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Algebraic periods and minimal number of periodic points for smooth self-maps of 1-connected 4-manifolds with definite intersection forms

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Abstract. Let M be a closed 1-connected smooth 4-manifolds, and let r be a non-negative integer. We study the problem of finding minimal number of r-periodic points in the smooth homotopy class of a given map $f\colon M \to M$. This task is related to determining a topological invariant $D_r^4[f]$, defined in Graff and Jezierski (Forum Math 21(3):491–509, 2009), expressed in terms of Lefschetz numbers of iterations and local fixed point indices of iterations. Previously, the invariant was computed for self-maps of some 3-manifolds. In this paper, we compute the invariants $D_r^4[f]$ for the self-maps of closed 1-connected smooth 4-manifolds with definite intersection forms (i.e., connected sums of complex projective planes). We also present some efficient algorithmic approach to investigate that problem

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1. Introduction

The problem of finding minimal number of fixed points in the homotopy class of a given self-map of a compact manifold goes back to the classical papers of Jacob Nielsen [33]. The same problem has also been studied for the more general setting of the r-periodic points, where $r \geq 2$ is a given integer. In 1983, B. Jiang introduced a topological invariant $NF_r(f)$, and showed that it is a lower bound for the number of r-periodic points in the homotopy class [26]. Later, it was proved that $NF_r(f)$ is the best such lower bound [24], so it provides the exact value of the minimal number of r-periodic points in the

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homotopy class. The invariant was determined during recent years for some special cases, see for example: [22, 27, 31, 32].

In this paper, we study the more complicated problem of minimizing the number of r-periodic points in a smooth homotopy class. The establishing mutual relations between continuous and smooth categories is one of the most important challenges in modern periodic point theory. That is, we consider the smooth version of the problem, asking about minimal number of r-periodic in the smooth homotopy class of f, i.e., for

$$\min\{\#\mathrm{Fix}(g^r)\colon g \overset{s}{\sim} f\}, \tag{1}$$

where $\overset{s}{\sim}$ means that the maps g and f are C^1-homotopic.

The smoothness imposes additional restrictions in the considered homotopy class, which results in the fact that the value of minimum (1) in smooth category is higher than in the continuous case.

In Refs. [13, 14], two counterparts of $NF_r(f)$ in smooth category were introduced: $D_r^m[f]$ for simply connected manifolds, and its generalization $NJD_r^m[f]$ in the non-simply connected case (where $m \geq 3$ denotes the dimension of the considered manifold).

The second invariant is much more difficult to compute, as it needs simultaneous control of the restrictions that come from both fundamental group and Lefschetz numbers of iterations. In this paper, we concentrate on simply connected case, when the first type of restriction disappears (notice that for simply connected manifolds $NF_r(f) \in \{0, 1\}$).

The determination of $D_r^m[f]$ and NJD_r^m depends on finding minimal decomposition of Lefschetz numbers of iterations into the local fixed point indices at periodic points. As the complexity of forms of indices for smooth maps grows with the dimension, for the first place, $D_r^m[f]$ was determined for 3-dimensional cases: for $\mathbb{S}^2 \times I$ [13], $\mathbb{S}^3$ [12], two-holed 3-dimensional closed ball [11] and so $NJD_r^3[f]$ for $\mathbb{R}P^3$ [17].

The next natural but more difficult step is to calculate the invariant $D_r^4[f]$ for self-maps of a 4-manifold M. This work is devoted to determining the invariant $D_r^4[f]$ for a closed 1-connected 4-manifold M whose intersection form is definite, hence homeomorphic to a connected sum of complex projective planes (cf. [6]).

In general, the process of finding the invariant may be divided into two parts. First part is related to determining the forms of $\{L(f^n)\}_{n|r}$—Lefschetz numbers of iterations and then identifying the set of so-called *algebraic periods* of periodic expansion of $\{L(f^n)\}_{n|r}$: $AP_r(f)$.

The second part is combinatorial in its nature: it is a decomposition of $AP_r(f)$ into the minimal number of sets that represent the sum of periodic expansion of the sequences of fixed point indices.

The first part, however, is interesting in itself, as the determination of $AP_r(f)$ (or generally $AP(f)$ for all natural r) is very important in many considerations in periodic point theory. It was used to study the minimal periods (especially for transversal self-maps) of various classes of manifolds cf. Refs. [9, 20, 21, 28, 29], see also Ref. [30] and the references therein.

The paper is organized in the following way. In Sect. 2, we introduce the language of periodic expansion, which allows one to write down the sequences of Lefschetz numbers of iteration in the convenient way. Section 3 is devoted to determining the Lefschetz numbers of iterations of a self-map of a simply connected 4-manifold obtained due to restrictions found by Duan and Wang [7] (Corollary 8).

In Sect. 4, we give the description of the set $AP(f)$ for manifolds with definite intersection forms (Theorem 13). Note that we work in this section without the assumption of smoothness, so the statements are valid in continuous category. We also show the application of the obtained results for finding minimal periods of transversal maps.

The definition of the invariant $D_r^4[f]$ is given in Sect. 6 and in the next section, we realize the combinatorial part related to finding the invariant. Section 7 provides the forms of local indices of iterations of smooth maps in dimension 4 (Theorem 23) and the way of using it to combinatorically compute $D_r^4[f]$ once we know the set $AP_r(f)$. In Sect. 8, we use the results on $AP_r(f)$ for finding $D_r^4[f]$. Although it is a demanding task in general case, we are able to achieve it for some particular values of r and also determine $D_r^4[f]$ using a computer program for manifolds having 2×2 intersection matrices and low values of r.

2. Periodic expansion of Lefschetz numbers of iterations

Definition 1. A sequence of integers $\{a_n\}_{n=1}^{\infty}$ is called a *Dold sequence* if the following Dold congruence relations are fulfilled:

$$\sum_{k|n} \mu(k) a_{\frac{n}{k}} \equiv 0 \pmod{n} \text{ for each } n \geq 1, \tag{2}$$

where $\mu\colon \mathbb{N} \to \mathbb{Z}$ is the classical Möbius function, given by the formula

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1; \\ (-1)^k & \text{if } n = p_1 p_2 \dots p_k \text{ for different primes } p_i; \\ 0 & \text{otherwise.} \end{cases}$$

Dold sequences play an important role both in dynamics and in topological number theory [3]. We recall from Ref. [25] a convenient way to express a Dold sequence as a combination of certain basic periodic sequences (so-called periodic expansion).

Definition 2. For a positive integer k, define the map $\mathrm{reg}_k : \mathbb{N} \to \mathbb{N}$ by

$$\mathrm{reg}_k(n) = \begin{cases} k & \text{if } k \mid n, \\ 0 & \text{if } k \nmid n, \end{cases}$$

which is the periodic sequence:

$$(0, \dots, 0, k, 0, \dots, 0, k, \dots),$$

where the non-zero entries k appear precisely at the places divisible by k.

Proposition 3. (cf. [3]) *Any integer valued sequence* $\{a_n\}_{n=1}^{\infty}$ *can be written uniquely in the following form of a periodic expansion:*

$$a_n = \sum_{k=1}^{\infty} b_k \operatorname{reg}_k(n), \text{ where } b_k = \frac{1}{k} \sum_{l|k} \mu\left(\frac{k}{l}\right) a_l \in \mathbb{Q}. \tag{3}$$

Moreover, the sequence $\{a_n\}_{n=1}^{\infty}$ *is integer valued and satisfies Dold congruences if and only if* $b_k \in \mathbb{Z}$ *for every* $k \in \mathbb{N}$.

Remark 4. By Proposition 3, every Dold sequence is an integral combination (perhaps infinite) of the basic sequences reg_k with so-called *Dold coefficients* b_k as coefficients.

For a given function $f\colon X \to X$, a point $x \in X$ such that $f^p(x) = x$ is called a periodic point of period $p \in \mathbb{N}$. If p is minimal with this property, then p is called minimal period. A point with a minimal period $p = 1$ is called a fixed point.

The fixed point index, which is one of the main tools used in this paper, is a well-known topological invariant which is the algebraic measure of the number of fixed points (for the precise definition, the reader may consult Ref. [25]).

A. Dold in 1985 proved a strong result stating that any sequence of fixed point indices of iterations is a Dold sequence.

Theorem 5. ([5]) *Any sequence of fixed point indices of iterations* $\{\operatorname{ind}(f^n)\}_{n=1}^{\infty}$ *is a Dold sequence (provided it is well-defined). As a result, it can be represented in the form of a periodic expansion (3) with integral coefficients.*

In particular, the sequence $\{L(f^n)\}_{n=1}^{\infty}$ of Lefschetz numbers is a Dold sequence. By Remark 4, we get

$$L(f^n) = \sum_{k=1}^{\infty} b_k \operatorname{reg}_k(n), \tag{4}$$

where b_k are integers.

3. Lefschetz numbers of iterations for self-maps of closed 1-connected 4-manifolds

An $n \times n$ integral matrix $A = (a_{ij})_{n\times n}$ is called *unimodular* if $\det A = \pm 1$; is called *symmetric* if $a_{ij} = a_{ji}$, $1 \le i, j \le n$. The classification of the closed 1-connected 4-dimensional manifolds has been done by Whitehead [34] and Freedman [8].

Theorem 6. (Theorem 1.5 [8]) *For any unimodular and symmetric integral matrix* $A = (a_{ij})_{n\times n}$, *there exists a closed 1-connected 4-dimensional manifold* M_A, *together with a basis*

$$\omega_A = \{\omega_1, \ldots, \omega_n\} \subset H^2(M_A)$$

of the cohomology $H^2(M_A)$, so that $\omega_i \smile \omega_j = a_{ij} \cdot \omega_M$, where $\omega_M \in H^4(M_A) = \mathbb{Z}$ is an orientation class on M_A.

Conversely, for any closed 1-connected 4-dimensional manifold M with a basis

$$\omega = \{\omega_1, \ldots, \omega_n\} \subset H^2(M)$$

of the cohomology $H^2(M)$, the integral matrix $A = (a_{ij})_{n\times n}$ defined by $\omega_i \smile \omega_j = a_{ij} \cdot \omega_M$ is unimodular and symmetric.

In view of Theorem 6, we may call the matrix A *the intersection form* of the 1-connected 4-dimensional manifold M with respect to the basis ω_A.

For a self-map $f\colon M \to M$ of the manifold M, the induced homomorphism $f^*\colon H^2(M) \to H^2(M)$ gives rise to an $n \times n$ integral matrix $P = (p_{ij})_{n\times n}$ so that

$$\begin{pmatrix} f^*(\omega_1) \\ \\ f^*(\omega_n) \end{pmatrix} = P \begin{pmatrix} \omega_1 \\ \\ \omega_n \end{pmatrix}.$$

In this way, we get a correspondence g from the homotopy set $[M, M]$ of self-maps of M into the set $M(n)$ of all $n \times n$ integral matrix:

$$g\colon [M, M] \to M(n) = \{(p_{ij})_{n\times n} : p_{ij} \in \mathbb{Z}\}.$$

The following result was proved by Duan and Wang [7] (Diagram (1.1) in Theorem A'). Let $\deg f$ denote the topological degree of the map f.

Theorem 7. *Let M be a closed 1-connected 4-dimensional manifold with a basis $\omega = \{\omega_1, \ldots, \omega_n\}$ of the cohomology $H^2(M)$. The map g induces a bijection*

$$[M, M] \longleftrightarrow \{P \in M(n)\colon PAP^T = k \cdot A,\ k \in \mathbb{Z}\}, \tag{5}$$

where A is the intersection form of M with respect to the basis ω, $P = g[f]$, $k = \deg(f)$, and where P^T denotes the transpose of the matrix P.

Example. Let M be the connected sum $\mathbb{C}P^2\#\ldots\#\mathbb{C}P^2$ of n-copies of the 2-dimensional complex projective space $\mathbb{C}P^2$. Then $H^2(M)$ is the free abelian group of rank n , while the intersection form A of M is the identity matrix I_n of rank n. In this case, Theorem 7 states that for each integer matrix P of rank n and integer k that satisfy the matrix equation $PP^T = kI_n$, there exists a self-map f of M such that $f^* = P$ on $H^2(M)$, and $\deg f = k$.

For the cases $n = 2$ and $k \in \{2, 3, 4\}$, all possible solutions (P, k) to the equation $PP^T = kI_n$ are given in the proof of Theorem 17, where the results are applied to calculate the algebraic periods $AP(f)$ of f introduced in Sect. 4.

By the definition of Lefschetz number, we get from Theorem 7 the following corollary.

Corollary 8. *For a homotopy class* $[f] \in [M, M]$, *we have*

(1) $L(f) = 1 + \mathrm{tr}(g([f])) + \deg(f)$. *In particular*

(2) $L(f^n) = 1 + \mathrm{tr}(g([f])^n) + (\deg(f))^n$.

Remark 9. Let M be a closed 1-connected 4-dimensional manifold with the intersection form A. Theorem 7 reduces the problem of homotopy classification of self-maps of M to the arithmetic problem of finding the solutions (P, k) to the quadratic system $PAP^T = k \cdot A$ over the integers.

Conversely, each solution (P, k) of the system $PAP^T = k \cdot A$ gives rise to a self-map $f\colon M \to M$ whose Lefschetz sequence is $L(f^n) = 1 + \mathrm{tr}(P^n) + k^n, n \geq 1$, by Corollary 8.

4. Algebraic periods of self-maps of closed 1-connected 4-manifolds with definite intersection forms

For a self-map f of a topological space, we define *the set of algebraic periods* of f, $AP(f)$, related to the representation of Lefschetz numbers in the form of the periodic expansion (cf. [9]):

$$AP(f) = \{k : b_k \neq 0 \text{ in the expansion } (4)\}.$$

Let us emphasize that in this section, we do not assume that f is a smooth map.

In this part of the paper, we describe the set $AP(f)$ of algebraic periods for self-maps of 1-connected 4-manifolds whose intersection forms A are definite. Obviously, we can assume that the forms A are positively definite with rank m. We begin with two useful lemmas. Let $\mathcal{M}_{m\times m}(\mathbb{R})$ denote the space of $m \times m$ real matrices.

Lemma 10. *If* $P \in \mathcal{M}_{m\times m}(\mathbb{R})$ *is a solution to the equation*

$$PAP^T = A, \tag{6}$$

then P *is similar to an orthogonal matrix.*

Proof. Since the form A is positively definite, we have the Cholesky decomposition $A = QQ^T$ for some $Q \in \mathcal{M}_{m\times m}(\mathbb{R})$. The equation $PAP^T = A$ then becomes

$$P \cdot QQ^T \cdot P^T = QQ^T \text{ i.e., } Q^{-1}PQQ^TP^T(Q^T)^{-1} = I.$$

We obtain $(Q^{-1}PQ) \cdot (Q^{-1}PQ)^T = I$, implying that the matrix $Q^{-1}PQ$ is orthogonal. □

Lemma 11. *If* $P \in \mathcal{M}_{m\times m}(\mathbb{R})$ *is a solution to the equation*

$$PAP^T = k \cdot A \tag{7}$$

for some $k \in \mathbb{N}$, *then*

$$|\operatorname{tr} P| \leq m \cdot \sqrt{k}. \tag{8}$$

Proof. The relation $PAP^T = kA$ is equivalent to $(\frac{1}{\sqrt{k}}P)A(\frac{1}{\sqrt{k}}P)^T = A$. By Lemma 10, the matrix $(\frac{1}{\sqrt{k}}P)$ is similar to an orthogonal matrix P'. On the other hand, since the eigenvalues of an orthogonal matrix have the modulus equal to 1, we get $|\operatorname{tr} P'| \leq m$. Finally, we obtain the desired inequality (8) from the formula:

$$\left|\operatorname{tr}\left(\frac{1}{\sqrt{k}}P\right)\right| = \frac{1}{\sqrt{k}}|\operatorname{tr} P| \leq m.$$

$\square$

Since $\deg(f) = k$ implies that $\deg(f^n) = k^n$, we get the following corollary.

Corollary 12. *Under the assumptions of Lemma 11 we have:*

$$|\operatorname{tr} P^n| \leq m \cdot \sqrt{k^n}. \tag{9}$$

Now we are ready to prove the main theorem of this section, which determines the set $AP(f)$ for self-maps of the 1-connected 4-manifolds whose intersection forms are positively definite.

Theorem 13. *Let M be a closed 1-connected 4-dimensional manifold whose intersection form $A = (a_{ij})_{m\times m}$ is positively definite, and let f be a self-map of M with $\deg(f) = k > 1$.*

(i) If $k \geq 6(m+1)^2$, then for arbitrary n, $n \in AP(f)$.
(ii) If $k < 6(m+1)^2$, but $n \geq 2\log_{\frac{2}{3}k}(2(1+m))$, then $n \in AP(f)$.

Proof. We fix k and m and first find the estimate of the form

$$|nb_n| \geq W(n,k,m),$$

and then examine when the map $W(n,k,m)$ is positive.

By Corollary 8 (the value of $L(f^l)$), we get:

$$\begin{aligned}
|nb_n| = \Big|\sum_{l|n}\mu(n/l)L(f^l)\Big| &\geq |L(f^n)| - \Big|\sum_{l|n,l\neq n}\mu(n/l)L(f^l)\Big| \\
&= |L(f^n)| - \Big|\sum_{l|n,l\neq n}\mu(n/l)(1+\operatorname{Tr}(P^l)+k^l)\Big| \\
&\geq |L(f^n)| - \Big|\sum_{l|n,l\neq n}\mu(n/l)(\operatorname{Tr}(P^l)+k^l) + \sum_{l|n,l\neq n}\mu(n/l)\Big| \\
&\geq |L(f^n)| - \sum_{l|n,l\neq n}|\mu(n/l)|(|\operatorname{Tr}(P^l)|+k^l) - 1,
\end{aligned} \tag{10}$$

where in the last inequality, we used the well-known fact that $\sum_{l|n}\mu(n/l) = 0$.

Now we apply Corollary 12 and the fact that μ takes values in the set $\{-1,0,1\}$. Continuing the sequence of inequalities from (10), we get

$$\begin{aligned}
|nb_n| \geq& |L(f^n)| - \sum_{l|n,l\neq n}(mk^{\frac{l}{2}}+k^l) - 1 \\
\geq& |L(f^n)| - (2\sqrt{n}-1)(mk^{\frac{n}{4}}+k^{\frac{n}{2}}) - 1.
\end{aligned} \tag{11}$$

The last inequality is a consequence of the fact that the number of different divisors of n is not greater than $2\sqrt{n}$ (cf. [4]) and $2|k|^{\frac{l}{2}} + |k|^l$ is increasing as a function of l.

On the other hand, again by the inequality (9) of Corollary 12, we get:

$$|L(f^n)| = |1 + \operatorname{Tr}(P^n) + k^n| \geq |k^n + 1| - |\operatorname{Tr}(P^n)| \geq k^n - mk^{\frac{n}{2}} + 1. \quad (12)$$

Finally, using (11) and (12), we get:

$$|nb_n| \geq W(n,k,m) = k^n - mk^{\frac{n}{2}} - (2\sqrt{n} - 1)(mk^{\frac{n}{4}} + k^{\frac{n}{2}}). \quad (13)$$

In the next part of the proof, we consider item (i) of the thesis. Recall that $k > 1$ and $n \in \mathbb{N}$. Obviously if $k^n - (m + (2\sqrt{n} - 1)(m+1))k^{\frac{n}{2}} > 0$, then $|nb_n| \geq W(n,k,m) > 0$. Solving the inequality, we get:

$$k^{\frac{n}{2}} > 2(1+m)\sqrt{n} - 1. \quad (14)$$

Now we are ready to specify the condition on k which imply that all b_i are non-zero. Observe that if $k^{\frac{n}{2}} > 2(1+m)\sqrt{n}$, then $|nb_n| = W(n,k,m) > 0$.

On the other hand, the inequality $k^{\frac{n}{2}} \geq 2(1+m)\sqrt{n}$ is equivalent to:

$$k \geq \sqrt[n]{(2(1+m))^2}\,\sqrt[n]{n}. \quad (15)$$

Applying the inequalities $\frac{3}{2} \geq \sqrt[n]{n}$ and $(2(1+m))^2 \geq \sqrt[n]{(2(1+m))^2}$ we obtain the estimate depending only on m: for all $k \geq k_0$ such that $k_0 = 6(m+1)^2$ we have $|nb_n| = W(n,k,m) > 0$, so $b_n \neq 0$ for all $n \in \mathbb{N}$. This completes the proof of item (i).

Finally, the last part of the proof will be devoted to the item (ii) of the thesis. Let $m \in \mathbb{Z}_+$ be fixed, for every integer $k > 1$ we have:

$$k \geq \frac{3}{2}\sqrt[n]{(2(1+m))^2} \Rightarrow k \geq \sqrt[n]{(2(1+m))^2}\,\sqrt[n]{n} \quad (16)$$

and

$$k \geq \frac{3}{2}\sqrt[n]{(2(1+m))^2} \Leftrightarrow n > 2\log_{\frac{2}{3}k}(2(1+m)) \quad (17)$$

so there is $n_0(k,m) = 2\log_{\frac{2}{3}k}(2(1+m))$ such that for every $n > n_0(k,m)$, the inequality (15) holds, and thus the expression $W(n,k,m)$ takes only positive values, so also $b_n \neq 0$, which proves (ii). $\square$

Corollary 14. *Let f be a self-map of a closed 1-connected manifold M with the intersection form represented by a positively definite 2×2 matrix A and $\deg(f) = k > 1$. Then*

- *$AP(f) = \mathbb{N}$ if $k \geq 5$;*
- *$AP(f) \supseteq \mathbb{N} \setminus \{1,2,3,4\}$ if $k = 2$;*
- *$AP(f) \supseteq \mathbb{N} \setminus \{1,2\}$ if $k = 3,4$.*

In particular, this statement holds for $M = \mathbb{C}P^2\#\mathbb{C}P^2$ for which A is 2×2 identity matrix.

Proof. By Theorem 13, we have $n \in AP(f)$ if either $k \geq 54$ or $k < 54$ and $n \geq \log_{\frac{2}{3}k}(6)$. It remains to consider the cases $2 \leq k < 54$ and $1 \leq n \leq \log_{\frac{2}{3}k}(6)$. To this end, we first make use of the inequality (11) obtained in the proof of Theorem 13 and then apply inequality (12) and (9) to obtain the desire estimate:

$$|nb_n| \geq |L(f^n)| - \left| \sum_{l|n, l\neq n} \mu(n/l)(1 + \mathrm{Tr}(P^l) + k^l) \right|$$
$$\geq k^n - mk^{\frac{n}{2}} - \left| \sum_{l|n, l\neq n} (1 + mk^{\frac{l}{2}} + k^l) \right|. \tag{18}$$

Facilitated with a computer program for $m = 2$, a case by case computation shows that (18) holds (and thus $n \in AP(f)$), unless $k = 2, n \in \{1, 2, 3, 4\}$ and $k = 3, 4, n \in \{1, 2\}$. $\square$

5. Smooth self-maps of closed 1-connected 4-manifolds with definite intersection forms

Let us remind that the classical Donaldson's theorem for a closed smooth simply connected 4-manifold states that its intersection form is diagonalisable. If the intersection form is positive (negative) definite, it can be diagonalized to the identity matrix (resp. negative identity matrix) over the integers [6].

In the remaining part of this paper, a considered manifold M whose intersection form A is definite, so due to the result of Donaldson, the intersection matrix A is $\pm I_m$, where $I_m = diag\{1, \ldots, 1\}$ is the identity matrix of rank m.

Thus, we get:

$$M = \mathbb{C}P^2 \# \mathbb{C}P^2 \# \ldots \# \mathbb{C}P^2 \text{ if } A = I_m. \tag{19}$$

$$M = \overline{\mathbb{C}P}^2 \# \overline{\mathbb{C}P}^2 \# \ldots \# \overline{\mathbb{C}P}^2 \text{ if } A = -I_m. \tag{20}$$

where $\overline{\mathbb{C}P}^2$ denotes the complex projective plane $\mathbb{C}P^2$ with opposite orientation, and $M \# N$ denotes the connected sum of two smooth manifolds M and N with the same dimension.

Remark 15. Theorem 7 provides a severe restriction on the possible degrees (denoted by k in (5)) for maps between 1-connected 4-manifolds, cf. [7]. In our case, the obvious restriction following Eq. (5) is that $k \geq 0$. In particular, Theorem 13 is applicable to all self-maps of the manifolds M with degree different from 0 and 1.

We remark that Baralić gave a more detailed discussion of possible degrees for self-maps of connected sum of complex projective planes ([2] Sect. 4), as well as maps between certain 4-manifolds.

Remark 16. Observe that the equation $PAP^T = k \cdot A$ in (5) is the same for both $P = \pm id$. For this reason, we can state our theorems for the connected sum of complex projective planes (19), keeping in mind that the same is true also for the connected sum (20).

Carrying on the discussion from Corollary 14, we determine the set $AP(f)$ of algebraic periods of a self-map f of the manifold $M = \mathbb{C}P^2 \# \mathbb{C}P^2$.

Theorem 17. *If* $f\colon \mathbb{C}P^2 \# \mathbb{C}P^2 \to \mathbb{C}P^2 \# \mathbb{C}P^2$ *is a smooth map with* $\deg(f) = k > 1$, *then* $AP(f) = \mathbb{N}$.

Proof. If $\deg(f) = k \notin \{2, 3, 4\}$ we have $b_n \neq 0$ by Corollary 14, implying that $AP(f) = \mathbb{N}$.

Assume next that $\deg(f) = k \in \{2, 3, 4\}$. As in Theorem 7, we set $P = g(f)$ to get by (5) the equations

$$PAP^T = k \cdot A \Leftrightarrow \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}^T = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (21)$$

implying in particular that $k = p_{11}^2 + p_{12}^2 = p_{21}^2 + p_{22}^2$. With $\deg(f) = k \in \{2, 3, 4\}$, the only possible solutions P to the system (21) are:

- $k = 2$

$$\pm \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \pm \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \pm \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \pm \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

- $k = 4$

$$\begin{bmatrix} 0 & \pm 2 \\ \pm 2 & 0 \end{bmatrix}, \begin{bmatrix} \pm 2 & 0 \\ 0 & \pm 2 \end{bmatrix},$$

By formula (3) of Proposition 3, the coefficient b_n of the periodic expansion of the Lefschetz numbers

$$a_n = L(f^n) = 1 + \mathrm{tr}(P^n) + (\deg(f))^n$$

satisfies that

$$b_n = \frac{1}{n} \sum_{l|n} \mu\left(\frac{n}{l}\right) L(f^l) \neq 0,$$

excluding perhaps the cases $k = 2, n \in \{1, 2, 3, 4\}$ or $k = 4, n \in \{1, 2\}$, see Corollary 14. We get $AP(f) = \mathbb{N}$ from the fact that also in these cases (computed straightforwardly) $b_n \neq 0$. This completes the proof of Theorem 17. ∎

5.1. Periodic points of transversal maps

In this subsection, we make use of the description of algebraic periods given in Theorem 13 to state the existence of periodic points with a given minimal periods for transversal maps. We consider here a map

$$f\colon \mathbb{C}P^2\#\ldots\#\mathbb{C}P^2 \to \mathbb{C}P^2\#\ldots\#\mathbb{C}P^2,$$

which is transversal. That is, it is a C^∞ map such that $1 \notin \sigma(Df^i(x))$ for any $i \in \mathbb{N}$ and x being a periodic point with minimal period i, where $\sigma(Df^i(x))$ denotes the spectrum of the derivative of f^i at x. In what follows, we use $\mathrm{Per}(f)$ to denote the set of minimal periods of a map f and by $P(f)$ the set of all periodic points.

The next theorem is a generalization in dimension 4 of the result of Guirao and Llibre ([21] Theorem 2) about the set of minimal periods for transversal self-maps of $\mathbb{C}P^n$ (for the idea of the proof see also [28]).

Theorem 18. *Let f be a transversal self-maps of connected sum of m complex projective spaces with topological degree* $\deg(f) = k$.

Assume that one of the below conditions hold:

(i) $k \geq 6(m+1)^2$,
(ii) $k > 1$ *is arbitrary and* $m = 2$,
(iii) $k > 1$ *and* $m > 1$ *are arbitrary and* $i \geq 2\log_{\frac{2}{3}k}(2(1+m))$.

Then $i \in \mathrm{Per}(f)$ *if i is odd;* $i \in \mathrm{Per}(f)$ *or* $i/2 \in \mathrm{Per}(f)$ *if i is even.*

Proof. For a point x_0 with minimal period i and orbit $O_{x_0} = \{x_0, f(x_0), \ldots, f^{i-1}(x_0)\}$, the index $\mathrm{ind}(f^n, O_{x_0})$ has only four possibilities (cf. [18]) written in terms of Definition 2 as follows:

$$\mathrm{ind}(f^n, O_{x_0}) = \begin{cases} \mathrm{reg}_i(n), \\ -\mathrm{reg}_i(n), \\ \mathrm{reg}_i(n) - \mathrm{reg}_{2i}(n), \\ -\mathrm{reg}_i(n) + \mathrm{reg}_{2i}(n). \end{cases} \tag{22}$$

On the other hand, by the Lefschetz–Hopf formula, we have:

$$L(f^n) = \sum_{x\in P(f)} \mathrm{ind}(f^n, x) = \sum_i \sum_{O\in \mathrm{Orb}_i(f)} \mathrm{ind}(f^n, O), \tag{23}$$

where $\mathrm{Orb}_i(f)$ denotes the set of i-orbits of f (i.e., orbits with points with minimal period i) and where each summand $\mathrm{ind}(f^n, O)$ has the form (22). Expressing $L(f^n)$ in terms of the periodic expansion (4) we get by (23) the equality

$$\sum_{i=1}^{\infty} b_i \,\mathrm{reg}_i(n) = \sum_i \sum_{O\in \mathrm{Orb}_i(f)} \mathrm{ind}(f^n, O). \tag{24}$$

If k, m and i satisfy one of the conditions (i)–(iii) of Theorem 18, we have by Theorem 13 and Theorem 17 that $b_i \neq 0$ in the left-hand side of (24).

Thus, taking into account the formula (22) of the terms $\text{ind}(f^n, O)$ on the right-hand side of (24), we get the desired result. $\square$

6. The invariant $D_r^4[f]$

In this section, we introduce, for a self-map f of a closed m-dimensional simply connected manifold, the topological invariant $D_r^m[f]$ of the minimal number of r-periodic points in the smooth homotopy class of f. The so-called *Differential Dold* sequences (called DD sequences in short) introduced in [13] will play a key role in the definition.

Definition 19. (*Definition 2.9* [13]) A sequence $\{c_n\}_{n=1}^{\infty}$ of integers is called a $DD^m(1)$ sequence if there is a C^1 map $\phi\colon U \to \mathbb{R}^m$ ($U \subset \mathbb{R}^m$) with an isolated fixed point p such that $c_n = \text{ind}(\phi^n, p)$. If $r \geq 1$ is a fixed integer and if the equality $c_n = \text{ind}(\phi^n, p)$ holds for every $n|r$, then the finite subsequence $\{c_n\}_{n|r}$ of $\{c_n\}_{n=1}^{\infty}$ will be called a $DD^m(1|r)$ sequence.

That is, by a $DD^m(1)$ sequence, we mean a sequence of integers that can be locally realized as the sequence of indices of an isolated fixed point of some C^1 map $\phi\colon U \to \mathbb{R}^m, U \subset \mathbb{R}^m$.

The strong restrictions for such sequences result in remarkable difference between smooth and continuous categories in periodic point theory.

Let us remark here that there are also the bounds for the indices of the iterations for other classes of maps, such as homeomorphisms and holomorphic maps cf. [1,23,35].

For a fixed integer $r \geq 1$, the minimal decomposition of the sequence of the Lefschetz numbers of iterations into $DD^m(1|r)$ sequences provides the value of $D_r^m[f]$.

Definition 20. (*Definition 3.4* [13]) Let $\{L(f^n)\}_{n|r}$ be a finite sequence of Lefschetz numbers. We decompose $\{L(f^n)\}_{n|r}$ into the sum of $DD^m(1|r)$ sequences $\{c_i\}_{n|r}$ for $i = 1, \ldots, s$.

Thus, for each $n|r$, the following equality holds:

$$L(f^n) = c_1(n) + \ldots + c_s(n). \tag{25}$$

Each such decomposition determines the number s. We define the number $D_r^m[f]$ as the smallest s which can be obtained in this way.

The following result tells that the invariant $D_r^m[f]$ is equal to the minimal number of r-periodic points in the smooth homotopy class of f.

Theorem 21. ([10]) *Let M be a closed smooth and simply connected manifold of dimension $m \geq 4$ and $r \in \mathbb{N}$ a fixed number. Then*

$$D_r^m[f] = \min\{\#\text{Fix}(g^r) : g \text{ is smoothly homotopic to } f\}.$$

Remark 22. According to Theorem 21, to minimize the number of r-periodic points within a smooth homotopy class, we need only to make use of the $DD^m(1)$ sequences related to the fixed points. This approach is applicable to the manifolds with dimensions at least 4, because the minimal number of

r-periodic points can then be realized by a map with all r-periodic points being fixed points (see [10] for details). We may interpret geometrically the result described in Theorem 21 in the following way. In the smooth homotopy class of a self-map f, one can create fixed points such that the sum of their indices of iterations is equal to the Lefschetz numbers of iterations and then remove all the other r-periodic points [13] (which is guaranteed by the powerful Canceling and Creating Procedures [24]). On the other hand, by definition, the created fixed points must have indices that are $DD^m(1)$ sequences. As a consequence, to determine the minimal number of r-periodic points, we have to find the minimal number of $DD^m(1|r)$ sequences $\{c_i\}_{n|r}$ in the decomposition (25).

7. Combinatorial methods of determination of $D_r^4[f]$.

For an integer $r > 0$, let $\mathrm{Div}(r)$ be the set of all divisors of r. We will consider the set $AP_r(f) := AP(f) \cap \mathrm{Div}(r)$.

To compute the invariant $D_r^4[f]$ by a use of Definition 20, we must find the minimal decomposition of $\{L(f^n)\}_{n|r}$ expressed by the periodic expansion:

$$\sum_{k|r} b_k \mathrm{reg}_k(n) = \sum_{k \in AP_r(f)} b_k \mathrm{reg}_k(n) \tag{26}$$

into the $DD^4(1|r)$ sequences. To this end, we need the following classification result on the $DD^4(1)$ sequences obtained in [15,16]. By $\mathrm{lcm}(d,l)$, we will denote the least common multiple of d and l. By $(d \bmod 2)$, we understand 0 if d is even or 1 if d is odd.

Theorem 23. (Theorem 3.1 [15]) *Any $DD^4(1)$ sequence has one of the following forms:*

(i) $b_1\mathrm{reg}_1(n) + b_d\mathrm{reg}_d(n)$;
(ii) $\varepsilon\mathrm{reg}_1(n) + b_2\mathrm{reg}_2(n) + b_d\mathrm{reg}_d(n) + \gamma_d b_{2d}\mathrm{reg}_{2d}(n)$;
(iii) $\mathrm{reg}_1(n) + b_d\mathrm{reg}_d(n) + b_l\mathrm{reg}_l(n) + b_{\mathrm{lcm}(d,l)}\mathrm{reg}_{\mathrm{lcm}(d,l)}(n)$,

where $\varepsilon \in \{0, \pm 1\}; b_i \in \mathbb{Z}; \gamma_d = d \bmod 2$, *and where* $d, l \geq 3$ *in (iii).*

We first find $D_r^4[f]$ *modulo* reg_1, i.e., we consider the decomposition (25) only for the divisors $k|r$ different from 1. In other words, we temporarily ignore the coefficients at reg_1.

Lemma 24. $D_r^4[f]$ *(mod* reg_1*) is equal h, defined in the following way. h is the minimal number determining the pairs of elements of $AP_r(f) \setminus \{1\}$*

$$\{d_1, l_1\}, \{d_2, l_2\}, \ldots, \{d_h, l_h\} \tag{27}$$

such that

$$\bigcup_{i=1}^{h}\{d_i, l_i, k_i\} = AP_r(f) \setminus \{1\}, \tag{28}$$

where $k_i = \mathrm{lcm}(d_i, l_i)$.

Proof. Let us observe that for $i > 2$ the coefficients, b_i in the list of the sequences in Theorem 23 may be arbitrary. Thus, to decompose the sum (26) into $DD(1|r)$ sequences, it is enough to decompose the set of natural numbers $AP_r(f)$ into the union of sets of indices of sequences "reg" from the list of Theorem 23. The minimal such realization takes the sequences of the form (ii) or (iii) from that list. □

Remark 25. Let us observe that

$$D_r^4[f](mod\,\mathrm{reg}_1) \leq D_r^4[f] \leq D_r^4[f](mod\,\mathrm{reg}_1) + 1. \tag{29}$$

Indeed, if needed, we may realize the sum $b_1\,\mathrm{reg}_1$ in (26) by one additional sequence of the type (i) in Theorem 23.

Summing up this section: we have a combinatorial method of finding our invariant $D_r[f]$ (Lemma 24) but to use, it is necessary to know the set $AP_r(f)$. The next section is devoted to the determination of this set for some special values of r.

8. Determining $D_r^4[f]$ for self-maps of closed smooth 1-connected 4-manifolds

8.1. The value of $D_r^4[f]$ for r being a product of different primes

In general finding, $D_r^4[f]$ is a challenging task; however, in some cases, the value of the invariant may be determined precisely. For H, a finite subset of natural numbers, by $\mathrm{LCM}(H)$ we mean the least common multiple of all elements in H with the convention that $\mathrm{LCM}(\emptyset) = 1$.

Theorem 26. *Let f be a smooth self-map of connected sum of m copies of complex projective planes with degree k and $r = p_1 \ldots p_s$ be a product of different odd prime numbers.*

Assume that one of the following conditions holds:

(i) $k \geq 6(m+1)^2$,

(ii) $k > 1$ *is arbitrary and* $m = 2$.

Then the value of $D_r^4[f] \bmod \mathrm{reg}_1$ *depends only on s and is equal:*

$$D_r^4[f] \bmod \mathrm{reg}_1 = \frac{2^s + (-1)^{s+1}}{3} = h(s). \tag{30}$$

If

(iii) $k > 1$ *and* $m > 1$ *are arbitrary and* $i \geq 2\log_{\frac{2}{3}k}(2(1+m)) =: r_0$, *then*

$$\frac{2^{s'} - 1}{3} + \left\lceil \frac{\#G(r_0)}{3} \right\rceil \leq D_r^4[f] \bmod \mathrm{reg}_1 \leq \frac{2^s + (-1)^{s+1}}{3}, \tag{31}$$

where:

s' denotes the number of elements in $\{p_1, \ldots, p_s\}$ which are $\geq r_0$,
$G(r_0) = \{\beta | r : \beta \geq r_0$ and $\exists_{\alpha|r, \alpha \neq 1}\ \alpha < r_0$ and $\alpha|\beta\}$,
$\lceil a \rceil$ stands for the least integer greater than or equal to a number a.

Proof. We recall that in dimension 4, each fixed point for a smooth map can realize at most 3 coefficients $b_l \neq 0$, i.e., by Theorem 23, each fixed point (identified with a $DD^4(1)$ sequence) may be represented as a combination of at most three basic sequences reg_l (not counting reg_1). Observe that by the assumptions (i) and (ii) and Theorem 13, we get that $AP(f) = \mathbb{N}$, which means that $b_l \neq 0$ for all $l \in \mathrm{LCM}(\{p_1, \ldots, p_s\})$. Since $\#\,\mathrm{LCM}(\{p_1, \ldots, p_s\}) = 2^s$, to realize all elements, different from 1, in $\mathrm{LCM}(\{p_1, \ldots, p_s\})$, we need at least $\dfrac{2^s - 1}{3}$ $DD^4(1)$ sequences. In Ref. [16] (Theorem 4.2), it is shown that all elements in $\mathrm{LCM}(\{p_1, \ldots, p_s\})$ may be realized by $\dfrac{2^s + (-1)^{s+1}}{3}$ $DD^4(1)$ sequences. This ends the proof of the formula (30).

To prove (31), observe that we have $b_l \neq 0$ for $l \geq r_0$ where r_0 is a fixed number. We assume that there are exactly s' numbers in the set $\{p_1, \ldots, p_s\}$ which are $\geq r_0$.

Now, by the same arguments as in the part (i), to realize $\mathrm{LCM}(\{p_1, \ldots, p_s\})$, we need at least $\dfrac{2^{s'} - 1}{3}$ $DD^4(1)$ sequences. On the other hand, again by Ref. [16] (Theorem 4.2), $\dfrac{2^s + (-1)^{s+1}}{3}$ sequences are enough.

As a consequence, we get the estimate:

$$\frac{2^{s'} - 1}{3} \leq D^4_r[f] \bmod \mathrm{reg}_1 \leq \frac{2^s + (-1)^{s+1}}{3}. \tag{32}$$

However, we may strengthen the estimate (32) by taking into account the elements in the set $G(r_0)$. As each $DD^4(1)$ sequence consists of at most 3 regs, to realize basic sequences reg_l with $l \in G(r_0)$ needs at least $\lceil \frac{\#G(r_0)}{3} \rceil$ $DD^4(1)$ sequences. This completes the proof. □

8.2. $D^4_r[f] \bmod \mathrm{reg}_1$ for self-maps of $\mathbb{C}P^2 \# \mathbb{C}P^2$ and small values of r

We established in Theorem 17 that the set of algebraic periods of self-maps of $\mathbb{C}P^2 \# \mathbb{C}P^2$ coincides with $\mathbb{N}$. As a consequence, $AP_r(f) = \mathrm{Div}(r)$, where $\mathrm{Div}(r)$ denotes the set of all divisors of r. Using this fact, in this section, we apply directly Lemma 24 to calculate $D^4_r[f] \bmod \mathrm{reg}_1$ for small values of r, by the application of the computer program. The pseudocode of the program is presented below as Algorithm 1.

The construction of the algorithm can be described as follows. The input data is the set $AP_r(f) = \mathrm{Div}(r)$ of algebraic periods and the output data is the value of the invariant $D^4_r[f] \bmod \mathrm{reg}_1$. In the line 1, we assign to the variable h the initial value equal to 0 and if $AP_r(f) \backslash \{1\}$ is nonempty (line 2), we start the proper part of the algorithm by assigning the value $h = 1$ (line 3). In the line 4, we create a family $\mathcal{B}$ containing pairs of elements of $AP_r(f) \backslash \{1\}$ (cf. (27) of Lemma 24). In the line 5, we define the variable $\overline{\mathcal{B}}$ equal to family sets of the form $\{d, l, k\}$, where $k = \mathrm{lcm}(d, l)$ (cf. (28) of

Algorithm 1: An algorithm for computation $D^4_{r[f]} \bmod \mathrm{reg}_1$

Data: $AP_r(f) = \mathrm{Div}(r)$
Result: $h = D^4_r[f] \bmod \mathrm{reg}_1$

1 $h \leftarrow 0$;
2 **if** $AP_r(f) \setminus \{1\} \neq \emptyset$ **then**
3     $h \leftarrow 1$;
4     $\mathcal{B} \leftarrow \{B : B \subset AP_r(f) \setminus \{1\} \wedge \# B = 2\}$;
5     $\overline{\mathcal{B}} \leftarrow \{\{d, l, \mathrm{lcm}(d, l)\} : d, l \in B, B \in \mathcal{B}\}$;
6     $\overline{\mathcal{B}_f}$, $\mathrm{SUM}_f \leftarrow \{F \in \overline{\mathcal{B}} : F$ is not a subset of any $G \in \overline{\mathcal{B}_f}\}$;
7   **while** $\# \mathrm{SUM}_f \neq 1$ **do**
8         $\mathrm{SUM} \leftarrow \{U \cup W : U \in \mathrm{SUM}_f \ \wedge W \in \overline{\mathcal{B}_f}\}$;
9         $\mathrm{SUM}_f \leftarrow \{F \in \mathrm{SUM} : F$ is not a subset of any $G \in \mathrm{SUM}\}$;
10         $h \leftarrow h + 1$;
11 **return** h;

Lemma 24). In the line 6, we define two equal variables $\overline{\mathcal{B}_f}$, SUM_f and assign them a value of filtering of $\overline{\mathcal{B}}$, leaving only the relevant sets, i.e., those that are not contained in any other set of $\overline{\mathcal{B}}$. The filtering to obtain $\overline{\mathcal{B}_f}$ allows us to limit the size of $\overline{\mathcal{B}}$ variable, and thus consider fewer cases in the rest of the algorithm. Of course, such filtering does not affect the value of the invariant $D^4_r[f] \bmod \mathrm{reg}_1$, indeed if $\bigcup_{i=1}^h B_i = AP_r(f) \setminus \{1\}$ where $B_i \in \mathcal{B}$ and for some number j the set $B_j \subset G$ for some $G \in \mathcal{B}$, then we can simply replace B_j by G. Now note that if $\# \mathrm{SUM}_f$ is equal to 1, then it must contain only the set $AP_r(f) \setminus \{1\}$, since the sum of all sets from $\mathcal{B}$, and hence also $\overline{\mathcal{B}}$ and $\overline{\mathcal{B}}_f$ (SUM_f), must be equal to $AP_r(f) \setminus \{1\}$ (which follows directly from their construction). Finally while $\# \mathrm{SUM}_f \neq 1$, we define a variable SUM containing a family of sets, where to each set of the family SUM_f, we add a set of the family $\overline{\mathcal{B}_f}$. After that SUM is filtered and h is increased by 1. When the algorithm is completed, it returns the value of the variable h. Now we present the results of the computations for small values of r. Note that complexity of computations grows rapidly as the number of divisors of r increases. The cardinality of the set SUM grows very fast, i.e., in each step $\# \mathrm{SUM} = \# \mathrm{SUM}_f \cdot \# \overline{\mathcal{B}}_f$ which involves high computation time of SUM_f of successive iterations and a memory filling problem. Let us notice that usually from the point of view of the applications, the values of considered periods are not very high; thus, our computations may turn out to be useful.

We introduce the following notation. Let $r = p_1^{a_1} \ldots p_k^{a_k}$ be a factorization of the number r and $M = \{a_i \colon i = 1, \ldots, k.\}$. Note that the same set M is shared by all r with the same number of primes and the set of powers appearing in the factorization of r.

Lemma 27. *Let* $r_1 = \prod_{i=1}^k p_i^{a_i}$ *and* $r_2 = \prod_{i=1}^k q_i^{a_i}$, $AP_{r_1}(f) = \mathrm{Div}(r_1)$, $AP_{r_2}(f) = \mathrm{Div}(r_2)$, *then* $D^4_{r_1}[f] \bmod \mathrm{reg}_1 = D^4_{r_2}[f] \bmod \mathrm{reg}_1$.

TABLE 1. The values of $D^4_r[f]$ mod reg_1 for $M = \{a_i : r = \prod_{i=1}^{k} p_i^{a_i}\}$

M	$D^4_r[f]$ mod reg_1
$\{1\}$	1
$\{1, 1\}$	1
$\{1, 1, 1\}$	2
$\{1, 1, 1, 1\}$	5
$\{2\}$	1
$\{2, 1\}$	2
$\{2, 2\}$	3
$\{2, 1, 1\}$	4
$\{2, 2, 1\}$	6
$\{3\}$	1
$\{3, 1\}$	2
$\{3, 2\}$	4
$\{3, 3\}$	5
$\{3, 1, 1\}$	5
$\{4\}$	2
$\{4, 1\}$	3
$\{5\}$	2
$\{5, 1\}$	4
$\{6\}$	3
$\{6, 1\}$	5
$\{7\}$	3

Proof. We define a map $\eta\colon \text{Div}(r_1) \to \text{Div}(r_2)$ as follows: if $x = \prod_{i=1}^{l} p_i^{\beta_i}$ is a factorization of the natural number x, where $l \leq k$, $\beta_i \leq \alpha_i$ then

$$\eta\left(\prod_{i=1}^{l} p_i^{\beta_i}\right) = \prod_{i=1}^{l} q_i^{\beta_i}.$$

Note that η is a bijection between $\text{Div}(r_1)$ and $\text{Div}(r_2)$ such that for a set of natural numbers B of the form $B = \{l, d, \text{lcm}(l, d)\}$, the image of B has the form $B' = \eta(B) = \{\eta(l), \eta(d), \eta(\text{lcm}(l, d))\} = \{\eta(l), \eta(d), \text{lcm}(\eta(l), \eta(d))\}$.

Let $h = D^4_{r_1}[f]$ mod reg_1 and $\mathcal{B} = \{B_1, \ldots, B_h\}$ be a family of sets of the form $B_i = \{d_i, l_i, \text{lcm}(d_i, l_i)\}$ satisfying the conditions (27) and (28) of Lemma 24. The family $\mathcal{B}' = \{B'_1, \ldots, B'_h\}$ where $B'_i = \eta(B_i)$ also satisfy the conditions (27) and (28) of Lemma 24 for $AP_{r_2}(f) = \text{Div}(r_2)$ so $h = D^4_{r_2}[f]$ mod reg_1. ☐

Table 1 contains the values of $D^4_r[f]$ mod reg_1 (i.e., under the assumption that the coefficient at reg_1 is equal to zero) for r such that $M = \{a_i : r = \prod_{i=1}^{k} p_i^{a_i}\}$ (considered as a multiset). In Lemma 27, we showed that the invariant $D^4_r[f]$ mod reg_1 depends only on the form of M. To illustrate the cases of M shown in the table, let us note that they cover all values of the invariant for $1 < r \leq 2000$ when r is odd and $1 < r \leq 250$ when r is even. However, considered cases are much more general since each M corresponds to infinitely many natural numbers, i.e., to $r_1 = 27$ and $r_2 = 12167$, we can assign the set $M = \{3\}$.

Example 28. Let $f\colon \mathbb{C}P^2\#\mathbb{C}P^2 \to \mathbb{C}P^2\#\mathbb{C}P^2$ be a smooth map of $\deg(f) = k > 1$ and $r = p_1^2 \cdot p_2^2$ be a natural number such that p_1, p_2 are prime numbers (e.g., $r = 36$ or $r = 1225$). We ask about the minimal number of r-periodic points in the smooth homotopy class of f. In our case $M = \{2, 2\}$ and by Table 1, we obtain that the minimum we search is (mod reg_1) equal to $D_r^4[f] \bmod \mathrm{reg}_1 = 3$. Note that in the case the coefficient b_1 of periodic expansion of Lefschetz numbers of iterations of f is known, we are able to determine the exact value of the invariant $D_r^4[f]$ which is equal to $D_r^4[f] \bmod \mathrm{reg}_1$ or $D_r^4[f] \bmod \mathrm{reg}_1 + 1$ (so either 3 or 4 in our case) cf. Theorem 4.8 in [16].

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