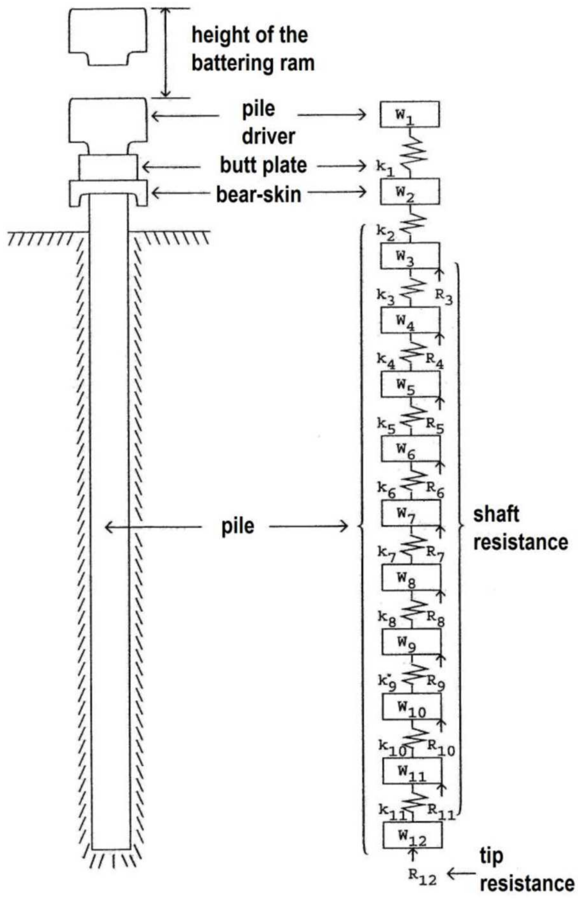

Fig. 3. Model of pile driving into the soil.
Rys. 3. Model pala wbijanego w grunt

This method substitutes the non-linear problem with an equivalent (and not approximate) linear problem. It is easy to explain all the details and also the way to create the useful functions on an example. Further considerations require precision and mathematical accuracy, so they are presented as definitions, theorems and corollaries, which are of the character of general laws, possible to be used in given geotechnical situations mentioned above!

In the literature, various types of the non-linear differential equations are analysed. A detailed analysis of the different classes of the non-linear equations can be found in e. g. [1, 2, 11].

In this paper, a special type of non-linear differential equation is considered, which in a sense corresponds to the special case presented in the form of an equation derived in [10].

The method by which a certain non-linear differential equation can be brought to a linear differential equation is presented. In this method some properties of non-linear operation G^n defined in [7, 8] and some properties of the operation S defined in [4, 5] are used. The above-mentioned method enables to determine an explicit solution of the analyzed non-linear differential equation, or an explicit solution of linear equation corresponding to it, or at any rate some representations of solutions of the differential equations being a member of a certain class.

2. The non-linear operation G and certain non-linear differential equation

Definition 1. *We denote*

$$\mathrm{Sx} \overset{\mathrm{def}}{=} \mathrm{b}\,(\mathrm{t})\,\frac{dx}{dt} \tag{2.1}$$

where $x \in C^1\,(< t_1, t_2 >, R)\,, b\,(t) \neq 0$ for $\mathrm{t} \in < t_1, t_2 >$.

Definition 2. *By induction we define a sequence of operations G^n, $n \in N$, such as*

$$G^n\,(x) = SG^{n-1}\,(x) + xG^{n-1}\,(x) \tag{2.2}$$

$$G^0\,(x) \overset{\mathrm{def}}{=} x \tag{2.3}$$

where $x \in C^n\,(< t_1, t_2 >, R)\,.$

Corollary 1. *For n=1 one obtains*

$$G^1\,(x) = Sx + x^2. \tag{2.4}$$

For n=2 we have

$$G^2\,(x) = S^2x + 3xSx + x^3. \tag{2.5}$$

For n=3 we have

$$G^3(x) = S^3x + 3(Sx)^2 + 4xS^2x + 6x^2Sx + x^4. \tag{2.6}$$

Remark. The operation G^n is a non-linear operation.
Definition 3. *We denote*

$$E^x \overset{\text{def}}{=} \exp\left(-\int_{t_0}^{t} \frac{x(\tau)}{b(\tau)} d\tau\right). \tag{2.7}$$

where

$$x \in C^0(<t_1, t_2>, R).$$

It can be proved by induction that the operation G^n formulated in definition 2 possesses the following properties:

$$G^n\left((Sx)x^{-1}\right) = \left(S^{n+1}x\right)x^{-1}, \quad x \in C^{n+1}(<t_1, t_2>, R), \quad x \in Inv \tag{2.8}$$

$$S^n(E^x) = E^x G^{n-1}(-x), \quad x \in C^n(<t_1, t_2>, R). \tag{2.9}$$

Let us consider a differential equation in the following form

$$a_{n+1}G^n(-x) + a_nG^{n-1}(-x) + a_{n-1}G^{n-2}(-x) + \cdots + a_1G^0(-x) + a_0 = 0, \tag{2.10}$$

where $a_i \in C^0(<t_1, t_2>, R)$ for $i = 0, 1, 2, \ldots, n+1, \quad x \in C^n(<t_1, t_2>, R)$.
Corollary 2. *For n=1 the non-linear first-order differential equation is obtained*

$$a_2Sx = a_2x^2 - a_1x + a_0. \tag{2.11}$$

For n=2 one gets the non-linear second-order differential equation

$$a_3\left(S^2x - 3xSx\right) + a_2Sx = -a_3x^3 + a_2x^2 - a_1x + a_0. \tag{2.12}$$

For n=3 one gets the non-linear third-order differential equation

$$\begin{aligned} &a_4\left(S^3x - 3(Sx)^2 - 4xS^2x + 6x^2Sx\right) + a_3\left(S^2x - 3xSx\right) + a_2Sx = \\ &= a_4x^4 - a_3x^3 + a_2x^2 - a_1x + a_0. \end{aligned} \tag{2.13}$$

Taking into account the above formulations it is possible to show, that using the operation G^1

$$Sx = d_1x^2 + d_2x + d_3 \tag{2.14}$$

where

$$x \in C^1(<t_1,t_2>,R), d_1,d_2 \in C^0(<t_1,t_2>,R), d_3 \in C^1(<t_1,t_2>,R), \quad d_3 \in Inv,$$
can be written in the form (2.11), with $\alpha_2 \stackrel{\text{def}}{=} 1$. Let us substitute

$$x = -d_3 y^{-1}, \quad y \in C^1(<t_1,t_2>,R), \quad y \in Inv \tag{2.15}$$

to the equation (2.14). From the substitution (2.15) and from definition of the operation S we get the following relation

$$Sx = [-(Sd_3)y + d_3(Sy)]\,y^{-2}, \tag{2.16}$$

so the equation (2.14) can be rewritten to the following form

$$[-(Sd_3)y + d_3(Sy)]\,y^{-2} = d_1 d_3^2 y^{-2} - d_2 d_3 y^{-1} + d_3, \tag{2.17}$$

so

$$Sy = y^2 + (Sd_3 - d_2 d_3)\,d_3^{-1} y + d_1 d_3. \tag{2.18}$$

Corollary 3. *We can write a differential equation (2.14) using the operation G^1, i.e. equation (2.14) can be transformed to the from (2.11). In this case $a_2 \stackrel{\text{def}}{=} 1$, $a_1 \stackrel{\text{def}}{=} -(Sd_3 - d_2 d_3)\,d_3^{-1}$, $a_0 \stackrel{\text{def}}{=} d_1 d_3$.*

3. Method of transformation of a linear equation

Theorem 4. *The non-linear differential equation (2.10) can be brought about to a linear differential equation*

$$a_{n+1}S^{n+1}y + a_n S^n y + a_{n-1}S^{n-1}y + \cdots + a_1 Sy + a_0 y = 0, \tag{3.1}$$

where $y \in C^{n+1}(<t_1,t_2>,R)$ and $x \stackrel{\text{def}}{=} -(Sy)\,y^{-1}, \quad y \in Inv$.
Proof. Substituting

$$x = -(Sy)\,y^{-1} \tag{3.2}$$

into the equation (2.10) we get

$$\begin{aligned} &a_{n+1}G^n\left((Sy)\,y^{-1}\right) + a_n G^{n-1}\left((Sy)\,y^{-1}\right) + a_{n-1}G^{n-2}\left((Sy)\,y^{-1}\right) + \cdots + \\ &+ a_1 G^0\left((Sy)\,y^{-1}\right) + a_0 = 0. \end{aligned} \tag{3.3}$$

Using the properties of the operations G^n we can write the previous equation in the following form

$$a_{n+1}\left(S^{n+1}y\right)y^{-1} + a_n\left(S^n y\right)y^{-1} + a_{n-1}\left(S^{n-1}y\right)y^{-1} + \cdots + a_1(Sy)\,y^{-1} + a_0 = 0, \tag{3.4}$$

so

$$a_{n+1}\left(S^{n+1}y\right) + a_n\left(S^n y\right) + a_{n-1}\left(S^{n-1}y\right) + \cdots + a_1(Sy) + a_0 = 0. \tag{3.5}$$

Theorem 5. *We can bring the differential equation (3.5) to the differential equation (2.10), where* $y \overset{\text{def}}{=} E^x$.

Proof. Substituting $y \overset{\text{def}}{=} E^x$ into the equation (3.5) we get

$$a_{n+1}\left(S^{n+1}E^x\right) + a_n\left(S^n E^x\right) + a_{n-1}\left(S^{n-1}E^x\right) + \cdots + a_1\left(SE^x\right) + a_0E^x = 0. \tag{3.6}$$

From properties of the operation G^n we get the equation

$$\begin{aligned} &a_{n+1}E^xG^n(-x) + a_nE^xG^{n-1}(-x) + a_{n-1}E^xG^{n-2}(-x) + \cdots + \\ &+a_1E^xG^0(-x) + a_0E^x = 0 \end{aligned} \tag{3.7}$$

Based on the previous equation one achieves the following equation

$$a_{n+1}G^n(-x) + a_nG^{n-1}(-x) + a_{n-1}G^{n-2}(-x) + \cdots + a_1G^0(-x) + a_0 = 0. \tag{3.8}$$

Corollary 6. *There exists an equivalence between the differential equation (3.5) and the non-linear differential equation (2.10). Thus, the differential equation (2.11), i.e.*

$$a_2b(t)x' = a_2x^2 - a_1x + a_0 \tag{3.9}$$

is an equivalent to the differential equation

$$a_2(b(t))^2y'' + b(t)\left(a_2b'(t) + a_1\right)y' + a_0y = 0, \tag{3.10}$$

where $y \overset{\text{def}}{=} E^x$.
Remark If

$$S^2y + \left(a_0c^{-1} + c\right)Sy + a_0y = 0, \quad c \in Ker\left(b(t)\frac{d}{dt}\right), \quad c \in Inv \tag{3.11}$$

or

$$S^2y + a_1Sy + \left(a_1c - c^2\right)y = 0, \quad c \in Ker\left(b(t)\frac{d}{dt}\right), \quad c \in Inv, \tag{3.12}$$

then the element

$$y = E^c$$

is a solution of the differential equation (3.11) or (3.12).
Theorem 7. *If there exists a solution* $\mu \in Inv$ *of the differential equation (2.10) then a differential equation*

$$a_{n+1}S^{n+1}y + a_nS^ny + a_{n-1}S^{n-1}y + \cdots + a_1Sy + a_0y = u, u \in C^0(< t_1, t_2 >, R)$$

is an equivalent to the system of equations

$$\bar{a}_n S^n u_1 + \bar{a}_{n-1} S^{n-1} u_1 + \cdots + \bar{a}_0 u_1 = u, \quad Sy + \mu y = u_1$$

(is an equivalent to the series connection of the dynamical systems illustrated in Fig. 4).

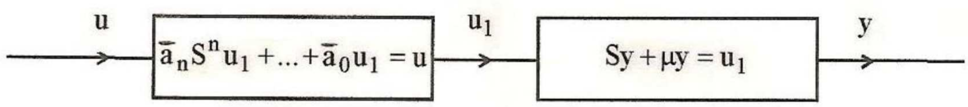

Fig. 4. Series connection of two dynamical systems.
Rys. 4. Szeregowe połączenie dwóch układów dynamicznych

Proof. From the previous system of equations we have

$$\left(\bar{a}_n S^n + \bar{a}_{n-1} S^{n-1} + \cdots + \bar{a}_0\right)(Sy + \mu y) = u.$$

In the case for n=3 the following equation

$$a_4 S^4 y + a_3 S^3 y + a_2 S^2 y + a_1 Sy + a_0 y = u$$

can be replaced by the equation below

$$\left(\bar{a}_3 S^3 + \bar{a}_2 S^2 + \bar{a}_1 S + \bar{a}_0\right)(Sy + \mu y) = u.$$

Based on the previous fact, it can be stated that the element $\mu \in Inv$ is the solution of the equation (13), i.e.

$$\begin{aligned} &a_4\left(S^3 x - 3(Sx)^2 - 4xS^2 x + 6x^2 Sx\right) + a_3\left(S^2 x - 3xSx\right) + a_2 Sx = \\ &= a_4 x^4 - a_3 x^3 + a_2 x^2 - a_1 x + a_0. \end{aligned}$$

Coefficients $\bar{a}_3, \bar{a}_2, \bar{a}_1, \bar{a}_0$ could be determined from the following dependencies

$$\begin{aligned} &\bar{a}_3 \overset{\text{def}}{=} a_4 \\ &\bar{a}_2 \overset{\text{def}}{=} a_3 - a_4\mu \\ &\bar{a}_1 \overset{\text{def}}{=} a_2 - 3a_4 S\mu - a_3\mu + a_4\mu^2 \\ &\bar{a}_0 \overset{\text{def}}{=} a_1 - 3a_4 S^2\mu - 2a_3 S\mu + 5a_4\mu S\mu - a_2\mu + a_3\mu^2 - a_4\mu^3. \end{aligned}$$

Remark. The proof of Theorem 7 is of practical importance because it shows how the unknown element μ and the coefficients $\bar{a}_i$ should be determined.

Remark. Non – linear problems can be replaced with a series connection of two or more simple systems, as presented in Fig. 5.

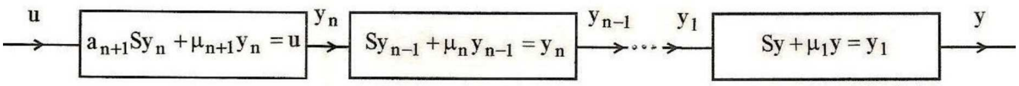

Fig. 5. Series connection of n+1 dynamical systems.
Rys. 5. Szeregowe połączenie n+1 układów dynamicznych

4. Example

Here the equation of beam deflection y(x) with variable stiffness is considered. A beam placed on an elastic foundation, loaded by axial force P and any transverse loading q(x), can be written in form of the following differential system of equations:

$$M''(x) + \frac{P}{B(x)} M(x) = q(x) \tag{4.1}$$

$$B(x)\, y''(x) = M(x) \tag{4.2}$$

where M(x) indicates moment, B(x) means variable beam stiffness understood as a variable foundation parameter.

In practical cases the equation (4.1) is solved based on approximation way by using the method of series, as well as the numerical methods. However, it often gives no answer to the following question: does the solution exist and if it exists, is the solution unique ?

If

$$B(x) = -\frac{Pc^2\left(c_1 + c_2 \exp\left(\frac{x}{c}\right)\right)}{c_2 \exp\left(\frac{x}{c}\right)}, \quad c_1, c_2, c \in R, c \neq 0, c_2 \neq 0,$$

Then, on the basis of the previous discussion, the equation (4.1) can be written as a series connection of the first order systems, i.e.

$$(S - \mu)(SM + \mu M) = q, \quad \text{where} \quad \mu = -\frac{c_2 \exp\left(\frac{x}{c}\right)}{c\left(c_1 + c_2 \exp\left(\frac{x}{c}\right)\right)}, \quad b = 1.$$

Thus, from the previous formula M(x) can be determined.

Afterwards, using the equation (4.2), one can determine y(x) by the double-integration.

Remark. In the purpose of determination of the moment M(x) in general cases the theorem 7 should be utilized, and as a consequence the equation (4.1) should be replaced by the following equivalent equation:

$$(S - \mu)(SM + \mu M) = q,$$

where the element μ is the solution of the non-linear differential equation shown below:

$$S\mu = \mu^2 + \frac{P}{B(x)}.$$

5. CONCLUSIONS

The analysis of non-linear differential equations is very important because of problems of practical use, particularly in geotechnics. The presented method enables transformation of the non-linear problems into the linear ones and vice versa with very effective results.

This method is effective for the reason that many non-linear differential equations created for geotechnical behaviors can be solved analytically with different conditions, i.e. initial conditions, boundary conditions or initial-boundary ones.

Non-linear geotechnical problems formulated in their differential form can be substituted with chain-connections with two or more simple systems. The analysis of the whole system can be therefore investigated with limitation to the analysis of the separate elements of the chain-connection, where only linear elements of the first-order are included. As a simple example of usage of this method, the stiffness of a beam placed on an elastic subground, described by variable quantities is considered.

This particular method presented in this paper evidently shows that future research can be based on the results presented here which can lead to the non-linear identification of the geotechnical differential systems.

REFERENCES

1. L.C. EVANS, *Partial Differential Equations*, Amer. Math. Soc. (1998).
2. H. HOLDEN, K.H. KARLSEN, N.H. RISEBRO, *A convergent finite-difference method for a nonlinear variational wave equation*, JMA Journal of Numerical Analysis, **39** (3), 539-572, (2009).
3. D. KOLYMBAS (red.), *Advances in Geotechnical Engineering and Tunneling*. Balkema, Rotterdam 2000.
4. E. MIELOSZYK, *Application of non-classical operational calculus to solving some boundary value problem*, Integral Transforms and Special Functions, **9** (4), 287-292 (2000).
5. E. MIELOSZYK, *Non-classical operational calculus at application to generalized dynamical systems* (in Polish), Publisher Polish Academy of Sciences, Institute of Fluid-Flow Machinery (2008).
6. E. MIELOSZYK, A. MILEWSKA, *Modelling of pile soil interaction by using generalized dynamical systems.* Proceedings of 11-th Baltic Sea Geotechnical Conference. Geotechnics in Maritime Engineering. Gdańsk, 15-18 September 2008, 735-742.
7. A. MILEWSKA, *Non-classical operational methods at dynamical systems* (in Polish), PhD Thesis, Gdansk University of Technology (2002).
8. A. MILEWSKA, Certain non-linear partial differential equation and linear partial differential equation corresponding to it. (in preparation and will be submitted to Integral Transforms and Special Functions).

9. J. MOREL, *Vibrations of machines and diagnostics of their technical conditions* [in Polish]. Polskie Towarzystwo Diagnostyki Technicznej, KBN Warszawa 1992.
10. R.G. PINSKY, *The Behaviour of the Life Span for Solutions to* $u_t = \Delta u + a(x)u^p$ *in* R^d, J. Differential Equations **147** (1) (1998) 30-57.
11. V. RAYSKIN, α*-Hölder Linearization*, J. Differential Equations, **147** (2), 271-284 (1998).
12. A. SAWICKI, A. WALTER, *Simple model of cooperation of geofiber with sub-soil* [in Polish]. Zesz. Nauk. PG, Bud. Ląd. XLIV, 165-181 (1987).
13. Z. SIKORA, Control conditions for geocomputing within general hypoplasticity. Proceedings of 10-th International Conference. Modern Building Materials, Structures and Techniques. Vilnius, 19–21 May 2010, 1161-1173.

PEWNE ROZWIĄZANIE NIELINIOWEGO PROBLEMU RÓŻNICZKOWEGO Z ZASTOSOWANIEM DO WYBRANYCH ZAGADNIEŃ GEOTECHNICZNYCH

S t r e s z c z e n i e

Pewne nieliniowe równanie różniczkowe jest analizowane pod kątem zastosowań do rozwiązywania wybranych problemów geotechnicznych, które modelowane są jako nieliniowe, bądź liniowe o współczynnikach funkcyjnych. Dane równanie różniczkowe jest sprowadzane analitycznie do równania różniczkowego liniowego i odwrotnie dzięki właściwościom iteracji operacji G oraz właściwościom operacji S.

Remarks on the paper should be sent to the Editorial Office no later than September 30, 2011

Received December 15, 2010
revised version
June 12, 2011