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Almost homoclinic solutions for a certain class of mixed type functional differential equations

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Abstract. We shall be concerned with the existence of almost homoclinic solutions for a class of second order functional differential equations of mixed type: $\ddot{q}(t)+V_q(t,q(t))+u(t,q(t),q(t-T),q(t+T))=f(t)$, where $t\in\mathbb{R}$, $q\in\mathbb{R}^n$ and $T>0$ is a fixed positive number.

By an almost homoclinic solution (to 0) we mean one that joins 0 to itself and $q\equiv 0$ may not be a stationary point. We assume that V and u are T-periodic with respect to the time variable, V is C^1-smooth and u is continuous. Moreover, f is non-zero, bounded, continuous and square-integrable. The main result provides a certain approximative scheme of finding an almost homoclinic solution.

1. Introduction. In this work, we shall be concerned with the existence of almost homoclinic solutions for second order functional differential equations of mixed type (sometimes known as forward-backward differential equations) of the form

$$\ddot{q}(t)+V_q(t,q(t))+u(t,q(t),q(t-T),q(t+T))=f(t), \tag{1.1}$$

where $t\in\mathbb{R}$, $q\in\mathbb{R}^n$, $T>0$ is a fixed positive number, under the following assumptions:

- $(H1)$ $V\colon \mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ is C^1-smooth, T-periodic with respect to t,
- $(H2)$ $u\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous, T-periodic in t,
- $(H3)$ $f\colon\mathbb{R}\to\mathbb{R}^n$ is non-zero, continuous, bounded and square-integrable.

Here and subsequently, V_q denotes the gradient of V with respect to q.

Definition 1.1. We will say that a solution $q\colon\mathbb{R}\to\mathbb{R}^n$ of (1.1) is *almost homoclinic* (to 0) if $q(t)\to 0$ as $t\to\pm\infty$.

Let us remark that $q\equiv 0$ may not satisfy (1.1). That is why we have decided to call solutions joining 0 to itself almost homoclinic.

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From now on, $(\cdot,\cdot)\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ stands for the standard inner product in $\mathbb{R}^n$ and $|\cdot|\colon \mathbb{R}^n\to[0,\infty)$ denotes the induced norm. Let $E=W^{1,2}(\mathbb{R},\mathbb{R}^n)$ be the Hilbert space of functions from $\mathbb{R}$ into $\mathbb{R}^n$ under the usual norm

$$\|q\|_E^2=\int_{-\infty}^{\infty}(|q(t)|^2+|\dot q(t)|^2)\,dt.$$

For each $k\in\mathbb{N}$, let $f_k\colon\mathbb{R}\to\mathbb{R}^n$ denote a $2kT$-periodic extension of $f_{|[-kT,kT)}$ over $\mathbb{R}$. Remark that f_k may not be continuous at $kT\pm 2kTj$, $j\in\mathbb{Z}$. Let us consider a sequence of functional differential equations

$$\ddot q(t)+V_q(t,q(t))+u(t,q(t),q(t-T),q(t+T))=f_k(t). \tag{1.2}$$

Let $E_k=W^{1,2}_{2kT}(\mathbb{R},\mathbb{R}^n)$ be the Hilbert space of $2kT$-periodic functions from $\mathbb{R}$ into $\mathbb{R}^n$ with the standard norm

$$\|q\|_{E_k}^2=\int_{-kT}^{kT}(|q(t)|^2+|\dot q(t)|^2)\,dt.$$

Let us denote by $C^l_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$, $l\in\mathbb{N}$, the space of functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the topology of almost uniform convergence of functions and all derivatives up to order l.

We will prove the following theorem.

Theorem 1.2. *Let V, u and f satisfy $(H1)$–$(H3)$. Assume that for each $k\in\mathbb{N}$ there is a solution $q_k\in E_k$ of (1.2). If the sequence $\{\|q_k\|_{E_k}\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}$ then there exist a subsequence $\{q_{k_j}\}_{j\in\mathbb{N}}$ and a function $q_0\in E$ such that*

$$q_{k_j}\to q_0\qquad\text{as } j\to\infty$$

in the topology of $C^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$ and q_0 is an almost homoclinic solution of (1.1).

Recently, differential equations involving both advanced and retarded arguments have appeared in an increasing number of models originating from a wide variety of scientific disciplines (see [HVL1, HVL2] and references given there).

Theorem 1.2 yields a certain approximative scheme of finding an almost homoclinic solution for (1.1). This method generalizes that of [J] for the Newtonian systems $\ddot q+V_q(t,q)=f(t)$, where V and f satisfy $(H1)$ and $(H3)$, respectively (see also [IJ1, IJ2]). Similar results for a class of Newtonian systems (with $f\equiv 0$) have been obtained by Rabinowitz in [R] and for a family of first order Hamiltonian systems by Tanaka in [T].

The paper is organized as follows. Theorem 1.2 will be proved in Section 2 by means of the Ascoli–Arzelà lemma. In Section 3, some applications of this theorem will be given for certain problems of variational nature.

2. Proof of Theorem 1.2. For each $k \in \mathbb{N}$, let $L^\infty_{2kT}(\mathbb{R}, \mathbb{R}^n)$ denote the space of $2kT$-periodic essentially bounded measurable functions from $\mathbb{R}$ into $\mathbb{R}^n$ with the norm

$$\|q\|_{L^\infty_{2kT}} = \operatorname{ess\,sup}\{|q(t)|: t \in [-kT, kT]\}.$$

FACT 2.1. *There is $C > 0$, independent of $k \in \mathbb{N}$, such that*

$$\|q\|_{L^\infty_{2kT}} \le C\|q\|_{E_k} \quad \text{for all } q \in E_k. \tag{2.1}$$

The proof of this fact may be found in [IJ1].

LEMMA 2.2. *Let V, u and f satisfy $(H1)$–$(H3)$. Assume that for each $k \in \mathbb{N}$ there is a solution $q_k \in E_k$ of (1.2). If the sequence $\{\|q_k\|_{E_k}\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}$ then there exist a subsequence $\{q_{k_j}\}_{j\in\mathbb{N}}$ and a function $q_0 \in E$ such that $q_{k_j} \to q_0$ as $j \to \infty$ in the topology of $C^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$.*

Proof. There is $M > 0$ such that for each $k \in \mathbb{N}$,

$$\|q_k\|_{E_k} \le M. \tag{2.2}$$

Combining (2.2) with (2.1), we get

$$\|q_k\|_{L^\infty_{2kT}} \le CM. \tag{2.3}$$

Since q_k is a solution of (1.2), from $(H1)$–$(H3)$ and (2.3) it follows that there is $M_1 > 0$, independent of $k \in \mathbb{N}$, such that

$$\|\ddot{q}_k\|_{L^\infty_{2kT}} \le M_1. \tag{2.4}$$

Fix $k \in \mathbb{N}$ and $t \in \mathbb{R}$. If $n = 1$ (i.e. $q_k : \mathbb{R} \to \mathbb{R}$) then there is $s_k \in (t-1, t)$ such that

$$\dot{q}_k(s_k) = \int_{t-1}^{t} \dot{q}_k(s)\,ds = q_k(t) - q_k(t-1)$$

and

$$\dot{q}_k(t) = \int_{s_k}^{t} \ddot{q}_k(s)\,ds + \dot{q}_k(s_k).$$

Consequently,

$$|\dot{q}_k(t)| \le \int_{t-1}^{t} |\ddot{q}_k(s)|\,ds + |q_k(t) - q_k(t-1)| \le M_1 + 2CM \equiv M_2.$$

Hence, if $n \ge 1$, we have

$$\|\dot{q}_k\|_{L^\infty_{2kT}} \le \sqrt{n}M_2. \tag{2.5}$$

To finish the proof, it suffices to show that $\{q_k\}_{k\in\mathbb{N}}$ and $\{\dot{q}_k\}_{k\in\mathbb{N}}$ are equicontinuous. For each $k \in \mathbb{N}$ and $t, s \in \mathbb{R}$, we have

$$|q_k(t) - q_k(s)| = \left|\int_{s}^{t} \dot{q}_k(\tau)\,d\tau\right| \le \sqrt{n}M_2|t-s|,$$

by (2.5), and

$$|\dot{q}_k(t) - \dot{q}_k(s)| \le M_1|t-s|,$$

by (2.4). Applying now the Ascoli–Arzelà lemma, we get the claim. ∎

Fact 2.3. *Let $q\colon \mathbb{R}\to\mathbb{R}^n$ be a continuous mapping. If a weak derivative $\dot{q}\colon \mathbb{R}\to\mathbb{R}^n$ is continuous at $t_0\in\mathbb{R}$, then q is differentiable at t_0 and*

$$\lim_{t\to t_0}\frac{q(t)-q(t_0)}{t-t_0} = \dot{q}(t_0).$$

Let $L^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$ be the space of functions from $\mathbb{R}$ into $\mathbb{R}^n$ that are locally square-integrable.

Fact 2.4. *Let $q\colon \mathbb{R}\to\mathbb{R}^n$ be a continuous map such that $\dot{q}\in L^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$. Then for each $t\in\mathbb{R}$,*

$$|q(t)| \le \sqrt{2}\Big(\int_{t-1/2}^{t+1/2}(|q(s)|^2+|\dot{q}(s)|^2)\,ds\Big)^{1/2}. \tag{2.6}$$

The proofs of Facts 2.3 and 2.4 can be found in [IJ1].

Lemma 2.5. *Let V, u and f satisfy $(H1)$–$(H3)$. If $\{q_{k_j}\}_{j\in\mathbb{N}}$ and q_0 are given by Lemma 2.2 then q_0 is an almost homoclinic solution of* (1.1) *and $q_{k_j}\to q_0$ as $j\to\infty$ in the topology of $C^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$.*

Proof. Fix $a,b\in\mathbb{R}$ such that $a<b$. There is $j_0\in\mathbb{N}$ such that $[a,b]\subset[-k_jT,k_jT)$ for all $j>j_0$. Thus

$$\ddot{q}_{k_j}(t) = f(t) - V_q(t,q_{k_j}(t)) - u(t,q_{k_j}(t),q_{k_j}(t-T),q_{k_j}(t+T))$$

for all $t\in[a,b]$ and $j>j_0$. Hence, if $j>j_0$ then the restriction of $\ddot{q}_{k_j}$ onto $[a,b]$ is continuous. From Fact 2.3 it follows that $\ddot{q}_{k_j}$ is a derivative of $\dot{q}_{k_j}$ in (a,b) for all $j>j_0$. Since $q_{k_j}\to q_0$ and $\dot{q}_{k_j}\to\dot{q}_0$ almost uniformly on $\mathbb{R}$,

$$\ddot{q}_{k_j}(t)\to f(t)-V_q(t,q_0(t))-u(t,q_0(t),q_0(t-T),q_0(t+T))$$

uniformly on $[a,b]$. Consequently,

$$\ddot{q}_0(t) = f(t)-V_q(t,q_0(t))-u(t,q_0(t),q_0(t-T),q_0(t+T))$$

in (a,b). By the above, we conclude that q_0 is a solution of (1.1) and $q_{k_j}\to q_0$ as $j\to\infty$ in the topology of $C^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$.

To finish the proof, we have to show that q_0 is an almost homoclinic solution. Take $l\in\mathbb{N}$. There is $j_0\in\mathbb{N}$ such that $[-lT,lT]\subset[-k_jT,k_jT]$ for all $j>j_0$. By (2.2), for all $j>j_0$ we get

$$\int_{-lT}^{lT}(|q_{k_j}(t)|^2+|\dot{q}_{k_j}(t)|^2)\,dt \le M^2.$$

Letting $j \to \infty$, we obtain

$$\int_{-lT}^{lT} (|q_0(t)|^2 + |\dot{q}_0(t)|^2)\, dt \le M^2.$$

Finally, since l is an arbitrary positive integer,

$$\int_{-\infty}^{\infty} (|q_0(t)|^2 + |\dot{q}_0(t)|^2)\, dt \le M^2,$$

which implies

$$\int_{|t|>r} (|q_0(t)|^2 + |\dot{q}_0(t)|^2)\, dt \to 0 \tag{2.7}$$

as $r \to \infty$. Combining (2.7) with (2.6), we get $q_0(t) \to 0$ as $t \to \pm\infty$. ■

3. Applications. In this section some applications of Theorem 1.2 are indicated. Let assumptions $(H1)$–$(H3)$ hold. From now on, we will also assume that

$(H4)$ there is a C^1-map $U\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of variables (t, x, y), T-periodic in $t \in \mathbb{R}$ such that

$$u(t, q(t), q(t-T), q(t+T)) = U_x(t, q(t), q(t-T)) + U_y(t, q(t+T), q(t)),$$

where U_x and U_y denote the gradients of U with respect to x and y, respectively.

For each $k \in \mathbb{N}$, let $I_k\colon E_k \to \mathbb{R}$ be given by

$$I_k(q) = \int_{-kT}^{kT} \left(\frac{1}{2}|\dot{q}(t)|^2 - V(t, q(t)) - U(t, q(t), q(t-T))\right) dt + \int_{-kT}^{kT} (f_k(t), q(t))\, dt.$$

The functional I_k is differentiable on E_k and it is easy to check that

$$\begin{aligned} I_k'(q)p = &\int_{-kT}^{kT} [(\dot{q}(t), \dot{p}(t)) - (V_q(t, q(t)), p(t))]\, dt \\ &- \int_{-kT}^{kT} [(U_x(t, q(t), q(t-T)), p(t)) + (U_y(t, q(t+T), q(t)), p(t))]\, dt \\ &+ \int_{-kT}^{kT} (f_k(t), p(t))\, dt, \end{aligned}$$

and by $(H4)$,

$$I_k'(q)p = \int_{-kT}^{kT} [(\dot{q}(t),\dot{p}(t)) - (V_q(t,q(t)),p(t))]\,dt$$
$$- \int_{-kT}^{kT} (u(t,q(t),q(t-T),q(t+T)),p(t))\,dt + \int_{-kT}^{kT} (f_k(t),p(t))\,dt.$$

Moreover, by the Fundamental Lemma (see [MW]), for a fixed $k \in \mathbb{N}$ the critical points of I_k are $2kT$-periodic solutions of (1.2).

Example 1. Let us consider second order functional differential equations of mixed type of the form

$$\ddot{q}(t) + u(t,q(t),q(t-T),q(t+T)) = f(t), \tag{3.1}$$

where $t \in \mathbb{R}$, $q \in \mathbb{R}^n$ and $T > 0$ is a fixed positive number. Suppose that $(H2)$–$(H4)$ hold, and furthermore,

$(H5)$ there are $a,b>0$ such that for all $t \in \mathbb{R}$ and $x,y \in \mathbb{R}^n$,
$$-U(t,x,y) \geq -U(t,0,0) + a|x|^2 + b|y|^2,$$

$(H6)$ $\int_0^T U(t,0,0)\,dt = 0.$

Theorem 3.1. *Under assumptions* $(H2)$–$(H6)$, *there is an almost homoclinic solution* $q_0\colon \mathbb{R} \to \mathbb{R}^n$ *of* (3.1) *such that* $\dot{q}_0(t) \to 0$ *as* $t \to \pm\infty$.

An approximative sequence of functional differential equations for (3.1) is as follows:

$$\ddot{q}(t) + u(t,q(t),q(t-T),q(t+T)) = f_k(t), \tag{3.2}$$

where $f_k\colon \mathbb{R} \to \mathbb{R}^n$ is a $2kT$-periodic extension of $f_{|[-kT,kT)}$ onto $\mathbb{R}$. In this case, $I_k\colon E_k \to \mathbb{R}$ is given by

$$I_k(q) = \int_{-kT}^{kT} \left(\frac{1}{2}|\dot{q}(t)|^2 - U(t,q(t),q(t-T)) + (f_k(t),q(t))\right) dt \tag{3.3}$$

and

$$I_k'(q)p = \int_{-kT}^{kT} (\dot{q}(t),\dot{p}(t))\,dt - \int_{-kT}^{kT} (U_x(t,q(t),q(t-T)),p(t))\,dt$$
$$- \int_{-kT}^{kT} (U_y(t,q(t+T),q(t)),p(t))\,dt + \int_{-kT}^{kT} (f_k(t),p(t))\,dt. \tag{3.4}$$

For each $k \in \mathbb{N}$, let $L^2_{2kT}(\mathbb{R},\mathbb{R}^n)$ denote the Hilbert space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norm

$$\|q\|^2_{L^2_{2kT}} = \int_{-kT}^{kT} |q(t)|^2\,dt.$$

Applying $(H3)$, $(H5)$ and $(H6)$, we get

$$\begin{aligned}
(3.5)\qquad I_k(q) &\geq \int_{-kT}^{kT} \left(\frac{1}{2}|\dot{q}(t)|^2 + a|q(t)|^2 + (f_k(t), q(t))\right) dt \\
&\geq \min\{1/2, a\} \int_{-kT}^{kT} (|\dot{q}(t)|^2 + |q(t)|^2)\,dt - \|f_k\|_{L^2_{2kT}} \|q\|_{E_k} \\
&\geq A\|q\|^2_{E_k} - B\|q\|_{E_k},
\end{aligned}$$

where $A = \min\{1/2, a\}$ and $B = \|f\|_{L^2(\mathbb{R},\mathbb{R}^n)}$.

LEMMA 3.2. *Under $(H2)$–$(H6)$, I_k defined by* (3.3) *satisfies the Palais–Smale condition.*

Proof. Let $\{p_j\}_{j\in\mathbb{N}} \subset E_k$ be a sequence such that $\{I_k(p_j)\}_{j\in\mathbb{N}}$ is bounded and $I_k'(p_j) \to 0$ as $j \to \infty$. We have to show that $\{p_j\}_{j\in\mathbb{N}}$ has a convergent subsequence. From (3.5) it follows that $\{p_j\}_{j\in\mathbb{N}}$ is bounded in the Hilbert space E_k. Therefore, along a subsequence, $\{p_j\}_{j\in\mathbb{N}}$ converges weakly in E_k and strongly in $L^\infty_{2kT}(\mathbb{R},\mathbb{R}^n)$ to $p_0 \in E_k$. Hence, passing to a subsequence if necessary, we have $p_j \to p_0$ in $L^2_{2kT}(\mathbb{R},\mathbb{R}^n)$, $(I_k'(p_j) - I_k'(p_0))(p_j - p_0) \to 0$,

$$\int_{-kT}^{kT} (U_x(t, p_j(t), p_j(t-T)) - U_x(t, p_0(t), p_0(t-T)), p_j(t) - p_0(t))\,dt \to 0$$

and

$$\int_{-kT}^{kT} (U_y(t, p_j(t+T), p_j(t)) - U_y(t, p_0(t+T), p_0(t)), p_j(t) - p_0(t))\,dt \to 0$$

as $j \to \infty$. By (3.4), we get

$$\begin{aligned}
&\int_{-kT}^{kT} |\dot{p}_j(t) - \dot{p}_0(t)|^2\,dt \\
&\quad= \int_{-kT}^{kT} (U_x(t, p_j(t), p_j(t-T)) - U_x(t, p_0(t), p_0(t-T)), p_j(t) - p_0(t))\,dt \\
&\qquad+ \int_{-kT}^{kT} (U_y(t, p_j(t+T), p_j(t)) - U_y(t, p_0(t+T), p_0(t)), p_j(t) - p_0(t))\,dt \\
&\qquad+ (I_k'(p_j) - I_k'(p_0))(p_j - p_0),
\end{aligned}$$

which implies $\int_{-kT}^{kT} |\dot{p}_j(t) - \dot{p}_0(t)|^2\,dt \to 0$ as $j \to \infty$. Consequently, $p_j \to p_0$ in E_k as $j \to \infty$. ∎

Proof of Theorem 3.1. By Ekeland's Variational Principle (see Theorems 4.1 and 4.2 in [MW]) and Lemma 3.2, for each $k \in \mathbb{N}$ there is $q_k \in E_k$ such that $I_k(q_k) = \inf_{q \in E_k} I_k(q)$ and $I_k'(q_k) = 0$. From (3.3) and $(H6)$ it follows that $I_k(q_k) \le 0 = I_k(0)$.

Set

$$\varrho = \frac{B + \sqrt{B^2 + 4A}}{2A}.$$

(3.5) implies that if $\|q\|_{E_k} \ge \varrho$ then $I_k(q) \ge 1$. Hence $\|q_k\|_{E_k} < \varrho$ for each $k \in \mathbb{N}$. By Theorem 1.2, we conclude that (3.1) has an almost homoclinic solution $q_0 \in E$.

To finish the proof of Theorem 3.1, we have to show that $\dot{q}_0(t) \to 0$ as $t \to \pm\infty$. From Fact 2.4 it follows that for each $t \in \mathbb{R}$,

$$|\dot{q}_0(t)|^2 \le 2 \int_{t-1/2}^{t+1/2} (|\dot{q}_0(s)|^2 + |\ddot{q}_0(s)|^2)\,ds.$$

By (2.7), it suffices to prove that

$$\int_r^{r+1} |\ddot{q}_0(s)|^2\,ds \to 0$$

as $r \to \pm\infty$. Since q_0 satisfies (3.1), we get

$$\int_r^{r+1} |\ddot{q}_0(s)|^2\,ds \le 2 \int_r^{r+1} |f(s)|^2\,ds + 2 \int_r^{r+1} |u(s, q_0(s), q_0(s-T), q_0(s+T))|^2\,ds.$$

From $(H4)$ and $(H5)$ we have $u(t, 0, 0, 0) = 0$ for each $t \in \mathbb{R}$. From this and $(H2)$ we deduce that

$$\int_r^{r+1} |u(s, q_0(s), q_0(s-T), q_0(s+T))|^2\,ds \to 0 \quad \text{as } r \to \pm\infty.$$

Finally, by $(H3)$,

$$\int_r^{r+1} |f(s)|^2\,ds \to 0 \quad \text{as } r \to \pm\infty. \ ∎$$

EXAMPLE 2. Consider now second order functional differential equations of mixed type

$$\ddot{q}(t) - V_q(t, q(t)) + u(t, q(t), q(t-T), q(t+T)) = f(t), \tag{3.6}$$

where $V\colon \mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, $u\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $f\colon\mathbb{R}\to\mathbb{R}^n$ satisfy assumptions $(H1)$–$(H4)$, and moreover,

$(H7)$ there exist constants $b_1, b_2 > 0$ such that for all $t\in\mathbb{R}$ and $q\in\mathbb{R}^n$,

$$b_1|q|^2 \le V(t,q) \le b_2|q|^2,$$

$(H8)$ for all $t\in\mathbb{R}$ and $q\in\mathbb{R}^n$,

$$V(t,q) \le (q, V_q(t,q)) \le 2V(t,q),$$

$(H9)$ there is $\mu > 2$ such that for all $t\in\mathbb{R}$ and $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{(0,0)\}$,

$$0 < \mu U(t,x,y) \le (U_x(t,x,y),x) + (U_y(t,x,y),y),$$

$(H10)$ the gradient of U with respect to (x,y) is equal to $o(\sqrt{|x|^2+|y|^2})$ as $|x|^2+|y|^2\to 0$ uniformly in t.

Let us remark that $(H9)$–$(H10)$ imply that $U(t,x,y) = o(|x|^2+|y|^2)$ as $|x|^2+|y|^2\to 0$ uniformly in t.

Set

$$\bar{U} = \sup\{U(t,x,y)\colon |x|^2+|y|^2 = 1,\ t\in[0,T]\}, \qquad \bar{b}_1 = \min\{2b_1, 1\}$$

and suppose that

$(H11)$ $\bar{b}_1 > 4\bar{U}$ and $\|f\|_{L^2(\mathbb{R},\mathbb{R}^n)} < \frac{\sqrt{2}}{4C}(\bar{b}_1 - 4\bar{U})$, where C is the positive constant given by (2.1).

THEOREM 3.3. *If* $(H1)$–$(H4)$ *and* $(H7)$–$(H11)$ *are satisfied, then* (3.6) *has an almost homoclinic solution* $q_0\colon\mathbb{R}\to\mathbb{R}^n$. *Moreover,* $\dot{q}_0(t)\to 0$ *as* $t\to\pm\infty$.

We will study an approximative sequence of functional differential equations for (3.6) given by

$$\ddot{q}(t) - V_q(t,q(t)) + u(t,q(t),q(t-T),q(t+T)) = f_k(t), \tag{3.7}$$

where $f_k\colon\mathbb{R}\to\mathbb{R}^n$ is a $2kT$-periodic extension of $f_{|[-kT,kT)}$ onto $\mathbb{R}$. Now, $I_k\colon E_k\to\mathbb{R}$ is defined by

$$I_k(q) = \int_{-kT}^{kT}\Big(\frac{1}{2}|\dot{q}(t)|^2 + V(t,q(t)) - U(t,q(t),q(t-T))\Big)\,dt + \int_{-kT}^{kT}(f_k(t),q(t))\,dt \tag{3.8}$$

and

$$I_k'(q)p = \int_{-kT}^{kT} [(\dot{q}(t), \dot{p}(t)) + (V_q(t, q(t)), p(t))]\, dt - \int_{-kT}^{kT} (U_x(t, q(t), q(t-T)), p(t))\, dt - \int_{-kT}^{kT} (U_y(t, q(t+T), q(t)), p(t))\, dt + \int_{-kT}^{kT} (f_k(t), p(t))\, dt. \tag{3.9}$$

LEMMA 3.4. *Let assumptions* $(H1)$–$(H4)$ *and* $(H7)$–$(H11)$ *hold. Then* I_k *defined by* (3.8) *satisfies the Palais–Smale condition.*

Proof. Let $\{p_j\}_{j\in\mathbb{N}} \subset E_k$ be a sequence such that $\{I_k(p_j)\}_{j\in\mathbb{N}}$ is bounded and $I_k'(p_j) \to 0$ as $j \to \infty$. We have to show that $\{p_j\}_{j\in\mathbb{N}}$ has a convergent subsequence.

There is $C_k > 0$ such that $|I_k(p_j)| \le C_k$ for each $j \in \mathbb{N}$. Moreover, there is $j_0 \in \mathbb{N}$ such that $\|I_k'(p_j)\|_{E_k^*} < \mu$ for all $j > j_0$. Using (3.8), (3.9) and $(H7)$–$(H9)$, we immediately check that

$$2I_k(p_j) - \frac{2}{\mu} I_k'(p_j)p_j \ge \left(1 - \frac{2}{\mu}\right)\bar{b}_1 \|p_j\|_{E_k}^2 + \left(2 - \frac{2}{\mu}\right) \int_{-kT}^{kT} (f_k(t), p_j(t))\, dt.$$

Hence, for all $j > j_0$,

$$\left(1 - \frac{2}{\mu}\right)\bar{b}_1 \|p_j\|_{E_k}^2 - \left(2 - \frac{2}{\mu}\right) \|f\|_{L^2(\mathbb{R},\mathbb{R}^n)} \|p_j\|_{E_k} - 2\|p_j\|_{E_k} - 2C_k \le 0.$$

Since $\mu > 2$, we conclude that $\{p_j\}_{j\in\mathbb{N}}$ is bounded in E_k.

Arguments similar to that in the proof of Lemma 3.2 show that $\{p_j\}_{j\in\mathbb{N}}$ has a convergent subsequence. ∎

FACT 3.5. *For each* $t \in [0, T]$ *and* $x, y \in \mathbb{R}^n$, *if* $|x|^2 + |y|^2 \le 1$ *then*

$$U(t, x, y) \le \bar{U} \cdot \left(\sqrt{|x|^2 + |y|^2}\right)^{\mu}. \tag{3.10}$$

To prove this fact, it is sufficient to notice that a real-valued function

$$(0, \infty) \ni \zeta \mapsto U(t, \zeta^{-1}x, \zeta^{-1}y)\zeta^{\mu}$$

is non-increasing for $t \in [0, T]$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0, 0)\}$. This is a trivial consequence of $(H9)$.

Proof of Theorem 3.3. Fix $k \in \mathbb{N}$. Set $\varrho = \sqrt{2}/(2C)$, where C is given by (2.1). Assume $0 < \|q\|_{E_k} \le \varrho$. From (2.1) it follows that $0 < \|q\|_{L^\infty_{2kT}} \le \sqrt{2}/2$. By (3.10), we have

$$\int_{-kT}^{kT} U(t, q(t), q(t-T))\,dt \le \bar{U} \int_{-kT}^{kT} (|q(t)|^2 + |q(t-T)|^2)^{\mu/2}\,dt \le 2\bar{U} \int_{-kT}^{kT} |q(t)|^2\,dt \le 2\bar{U}\|q\|^2_{E_k}. \tag{3.11}$$

Using (3.8), $(H7)$ and (3.11), we get

$$I_k(q) \ge \frac{1}{2}\bar{b}_1\|q\|^2_{E_k} - 2\bar{U}\|q\|^2_{E_k} - \|f_k\|_{L^2_{2kT}}\|q\|_{E_k} \ge \frac{\bar{b}_1 - 4\bar{U}}{2}\|q\|^2_{E_k} - \|f\|_{L^2(\mathbb{R},\mathbb{R}^n)}\|q\|_{E_k}. \tag{3.12}$$

Moreover, if $\|q\|_{E_k} = \varrho$, then

$$I_k(q) \ge \frac{\bar{b}_1 - 4\bar{U}}{4C^2} - \frac{\sqrt{2}}{2C}\|f\|_{L^2(\mathbb{R},\mathbb{R}^n)} \equiv \alpha > 0 = I_k(0), \tag{3.13}$$

by $(H11)$. Note that both ϱ and α are independent of k. Applying Ekeland's Variational Principle and Lemma 3.4, we obtain a sequence $\{q_k\}_{k\in\mathbb{N}}$ such that $I_k(q_k) = \inf_{\|q\|_{E_k}\le\varrho} I_k(q)$ and $I'_k(q_k) = 0$. By Theorem 1.2, we conclude that (3.6) has an almost homoclinic solution $q_0 \in E$. Analysis similar to that in the proof of Theorem 3.1 shows that $\dot{q}_0(t) \to 0$ as $t \to \pm\infty$. ∎

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