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An analogue of the Pöschl–Teller anharmonic oscillator on an N-dimensional sphere

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Abstract A Schrödinger particle on an N-dimensional ($N \geqslant 2$) hypersphere of radius R is considered. The particle is subjected to the action of a force characterized by the potential $V(\theta) = 2m\omega_1^2 R^2 \tan^2(\theta/2) + 2m\omega_2^2 R^2 \cot^2(\theta/2)$, where $0 \leqslant \theta \leqslant \pi$ is the hyperlatitude angular coordinate. In the general case when $\omega_1 \neq \omega_2$, this is a model of a hyperspherical analogue of the Pöschl–Teller anharmonic oscillator. Energy eigenvalues and normalized eigenfunctions for this system are found in closed analytical forms. For $N = 2$, our results reproduce those obtained by Kazaryan *et al*. (Physica E 52:122, 2013). For $N \geqslant 2$ arbitrary and for $\omega_2 = 0$, the results of Mardoyan and Petrosyan (J. Contemp. Phys. 48:70, 2013) for their model of an isotropic hyperspherical harmonic oscillator are recovered. The Euclidean limit for the anharmonic oscillator in question is also discussed.

1 Introduction

The search for closed-form analytical solutions to quantum-mechanical wave equations in spherical geometry dates back at least to the early 1940s. It was then that Schrödinger [1] and Stevenson [2] considered an analogue of the Schrödinger–Coulomb energy eigenvalue problem on the hypersphere $\mathbb{S}_R^3$, with the potential $V(\theta) = V_0 \cot\theta$, where $0 \leqslant \theta \leqslant \pi$ is the hyperlatitude angle on $\mathbb{S}_R^3$. At the end of the 1970s, Higgs [3] and Leemon [4] solved the spherical Schrödinger equation for the hemispherical analogue of the isotropic harmonic oscillator with the latitudinal potential $V(\theta) = V_0 \tan^2\theta$ ($V_0 \geqslant 0$, $0 \leqslant \theta \leqslant \pi/2$). The two systems mentioned, like their counterparts in the Euclidean space, possess hidden dynamical symmetries. For this reason, they appeared to be attractive objects of research and in the following years their various properties were comprehensively studied in a number of works. The most important results obtained in the course of research conducted up to the mid- and late-2000s were reviewed in the monographs by Shchepetilov [5] and by Redkov and Ovsiyuk [6], respectively.

In 2013, a paper by Mardoyan and Petrosyan [7] was published, in which they found analytical solutions to the energy eigenproblem on the hypersphere $\mathbb{S}_R^N$ for a Schrödinger particle in the latitudinal potential $V(\theta) = V_0 \tan^2(\theta/2)$, ($V_0 \geqslant 0$, $0 \leqslant \theta \leqslant \pi$). These authors also showed that in the Euclidean limit the system considered by them, similarly to the one discussed earlier by Higgs [3] and Leemon [4], passes into an N-dimensional isotropic harmonic oscillator.

Also in 2013, Kazaryan *et al.* [8] considered a Schrödinger particle moving on a two-dimensional sphere $\mathbb{S}_R^2$, in the potential $V(\theta) = V_1 \tan^2(\theta/2) + V_2 \cot^2(\theta/2)$, ($V_1 \geqslant 0$, $V_2 \geqslant 0$, $0 \leqslant \theta \leqslant \pi$), and found analytical solutions to the corresponding energy eigenproblem. In the case of $V_2 = 0$ and $V_1 > 0$ (and, due to the invariance of the potential in question with respect to the combined transformations $\theta \leftrightarrow \pi - \theta$ and $V_1 \leftrightarrow V_2$, also in the case of $V_1 = 0$ and $V_2 > 0$), the system considered in Ref. [8] evidently reduces to the two-dimensional example of a spherical isotropic harmonic oscillator of Mardoyan and Petrosyan [7]. On the other hand, in the case when both V_1 and V_2 are different from zero, the system from Ref. [8] may be seen as a kind of a (two-dimensional) spherical analogue of the anharmonic Pöschl–Teller oscillator [9].

The purpose of this work is to generalize the results from Ref. [8] to the case when the domain is an N-dimensional hypersphere $\mathbb{S}_R^N$, ($N \geqslant 2$). The paper is structured as follows. In Sect. 2, we recall the basic relevant facts from the methodology of treating the Schrödinger equation in the spherical geometry. In particular, we give a quasi-radial Schrödinger equation for potentials which depend on the hyperlatitudinal angle only. In Sect. 3, the Sturm–Liouville problem involving such an equation with a potential describing a spherical analogue of the Pöschl–Teller anharmonic oscillator is solved, yielding energy eigenvalues and associated normalized quasi-radial eigenfunctions in closed analytical forms. Some special cases are then considered in more detail in Sect. 4. In Sect. 5, we study the Euclidean limit for the oscillator in question by stereographically projecting the system onto a suitably chosen tangent space and then by going with the radius of the sphere to infinity. The main results of the work are summarized in Sect. 6.

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2 The Schrödinger equation in a spherical geometry

Let $\mathbb{S}_R^N \subset \mathbb{R}^{N+1}$ be an N-dimensional hypersphere with its center at the point $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^{N+1}$ and of radius R. Its equation is

$$\sum_{n=1}^{N+1} x_n^2 = R^2, \tag{2.1}$$

where $\{x_n\}_{n=1}^{N+1}$ are coordinates in some Cartesian system in $\mathbb{R}^{N+1}$ with its origin at $\mathbf{0}$. The position of a point on $\mathbb{S}_R^N$ may be uniquely determined by giving its hyperspherical angular coordinates $\{\theta_n\}_{n=1}^{N}$ related to the Cartesian coordinates $\{x_n\}_{n=1}^{N+1}$ as follows:

$$\begin{cases} x_1 = R \sin\theta_N \sin\theta_{N-1} \ldots \sin\theta_3 \sin\theta_2 \cos\theta_1, \\ x_2 = R \sin\theta_N \sin\theta_{N-1} \ldots \sin\theta_3 \sin\theta_2 \sin\theta_1, \\ x_3 = R \sin\theta_N \sin\theta_{N-1} \ldots \sin\theta_3 \cos\theta_2, \\ x_4 = R \sin\theta_N \sin\theta_{N-1} \ldots \cos\theta_3, \\ \vdots \\ x_N = R \sin\theta_N \cos\theta_{N-1}, \\ x_{N+1} = R \cos\theta_N, \end{cases} \tag{2.2a}$$

with

$$0 \leqslant \theta_1 < 2\pi \tag{2.2b}$$

and

$$0 \leqslant \theta_n \leqslant \pi \qquad (2 \leqslant n \leqslant N). \tag{2.2c}$$

The coordinate θ_N is termed the hyperlatitude or the quasi-radial variable. The points for which $\theta_N = 0$ or $\theta_N = \pi$ are sometimes called the *north* or the *south* poles of the hypersphere, respectively. In what follows, the set of the coordinates $\{\theta_n\}_{n=1}^{N}$ as an entity will be denoted as Ω_N; similarly, hereafter the symbol Ω_{N-1} will stand for the set $\{\theta_n\}_{n=1}^{N-1}$.

The Laplace–Beltrami operator on $\mathbb{S}_R^N$ is related to its counterpart on the unit sphere $\mathbb{S}^N$ simply through

$$\Delta_{\mathbb{S}_R^N} = \frac{1}{R^2} \Delta_{\mathbb{S}^N}. \tag{2.3}$$

The latter operator may be conveniently defined by the chain of recurrence relations

$$\begin{aligned} \Delta_{\mathbb{S}^n} &= \frac{1}{\sin^{n-1}\theta_n} \frac{\partial}{\partial\theta_n} \sin^{n-1}\theta_n \frac{\partial}{\partial\theta_n} + \frac{1}{\sin^2\theta_n} \Delta_{\mathbb{S}^{n-1}} \\ &= \frac{\partial^2}{\partial\theta_n^2} + (n-1)\cot\theta_n \frac{\partial}{\partial\theta_n} + \frac{1}{\sin^2\theta_n} \Delta_{\mathbb{S}^{n-1}} \qquad (2 \leqslant n \leqslant N), \end{aligned} \tag{2.4a}$$

with

$$\Delta_{\mathbb{S}^1} = \frac{\partial^2}{\partial\theta_1^2}. \tag{2.4b}$$

The time-independent Schrödinger partial differential equation for a particle moving on the sphere $\mathbb{S}_R^N$, $(N \geqslant 2)$, under the action of a force derivable from the potential $V(\Omega_N)$, is

$$\left[-\frac{\hbar^2}{2mR^2} \Delta_{\mathbb{S}^N} + V(\Omega_N)\right] \Psi(\Omega_N) = E\Psi(\Omega_N). \tag{2.5}$$

In the special case when the potential V is a function of the hyperlatitude only, i.e., when it holds that

$$V(\Omega_N) = V(\theta_N), \tag{2.6}$$

Equation (2.5) is separable and admits particular solutions that are of the form

$$\Psi_{L\eta}(\Omega_N) = F_L(\theta_N) Y_{L\eta}^{(N-1)}(\Omega_{N-1}). \tag{2.7}$$

Here $Y_{L\eta}^{(N-1)}(\Omega_{N-1})$ are the hyperspherical harmonics, i.e., the eigenfunctions of the Laplace–Beltrami operator on $\mathbb{S}^{N-1}$,

$$\Delta_{\mathbb{S}^{N-1}} Y_{L\eta}^{(N-1)}(\Omega_{N-1}) = -L(L+N-2) Y_{L\eta}^{(N-1)}(\Omega_{N-1}), \tag{2.8}$$

which are normalizable in the sense of

$$\oint_{\mathbb{S}^{N-1}} \mathrm{d}^{N-1}\Omega_{N-1}\,|Y_{L\eta}^{(N-1)}(\Omega_{N-1})|^2 = 1. \tag{2.9}$$

We take $L \in \mathbb{Z}$ for $N = 2$ and $L \in \mathbb{N}_0$ for $N \geqslant 3$. If $N = 2$, then η is redundant and is to be suppressed, while if $N \geqslant 3$, then η stands collectively for all quantum numbers necessary to distinguish between the hyperspherical harmonics that belong to the (degenerate for $L \neq 0$) eigenvalue $-L(L+N-2)$ of $\Delta_{\mathbb{S}^{N-1}}$. The function $F_L(\theta_N)$ obeys the quasi-radial Schrödinger equation

$$\left\{-\frac{\hbar^2}{2mR^2}\left[\frac{\mathrm{d}^2}{\mathrm{d}\theta_N^2} + (N-1)\cot\theta_N\frac{\mathrm{d}}{\mathrm{d}\theta_N} - \frac{L(L+N-2)}{\sin^2\theta_N}\right] + V(\theta_N) - E\right\}F_L(\theta_N) = 0. \tag{2.10}$$

On the following pages, we shall derive and analyze solutions to Eq. (2.10) with arbitrary $N \geqslant 2$ and with the potential

$$V(\theta_N) = 2m\omega_1^2R^2\tan^2\frac{\theta_N}{2} + 2m\omega_2^2R^2\cot^2\frac{\theta_N}{2} \qquad (0 \leqslant \theta_N \leqslant \pi), \tag{2.11a}$$

where, for definiteness, we shall be assuming that $\omega_1 \geqslant 0$, $\omega_2 \geqslant 0$. The system characterized by the above potential is essentially a hyperspherical analogue of the Pöschl–Teller [9] anharmonic oscillator. This is justified by the fact that after a bit of trigonometry Eq. (2.11a) may be rewritten in the form

$$V(\theta_N) = \frac{2m\omega_1^2R^2}{\cos^2(\theta_N/2)} + \frac{2m\omega_2^2R^2}{\sin^2(\theta_N/2)} - 2m(\omega_1^2+\omega_2^2)R^2 \qquad (0 \leqslant \theta_N \leqslant \pi). \tag{2.11b}$$

Once Eq. (2.11b) is plugged into Eq. (2.10) and the formal replacements $N \mapsto 1$, $L \mapsto 0$ and $E \mapsto E - 2m(\omega_1^2+\omega_2^2)R^2$ are made therein, the resulting equation is found to coincide, up to notational differences, with the one-dimensional Schrödinger equation following from Eqs. (2a) and (3) of Ref. [9].

Before moving on, it should be emphasized that the *spherical* analogue of the Pöschl–Teller oscillator considered in this work differs from the system with the potential

$$\begin{aligned} V(\theta_N) &= \frac{1}{2}m\omega_1^2R^2\tan^2\theta_N + \frac{1}{2}m\omega_2^2R^2\cot^2\theta_N \\ &= \frac{m\omega_1^2R^2}{2\cos^2\theta_N} + \frac{m\omega_2^2R^2}{2\sin^2\theta_N} - \frac{1}{2}m(\omega_1^2+\omega_2^2)R^2 \qquad (0 \leqslant \theta_N \leqslant \pi/2), \end{aligned} \tag{2.12}$$

which also admits analytical solutions to the quasi-radial Schrödinger equation (2.10), albeit on a different domain, and which in turn can be viewed as a *hemispherical* analogue of the Pöschl–Teller oscillator.

For the sake of notational clarity, from now on we shall be dropping the index N at θ_N.

3 Energy levels and normalized eigenfunctions for the potential (2.11a) on an *N*-dimensional sphere

The quasi-radial Eq. (2.10) with the potential (2.11a) is

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + (N-1)\cot\theta\frac{\mathrm{d}}{\mathrm{d}\theta} - \frac{L(L+N-2)}{\sin^2\theta} - \frac{4m^2\omega_1^2R^4}{\hbar^2}\tan^2\frac{\theta}{2} - \frac{4m^2\omega_2^2R^4}{\hbar^2}\cot^2\frac{\theta}{2} + \frac{2mR^2}{\hbar^2}E\right]F_L(\theta) = 0. \tag{3.1}$$

We wish to solve it under the constraint that $F_L(\theta)$ remains bounded as $\theta \to 0+0$ and as $\theta \to \pi - 0$, considering the energy parameter E as an eigenvalue.

In the first step, we switch from the variable θ to the new one:

$$\rho = \cos\theta \qquad (-1 \leqslant \rho \leqslant 1) \tag{3.2}$$

and make the substitution

$$F_L(\theta) \equiv F_L(\rho) = \frac{f_L(\rho)}{(1-\rho^2)^{N/4-1/2}} \tag{3.3}$$

This casts Eq. (3.1) into the following one:

$$\left[\frac{\mathrm{d}}{\mathrm{d}\rho}(1-\rho^2)\frac{\mathrm{d}}{\mathrm{d}\rho} + \lambda(\lambda+1) - \frac{\mu_{L2}^2}{2(1-\rho)} - \frac{\mu_{L1}^2}{2(1+\rho)}\right]f_L(\rho) = 0, \tag{3.4}$$

with

$$\lambda = -\frac{1}{2} + \frac{1}{2}\sqrt{(N-1)^2 + \frac{16m^2(\omega_1^2+\omega_2^2)R^4}{\hbar^2} + \frac{8mR^2}{\hbar^2}E} \tag{3.5}$$

and

$$\mu_{Lk} = \sqrt{\left(L + \frac{N}{2} - 1\right)^2 + \left(\frac{4m\omega_k R^2}{\hbar}\right)^2} \qquad (k = 1, 2). \tag{3.6}$$

Equation (3.4) is the generalized associated Legendre equation [10–12]. Its general solution is

$$f_L(\rho) = A_L P_\lambda^{-\mu_{L2},\mu_{L1}}(\rho) + B_L P_\lambda^{\mu_{L2},\mu_{L1}}(\rho), \tag{3.7}$$

where

$$P_\lambda^{\mu,\nu}(\rho) = \frac{1}{\Gamma(1-\mu)}\frac{(1+\rho)^{\nu/2}}{(1-\rho)^{\mu/2}}\,{}_2F_1\left(-\lambda - \frac{\mu-\nu}{2}, \lambda + 1 - \frac{\mu-\nu}{2}; 1-\mu; \frac{1-\rho}{2}\right) \tag{3.8}$$

is the generalized associated Legendre function of the first kind, while A_L and B_L are arbitrary constants. Combining Eqs. (3.3), (3.7) and (3.8) yields the quasi-radial function $F_L(\rho)$ in terms of the hypergeometric function ${}_2F_1$:

$$\begin{aligned} F_L(\rho) &= A_L'(1-\rho)^{\mu_{L2}/2-N/4+1/2}(1+\rho)^{\mu_{L1}/2-N/4+1/2} \\ &\quad\times {}_2F_1\left(-\lambda + \frac{\mu_{L1}+\mu_{L2}}{2}, \lambda + 1 + \frac{\mu_{L1}+\mu_{L2}}{2}; 1+\mu_{L2}; \frac{1-\rho}{2}\right) \\ &\quad+ B_L'(1-\rho)^{-\mu_{L2}/2-N/4+1/2}(1+\rho)^{\mu_{L1}/2-N/4+1/2} \\ &\quad\times {}_2F_1\left(-\lambda + \frac{\mu_{L1}-\mu_{L2}}{2}, \lambda + 1 + \frac{\mu_{L1}-\mu_{L2}}{2}; 1-\mu_{L2}; \frac{1-\rho}{2}\right), \end{aligned} \tag{3.9}$$

with A_L' and B_L' being arbitrary constants. The presence of the factor $(1-\rho)^{-\mu_{L2}/2-N/4+1/2}$ implies that the second term on the right-hand side of Eq. (3.9) diverges as $\rho \to 1-0$. To get rid of this term, we set $B_L' = 0$, thus obtaining

$$\begin{aligned} F_L(\rho) &= A_L'(1-\rho)^{\mu_{L2}/2-N/4+1/2}(1+\rho)^{\mu_{L1}/2-N/4+1/2} \\ &\quad\times {}_2F_1\left(-\lambda + \frac{\mu_{L1}+\mu_{L2}}{2}, \lambda + 1 + \frac{\mu_{L1}+\mu_{L2}}{2}; 1+\mu_{L2}; \frac{1-\rho}{2}\right). \end{aligned} \tag{3.10}$$

The hypergeometric function in Eq. (3.10) diverges as $\rho \to -1+0$ unless we stipulate that

$$-\lambda + \frac{\mu_{L1}+\mu_{L2}}{2} = -n_\theta \qquad (n_\theta \in \mathbb{N}_0). \tag{3.11}$$

Merging this condition with the definition (3.5) of λ gives the quantized energy levels for the system under the consideration:

$$\begin{aligned} E_{n_\theta L} &= \frac{\hbar^2}{2mR^2}\left[n_\theta + \frac{N}{2} + \frac{1}{2}\sqrt{\left(L+\frac{N}{2}-1\right)^2 + \left(\frac{4m\omega_1R^2}{\hbar}\right)^2} + \frac{1}{2}\sqrt{\left(L+\frac{N}{2}-1\right)^2 + \left(\frac{4m\omega_2R^2}{\hbar}\right)^2}\right] \\ &\quad\times\left[n_\theta - \frac{N}{2} + 1 + \frac{1}{2}\sqrt{\left(L+\frac{N}{2}-1\right)^2 + \left(\frac{4m\omega_1R^2}{\hbar}\right)^2} + \frac{1}{2}\sqrt{\left(L+\frac{N}{2}-1\right)^2 + \left(\frac{4m\omega_2R^2}{\hbar}\right)^2}\right] \\ &\quad- 2m(\omega_1^2+\omega_2^2)R^2, \end{aligned} \tag{3.12a}$$

or equivalently

$$\begin{aligned} E_{n_\theta L} = \frac{\hbar^2}{2mR^2}\Bigg\{&\left(n_\theta + \frac{N}{2}\right)\left(n_\theta - \frac{N}{2} + 1\right) + \frac{1}{2}\left(L + \frac{N}{2} - 1\right)^2 \\ &+ \left(n_\theta + \frac{1}{2}\right)\left[\sqrt{\left(L+\frac{N}{2}-1\right)^2 + \left(\frac{4m\omega_1R^2}{\hbar}\right)^2} + \sqrt{\left(L+\frac{N}{2}-1\right)^2 + \left(\frac{4m\omega_2R^2}{\hbar}\right)^2}\right] \\ &+ \frac{1}{2}\sqrt{\left[\left(L+\frac{N}{2}-1\right)^2 + \left(\frac{4m\omega_1R^2}{\hbar}\right)^2\right]\left[\left(L+\frac{N}{2}-1\right)^2 + \left(\frac{4m\omega_2R^2}{\hbar}\right)^2\right]}\Bigg\}. \end{aligned} \tag{3.12b}$$

For $N = 2$, either of Eqs. (3.12) may be shown to reduce, up to notational differences, to Eq. (25) in Ref. [8].

To obtain the most suitable form of the quasi-radial eigenfunctions associated with the eigenenergies (3.12), we observe that insertion of Eq. (3.11) into Eq. (3.10) gives

$$F_{n_\theta L}(\rho) = A'_L(1-\rho)^{\mu_{L2}/2-N/4+1/2}(1+\rho)^{\mu_{L1}/2-N/4+1/2}\,{}_2F_1\left(-n_\theta, n_\theta+\mu_{L1}+\mu_{L2}+1; \mu_{L2}+1; \frac{1-\rho}{2}\right). \tag{3.13}$$

The function ${}_2F_1$ appearing in the above equation is closely related to the Jacobi polynomial [13, p. 212]:

$$P_{n_\theta}^{(\mu_{L2},\mu_{L1})}(\rho) = \frac{\Gamma(n_\theta+\mu_{L2}+1)}{n_\theta!\,\Gamma(\mu_{L2}+1)}\,{}_2F_1\left(-n_\theta, n_\theta+\mu_{L1}+\mu_{L2}+1; \mu_{L2}+1; \frac{1-\rho}{2}\right) \tag{3.14}$$

[not to be confused with the generalized associated Legendre function of the first kind defined in Eq. (3.8)]. Hence, we may write

$$F_{n_\theta L}(\rho) = A''_L(1-\rho)^{\mu_{L2}/2-N/4+1/2}(1+\rho)^{\mu_{L1}/2-N/4+1/2}P_{n_\theta}^{(\mu_{L2},\mu_{L1})}(\rho), \tag{3.15}$$

where A''_L is a nonzero constant. We choose the value of the latter as to have the complete wave function (2.7) normalized to unity in the sense of

$$R^N \oint_{\mathbb{S}^N} \mathrm{d}^N\Omega_N\,|\Psi_{n_\theta L\eta}(\Omega_N)|^2 = 1, \tag{3.16}$$

where $\mathrm{d}^N\Omega_N$ is an infinitesimal surface element of the unit hypersphere $\mathbb{S}^N$. By virtue of the identity

$$\mathrm{d}^N\Omega_N = \sin^{N-1}\theta\,\mathrm{d}\theta\,\mathrm{d}^{N-1}\Omega_{N-1} \tag{3.17}$$

and the normalization condition (2.9) for the hyperspherical harmonics, Eq. (3.16) yields the integral constraint

$$R^N\int_0^\pi \mathrm{d}\theta\,\sin^{N-1}\theta\,|F_{n_\theta L}(\theta)|^2 = 1. \tag{3.18}$$

On plugging Eq. (3.15) into the left-hand side of Eq. (3.18) and invoking the formula [13, p. 212]

$$\begin{aligned}&\int_{-1}^{1}\mathrm{d}x\,(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)P_{n'}^{(\alpha,\beta)}(x)\\&=\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\,(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}\delta_{nn'}\quad(\alpha>-1,\beta>-1),\end{aligned} \tag{3.19}$$

we finally obtain that, up to an arbitrary phase factor which we set equal to unity, the normalized quasi-radial eigenfunctions, expressed in terms of the Jacobi polynomials, are

$$\begin{aligned}F_{n_\theta L}(\theta) &= \sqrt{\frac{n_\theta!\,(2n_\theta+\mu_{L1}+\mu_{L2}+1)\Gamma(n_\theta+\mu_{L1}+\mu_{L2}+1)}{R^N 2^{\mu_{L1}+\mu_{L2}+1}\Gamma(n_\theta+\mu_{L1}+1)\Gamma(n_\theta+\mu_{L2}+1)}}\\&\quad\times(1-\cos\theta)^{\mu_{L2}/2-N/4+1/2}(1+\cos\theta)^{\mu_{L1}/2-N/4+1/2}P_{n_\theta}^{(\mu_{L2},\mu_{L1})}(\cos\theta),\end{aligned} \tag{3.20a}$$

or equivalently

$$\begin{aligned}F_{n_\theta L}(\theta) &= \sqrt{\frac{n_\theta!\,(2n_\theta+\mu_{L1}+\mu_{L2}+1)\Gamma(n_\theta+\mu_{L1}+\mu_{L2}+1)}{R^N 2^{N-1}\Gamma(n_\theta+\mu_{L1}+1)\Gamma(n_\theta+\mu_{L2}+1)}}\\&\quad\times\left(\sin\frac{\theta}{2}\right)^{\mu_{L2}-N/2+1}\left(\cos\frac{\theta}{2}\right)^{\mu_{L1}-N/2+1}P_{n_\theta}^{(\mu_{L2},\mu_{L1})}(\cos\theta).\end{aligned} \tag{3.20b}$$

If Eqs. (3.20b) and (3.14) are combined to yield the normalized $F_{n_\theta L}(\theta)$ in terms of the hypergeometric function ${}_2F_1$ rather than in terms of the Jacobi polynomial, in the case of $N=2$ the resulting expression may be shown to coincide, up to notational differences and up to the factor R^{-1} due to slightly different normalization conditions used, with what follows from Eqs. (22) and (24) of Ref. [8].

4 Special cases

With the solutions corresponding to the general form of the potential (2.11a) being found, we proceed to the analysis of some particular cases when the functional form of $V(\theta)$ simplifies.

4.1 The case of $\omega_1 = \omega$ and $\omega_2 = 0$

The potential $V(\theta)$ is then

$$V(\theta) = 2m\omega^2 R^2 \tan^2\frac{\theta}{2} \qquad (0 \leqslant \theta \leqslant \pi) \tag{4.1}$$

and is the one considered by Mardoyan and Petrosyan [7]. From Eqs. (3.12), we obtain

$$\begin{aligned} E_{n_\theta L} = \frac{\hbar^2}{2mR^2}\left[n_\theta + \frac{L}{2} + \frac{3N}{4} - \frac{1}{2} + \frac{1}{2}\sqrt{\left(L + \frac{N}{2} - 1\right)^2 + \left(\frac{4m\omega R^2}{\hbar}\right)^2}\,\right] \\ \times \left[n_\theta + \frac{L}{2} - \frac{N}{4} + \frac{1}{2} + \frac{1}{2}\sqrt{\left(L + \frac{N}{2} - 1\right)^2 + \left(\frac{4m\omega R^2}{\hbar}\right)^2}\,\right] - 2m\omega^2 R^2, \end{aligned} \tag{4.2a}$$

or equivalently

$$\begin{aligned} E_{n_\theta L} = \frac{\hbar^2}{2mR^2}\Bigg[\left(n_\theta + \frac{L}{2} + \frac{3N}{4} - \frac{1}{2}\right)\left(n_\theta + \frac{L}{2} - \frac{N}{4} + \frac{1}{2}\right) + \frac{1}{4}\left(L + \frac{N}{2} - 1\right)^2 \\ + \left(n_\theta + \frac{L}{2} + \frac{N}{4}\right)\sqrt{\left(L + \frac{N}{2} - 1\right)^2 + \left(\frac{4m\omega R^2}{\hbar}\right)^2}\,\Bigg]. \end{aligned} \tag{4.2b}$$

Save for notational differences, the expression for $E_{n_\theta L}$ given in Eq. (4.2b) is identical to the one displayed in Eq. (8) of Ref. [7]. Furthermore, in the case discussed here our Eq. (3.20b) simplifies to

$$\begin{aligned} F_{n_\theta L}(\theta) = \sqrt{\frac{n_\theta!\,\big(2n_\theta + L + \frac{N}{2} + \mu_L\big)\Gamma\big(n_\theta + L + \frac{N}{2} + \mu_L\big)}{R^N 2^{N-1}\Gamma\big(n_\theta + L + \frac{N}{2}\big)\Gamma(n_\theta + \mu_L + 1)}} \\ \times \left(\sin\frac{\theta}{2}\right)^L \left(\cos\frac{\theta}{2}\right)^{\mu_L - N/2+1} P_{n_\theta}^{(L+N/2-1,\mu_L)}(\cos\theta), \end{aligned} \tag{4.3}$$

with

$$\mu_L = \sqrt{\left(L + \frac{N}{2} - 1\right)^2 + \left(\frac{4m\omega R^2}{\hbar}\right)^2}. \tag{4.4}$$

After being combined with Eq. (3.14), the quasi-radial eigenfunction in Eq. (4.3) coincides, again up to notational differences, with the corresponding expression resulting from Eqs. (7) and (9) in Ref. [7].

4.2 The case of $\omega_1 = 0$ and $\omega_2 = \omega$

This case corresponds to the potential $V(\theta)$ of the form

$$V(\theta) = 2m\omega^2 R^2 \cot^2\frac{\theta}{2} \qquad (0 \leqslant \theta \leqslant \pi). \tag{4.5}$$

The expressions for $E_{n_\theta L}$ resulting from Eqs. (3.12) are the same as in Eqs. (4.2), whereas Eq. (3.20b) yields

$$\begin{aligned} F_{n_\theta L}(\theta) = \sqrt{\frac{n_\theta!\,\big(2n_\theta + L + \frac{N}{2} + \mu_L\big)\Gamma\big(n_\theta + L + \frac{N}{2} + \mu_L\big)}{R^N 2^{N-1}\Gamma\big(n_\theta + L + \frac{N}{2}\big)\Gamma(n_\theta + \mu_L + 1)}} \\ \times \left(\sin\frac{\theta}{2}\right)^{\mu_L - N/2+1} \left(\cos\frac{\theta}{2}\right)^{L} P_{n_\theta}^{(\mu_L, L+N/2-1)}(\cos\theta). \end{aligned} \tag{4.6}$$

Since Eq. (4.5) results from Eq. (4.1) after the replacement $\theta \to \pi - \theta$ is made in the latter, the function (4.6) coincides, up to an unimportant phase factor, with the one that emerges once the same replacement is made on the right-hand side of Eq. (4.3) and then the use is made of the identity [13, p. 210]

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x). \tag{4.7}$$

4.3 The case of $\omega_1 = \omega_2 = \omega$

In this case the potential (2.11) reduces to the form

$$V(\theta) = \frac{8m\omega^2 R^2}{\sin^2\theta} - 4m\omega^2 R^2 = 8m\omega^2 R^2 \cot^2\theta + 4m\omega^2 R^2 \qquad (0 \leqslant \theta \leqslant \pi). \tag{4.8}$$

Then the energy eigenvalues (3.12) become

$$E_{n_\theta L} = \frac{\hbar^2}{2mR^2}\left[n_\theta + \frac{N}{2} + \sqrt{\left(L + \frac{N}{2} - 1\right)^2 + \left(\frac{4m\omega R^2}{\hbar}\right)^2}\right] \times \left[n_\theta - \frac{N}{2} + 1 + \sqrt{\left(L + \frac{N}{2} - 1\right)^2 + \left(\frac{4m\omega R^2}{\hbar}\right)^2}\right] - 4m\omega^2 R^2, \tag{4.9a}$$

or equivalently

$$E_{n_\theta L} = \frac{\hbar^2}{2mR^2}\left[\left(n_\theta + \frac{N}{2}\right)\left(n_\theta - \frac{N}{2} + 1\right) + \left(L + \frac{N}{2} - 1\right)^2 + (2n_\theta + 1)\sqrt{\left(L + \frac{N}{2} - 1\right)^2 + \left(\frac{4m\omega R^2}{\hbar}\right)^2}\right] + 4m\omega^2 R^2. \tag{4.9b}$$

The quasi-radial eigenfunctions (3.20) are

$$F_{n_\theta L}(\theta) = \sqrt{\frac{n_\theta!\,(2n_\theta + 2\mu_L + 1)\Gamma(n_\theta + 2\mu_L + 1)}{R^N 2^{2\mu_L+1}\Gamma^2(n_\theta + \mu_L + 1)}}(\sin\theta)^{\mu_L - N/2+1} P_{n_\theta}^{(\mu_L,\mu_L)}(\cos\theta), \tag{4.10}$$

with μ_L defined as in Eq. (4.4). By virtue of the relationship [13, p. 219]

$$P_n^{(\mu,\mu)}(x) = \frac{2^{2\mu}\Gamma(\mu + \frac{1}{2})\Gamma(n + \mu + 1)}{\sqrt{\pi}\,\Gamma(n + 2\mu + 1)} C_n^{\mu+1/2}(x), \tag{4.11}$$

linking the Jacobi polynomial $P_n^{(\mu,\mu)}(x)$ to the Gegenbauer polynomial $C_n^{\mu+1/2}(x)$, $F_{n_\theta L}(\theta)$ may be cast into the form

$$F_{n_\theta L}(\theta) = \sqrt{\frac{2^{2\mu_L-1} n_\theta!\,(2n_\theta + 2\mu_L + 1)\Gamma^2(\mu_L + \frac{1}{2})}{R^N \pi\,\Gamma(n_\theta + 2\mu_L + 1)}}(\sin\theta)^{\mu_L - N/2+1} C_{n_\theta}^{\mu_L+1/2}(\cos\theta). \tag{4.12}$$

In the limit $\omega \to 0$, corresponding to the case of a free particle on $\mathbb{S}_R^N$, Eqs. (4.9) and (4.12) go over, as they should, into

$$E_{n_\theta L} = \frac{\hbar^2}{2mR^2}(n_\theta + L)(n_\theta + L + N - 1) \tag{4.13}$$

and

$$F_{n_\theta L}(\theta) = \sqrt{\frac{2^{2L+N-3} n_\theta!\,(2n_\theta + 2L + N - 1)\Gamma^2(L + \frac{N-1}{2})}{R^N \pi\,\Gamma(n_\theta + 2L + N - 1)}} \sin^L\theta\, C_{n_\theta}^{L+N/2-1/2}(\cos\theta), \tag{4.14}$$

respectively.

5 The Euclidean limit ($R \to \infty$)

5.1 General considerations

Consider a Schrödinger particle moving on the hypersphere $\mathbb{S}_R^N$ in a reasonably arbitrary longitudinal potential $V(\theta)$. On the route to the Euclidean limit for this system, we first project stereographically the hypersphere $\mathbb{S}_R^N$ from its *south* pole onto the hyperplane tangent to $\mathbb{S}_R^N$ at the *north* pole [with regard to the nomenclature used here, cf. the pertinent remark under Eq. (2.2c)]. In this tangent hyperplane, we introduce the hyperspherical coordinates $r, \theta_{N-1}, \ldots, \theta_1$. The angles $\{\theta_n\}_{n=1}^{N-1}$ are defined as in Sect. 2, whereas the radial variable r is related to the hyperlatitude $\theta \equiv \theta_N$ on $\mathbb{S}_R^N$ through

$$r = 2R\tan\frac{\theta}{2} \quad \Rightarrow \quad \theta = 2\arctan\frac{r}{2R}. \tag{5.1}$$

Clearly, it holds that

$$\theta \overset{R\to\infty}{\to} \frac{r}{R}. \tag{5.2}$$

Next we define

$$U(r) = V(\theta) \tag{5.3}$$

and put[1,2]

$$F_{n_\theta L}(\theta) = \left(1 + \frac{r^2}{4R^2}\right)^{N/2-1} f_{n_r L}(r), \tag{5.4}$$

where for convenience we define $n_r \equiv n_\theta$. On inserting Eqs. (5.3) and (5.4) into the quasi-radial Schrödinger equation (2.10) (with E identified with the energy eigenvalue $E_{n_r L}$), we find that the radial function $f_{n_r L}(r)$ solves the equation

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{L(L+N-2)}{r^2}\right] + \left(\frac{4R^2}{r^2+4R^2}\right)^2\left[U(r) - \frac{\hbar^2 N(N-2)}{8mR^2} - E_{n_r L}\right]\right\} f_{n_r L}(r) = 0, \tag{5.5}$$

the normal (i.e., without the first derivative) form of which is

$$\left\{-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\hbar^2\left(L+\frac{N-3}{2}\right)\left(L+\frac{N-1}{2}\right)}{2mr^2} + \left(\frac{4R^2}{r^2+4R^2}\right)^2\left[U(r) - \frac{\hbar^2 N(N-2)}{8mR^2} - E_{n_r L}\right]\right\} r^{(N-1)/2} f_{n_r L}(r) = 0. \tag{5.6}$$

It is easy to see from Eqs. (5.4) and (5.1) that if $F_{n_\theta L}(\theta)$ is forced to obey the normalization constraint

$$R^N \int_0^\pi \mathrm{d}\theta\ \sin^{N-1}\theta\, |F_{n_\theta L}(\theta)|^2 = 1 \tag{5.7}$$

[a special case of which, for the potential (2.11), has been given in Eq. (3.18)], then $f_{n_r L}(r)$ is normalized in the sense of

$$\int_0^\infty \mathrm{d}r\, r^{N-1}\left(\frac{4R^2}{r^2+4R^2}\right)^2 |f_{n_r L}(r)|^2 = 1. \tag{5.8}$$

The presence of the weight function $\left(\frac{4R^2}{r^2+4R^2}\right)^2$ in Eqs. (5.5), (5.6) and (5.8) is a manifestation of the fact that we still remain outside the realm of the Euclidean geometry. The transition to the latter is achieved only in the next step, by going to infinity with the value of the radius R of the hypersphere. On defining

$$\widetilde{U}(r) = \lim_{R\to\infty} U(r), \qquad \widetilde{E}_{n_r L} = \lim_{R\to\infty} E_{n_r L}, \qquad \widetilde{f}_{n_r L}(r) = \lim_{R\to\infty} f_{n_r L}(r) \tag{5.9}$$

and making the limiting passage $R \to \infty$ in Eqs. (5.5), (5.6) and (5.8), we find that they take the desired Euclidean forms

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{L(L+N-2)}{r^2}\right] + \widetilde{U}(r) - \widetilde{E}_{n_r L}\right\}\widetilde{f}_{n_r L}(r) = 0, \tag{5.10}$$

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\hbar^2\left(L+\frac{N-3}{2}\right)\left(L+\frac{N-1}{2}\right)}{2mr^2} + \widetilde{U}(r) - \widetilde{E}_{n_r L}\right] r^{(N-1)/2}\widetilde{f}_{n_r L}(r) = 0 \tag{5.11}$$

and

$$\int_0^\infty \mathrm{d}r\, r^{N-1}|\widetilde{f}_{n_r L}(r)|^2 = 1, \tag{5.12}$$

respectively.

[1] For the general potential $V(\theta)$ considered here, the quasi-radial quantum number n_θ equals the number of zeros of the eigenfunction $F_{n_\theta L}(\theta)$ in the *open* interval $0 < \theta < \pi$.

[2] The function $f_{n_r L}(r)$ used in this section should not be confused with the function $f_L(\rho)$ that appeared in Eqs. (3.3), (3.4) and (3.7).

5.2 Application to the hyperspherical Pöschl–Teller oscillator

In the case of the hyperspherical Pöschl–Teller oscillator, the procedure described above will certainly be well-defined if we assume that of the two strength parameters characterizing the potential, the parameter ω_1 is independent of the radius R, while the parameter ω_2 is inversely proportional to R squared. By making the adequate substitution

$$\omega_2 = \frac{\hbar\chi}{4mR^2}, \tag{5.13}$$

where $\chi > 0$ is independent of R, and dropping from now on, for clarity of notation, the index 1 at ω_1, we get

$$V(\theta) = 2m\omega^2 R^2 \tan^2\frac{\theta}{2} + \frac{\hbar^2\chi^2}{8mR^2}\cot^2\frac{\theta}{2}. \tag{5.14}$$

Hence, with the use of Eqs. (5.3) and (5.1), as well as of the first of Eqs. (5.9), it follows that

$$U(r) = \frac{m\omega^2 r^2}{2} + \frac{\hbar^2\chi^2}{2mr^2} = \widetilde{U}(r). \tag{5.15}$$

This potential describes the isotropic harmonic oscillator subjected to an additional centrifugal-like force. With $\widetilde{U}(r)$ in the form (5.15), Eq. (5.11) becomes

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\hbar^2\Lambda_L(\Lambda_L+1)}{2mr^2} + \frac{m\omega^2 r^2}{2} - \widetilde{E}_{n_r L}\right] r^{(N-1)/2}\widetilde{f}_{n_r L}(r) = 0, \tag{5.16}$$

where

$$\Lambda_L = \sqrt{\left(L + \frac{N}{2} - 1\right)^2 + \chi^2} - \frac{1}{2} \qquad \left(= \mu_{L2} - \frac{1}{2}\right). \tag{5.17}$$

Solving Eq. (5.16) subject to the boundary conditions

$$\widetilde{f}_{n_r L}(r) \text{ bounded for } r \to 0, \qquad \widetilde{f}_{n_r L}(r) \overset{r\to\infty}{\longrightarrow} 0 \tag{5.18}$$

belongs to the class of standard exercises in intermediate quantum mechanics (cf., for instance, Ref. [14, Problem 65]). One finds that

$$\widetilde{E}_{n_r L} = \hbar\omega\left[2n_r + 1 + \sqrt{\left(L + \frac{N}{2} - 1\right)^2 + \chi^2}\right] \qquad (n_r \in \mathbb{N}_0) \tag{5.19}$$

and, once the normalization constraint (5.12) is imposed and phase factors in $\widetilde{f}_{n_r L}(r)$ are suitably adjusted, that[3]

$$\widetilde{f}_{n_r L}(r) = \sqrt{\frac{2n_r!}{\Gamma\left(n_r + \Lambda_L + \frac{3}{2}\right)}}\left(\frac{m\omega}{\hbar}\right)^{N/4}\left(\frac{m\omega r^2}{\hbar}\right)^{\Lambda_L/2 - N/4 + 3/4} \mathrm{e}^{-m\omega r^2/2\hbar} L_{n_r}^{(\Lambda_L+1/2)}\left(\frac{m\omega r^2}{\hbar}\right), \tag{5.20}$$

where $L_n^{(\alpha)}(x)$ is the generalized Laguerre polynomial [13, Sec. 5.5].

Of course, one should expect to be able to arrive at Eq. (5.19) also in another way, namely after combining Eqs. (3.12) and (5.13), and taking then the limit $R \to \infty$; it presents no difficulties to show that this is indeed the case. Similarly, Eq. (5.20) should follow (possibly up to a sign factor) from Eqs. (5.4) and (3.20), after the aforementioned limit is taken. However, in this case the proof appears to be a bit more complex. On exploiting Eqs. (5.4), (3.20b), (5.1) and (5.17), we may write

$$f_{n_r L}(r) = \sqrt{\frac{n_r!\left(2n_r + \mu_{L1} + \Lambda_L + \frac{3}{2}\right)\Gamma\left(n_r + \mu_{L1} + \Lambda_L + \frac{3}{2}\right)}{R^N 2^{N-1}\Gamma(n_r + \mu_{L1} + 1)\Gamma\left(n_r + \Lambda_L + \frac{3}{2}\right)}}$$
$$\times \frac{\left(\frac{r}{2R}\right)^{\Lambda_L - N/2 + 3/2}}{\left(1 + \frac{r^2}{4R^2}\right)^{\mu_{L1}/2 + \Lambda_L/2 + 1/4}} P_{n_r}^{(\Lambda_L+1/2,\mu_{L1})}\left(\frac{1 - \frac{r^2}{4R^2}}{1 + \frac{r^2}{4R^2}}\right). \tag{5.21}$$

Now, as it holds that

$$\mu_{L1} \overset{R\to\infty}{\longrightarrow} \frac{4m\omega R^2}{\hbar} \tag{5.22}$$

[3] In Ref. [14, Problem 65] the radial wave functions were expressed in terms of the confluent hypergeometric function. In the present work, we prefer to use the generalized Laguerre polynomials instead.

[cf. Eq. (3.6)], we obviously have

$$\lim_{R\to\infty}\left(1+\frac{r^2}{4R^2}\right)^{\mu_{L1}/2+\Lambda_L/2+1/4}=\mathrm{e}^{m\omega r^2/2\hbar}. \tag{5.23}$$

Furthermore, from Eq. (5.22) and from the following two asymptotic relationships:

$$\frac{\Gamma(x+a)}{\Gamma(x+b)}\xrightarrow{x\to\infty}x^{a-b} \tag{5.24}$$

(cf. Ref. [13, p. 12]) and

$$\lim_{\beta\to\infty}P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right)=L_n^{(\alpha)}(x) \tag{5.25}$$

(cf. Ref. [13, p. 247]), it follows that

$$\frac{\Gamma\left(n_r+\mu_{L1}+\Lambda_L+\frac{3}{2}\right)}{\Gamma(n_r+\mu_{L1}+1)}\xrightarrow{R\to\infty}\left(\frac{4m\omega R^2}{\hbar}\right)^{\Lambda_L+1/2} \tag{5.26}$$

and

$$\lim_{R\to\infty}P_{n_r}^{(\Lambda_L+1/2,\mu_{L1})}\left(\frac{1-\dfrac{r^2}{4R^2}}{1+\dfrac{r^2}{4R^2}}\right)=\lim_{R\to\infty}P_{n_r}^{(\Lambda_L+1/2,\mu_{L1})}\left(1-\frac{r^2}{2R^2}\right)=L_{n_r}^{(\Lambda_L+1/2)}\left(\frac{m\omega r^2}{\hbar}\right), \tag{5.27}$$

respectively. On taking the limit $R\to\infty$ on the right-hand side of Eq. (5.21) and employing then Eqs. (5.23), (5.26) and (5.27), after some algebra one indeed arrives at Eq. (5.20).

6 Conclusions

In this work, we have arrived at analytical solutions to an energy eigenvalue problem for a Schrödinger particle moving on an ($N\geqslant 2$)-dimensional hypersphere $\mathbb{S}_R^N$ in the field of the hyperlatitude-dependent potential $V(\theta)=2m\omega_1^2R^2\tan^2(\theta/2)+2m\omega_2^2R^2\cot^2(\theta/2)$. This system may be viewed as an N-dimensional generalization of the one-dimensional anharmonic trigonometric oscillator of Pöschl and Teller [9].

In the special case of $N=2$, our results coincide with those presented in Ref. [8] by Kazaryan *et al*. On the other hand, in the limit $\omega_2=0$ our findings reproduce those derived by Mardoyan and Petrosyan [7] for their model of an isotropic harmonic oscillator on $\mathbb{S}_R^N$.

If the parameter ω_2 in $V(\theta)$ depends on the radius of $\mathbb{S}_R^N$ in the inverse-square manner and if ω_1 is independent of R, then the Euclidean limit for the system considered here does exist. This limit may be achieved by performing a suitable stereographic projection of $\mathbb{S}_R^N$ onto a tangent space, followed by the passage with R to infinity. In result, our hyperspherical Pöschl–Teller oscillator goes over into an isotropic harmonic oscillator in $\mathbb{R}^N$ perturbed by a centrifugal-type force.

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Declarations

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