Analytical study of sliding instability due to velocity- and temperature-dependent friction

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The instability of sliding causes deterioration of performance characteristics of tribosystems and is undesired. To predict its occurrence, the motion of a body of a one-degree-of-freedom system with friction is investigated about the steady sliding equilibrium position. The motion equation is formulated with the friction coefficient dependent on the sliding velocity and contact temperature changing due to transient heat conduction in the body. An analytical expression for the body motion is derived using the Laplace integral transform. It is shown that the sliding instability can manifest in the form of deviation of the body from the equilibrium position or in the form of oscillation. The instability conditions containing the friction–velocity and friction–temperature slope coefficients are obtained. Positive friction–temperature slope results in the deviation of the body from the equilibrium position. At negative friction–temperature slope, both types of the sliding instability can occur. The proposed instability conditions agree well with existing theoretical concepts and can be useful when designing tribosystems.

Keywords: sliding instability; frictional oscillation; frictional heating; instability condition.

1. Introduction.

The instability of sliding is a common phenomenon which may occur in the process of relative motion of contacting surfaces and is usually accompanied by uncontrollable changes in the sliding velocity v_s and friction characteristics. In tribosystems such as brakes, clutches, or bearings, these changes lead to performance deterioration, noise emission, and premature failures [1]. Therefore, prediction of the sliding instability occurrence is often required at the design stage.

It is common knowledge that the relationship between the friction coefficient μ and v_s affects essentially the character of sliding. If μ decreases with v_s, i.e. the μ–v_s slope is negative, the negative damping and associated sliding instability take place [2, 3]. The difference between the kinetic friction force and the maximum tangential force in static contact is the reason for excitation of stick−slip oscillations [4, 5]. The sliding instability in the form of oscillation can also occur due to the sprag–slip mechanism [6], thermoelastic contact instability [7, 8], self-excited elastic waves at the sliding surface [9], etc.

Notations

a	friction–velocity slope coefficient	X	dimensionless displacement, $X=(kx+\mu_0 N)/\left(v\sqrt{km}\right)$
b	friction–temperature slope coefficient	α	heat-partition coefficient
c	damping coefficient	ε_0	velocity perturbation
i	summation index	ζ	reduced damping ratio, $\zeta=(c+aN)/\left(2\sqrt{km}\right)$
j	summation index	η	dimensionless coefficient, $\eta=\alpha bNv\sqrt{\varkappa}/(KS)$
k	stiffness coefficient	μ	friction coefficient
m	body mass	μ_0	steady sliding friction coefficient
n	summation index	ς	integration variable
q	friction heat flux rise	τ	dimensionless time variable, $\tau=t\sqrt{k/m}$
s	Laplace integral transform parameter	ψ	dimensionless friction–temperature slope coefficient, $\psi=\eta\sqrt[4]{m/k}$
t	time variable	Θ	dimensionless friction force
v	steady sliding velocity	$\varkappa$	thermal diffusivity coefficient
v_s	sliding velocity, $v_s=v+dx/dt$	$\mathcal{L}$	Laplace integral transform operator
x	displacement	$\mathrm{erf}(\cdot)$	Gauss error function
F	dimensionless parameter, $F=N(\mu_0+av)/\left(v\sqrt{km}\right)$	$\mathbf{C}(\cdot)$	Fresnel cosine integral function
K	thermal conductivity coefficient	$E_{1/2,\beta/2}(\cdot)$	Mittag-Leffler function with two parameters $1/2$ and $\beta/2$
N	normal force	$\mathbf{S}(\cdot)$	Fresnel sine integral function
S	contact area	$\Gamma(\cdot)$	gamma function
T	contact temperature rise		

Frictional heating of the sliding surface leads to a rise T of its temperature. Since μ is generally temperature dependent, T has influence on the dynamics of sliding. In most cases this influence is of quantitative character. For example, it is known that heating of the friction pair of brake affects the deceleration rate and braking duration [10]. At the same time, there have been found evidences for qualitative influence of the dependency $\mu(T)$ on the sliding dynamics. Kokonin et al. [11] showed experimentally that carbonic friction discs used in aircraft brakes exhibit an oscillatory sliding when the μ–T slope is negative. Temperature-related instabilities of sliding were also noted in [12–15].

Several theoretical studies were done on the sliding instability caused by the dependency $\mu(T)$. Maksimov and Rakhmanov [16, 17] investigated the motion of a body on a counterbody with μ dependent on v_s and T. Assuming a uniform temperature distribution in the body, it was shown that the sliding instability can arise in different forms: joint exponential increase in v_s and T, growing oscillations of v_s and T, periodic oscillations of v_s and T. The instability conditions containing the μ–v_s and μ–T slope coefficients were derived for the mentioned cases.

Nosonovsky and Bhushan [18] investigated the thermodynamics of friction and self-organization during friction, relying on the study [19] by Fox-Rabinovich et al. It was assumed that μ depends on v_s and T, while the thermal conductivity depends on v_s. The stability condition was obtained in variational form, according to which the friction system loses stability at positive μ–T slope.

Mortazavi et al. [20] analysed the sliding instability excited by positive μ–T slope. The friction body was represented as a one-dimensional slab. The temperature distribution in it was described analytically with account of the influence of the μ–T slope on the friction heat generation. The instability condition was derived, stating that T would increase exponentially if the μ–T slope coefficient exceeds a critical value dependent on the sizes and thermal conductivity coefficient of the body.

Nosko et al. [21] investigated stick−slip oscillations in a one-degree-of-freedom system with friction, induced by negative μ–T slope. The temperature in the body was simulated allowing for the contact heat transfer between the body and counterbody. The instability threshold curve was obtained, dividing the parameter plane into the region of stable sliding and the region of stick−slip oscillation. It was shown that an increase in the μ–T slope coefficient in modulus leads to a decrease in the stability margin for a stable system or an oscillation amplification for an unstable one.

It follows from the studies above that the sliding instability can manifest in the form of exponential increase in v_s or in the form of non-decaying oscillation. The former instability type is likely to occur when the μ–T slope is positive [16−18, 20], while the latter is caused by negative μ–v_s slope [2, 3] or negative μ–T slope [11, 21]. The aim of this study is to investigate both types of the sliding instability and obtain the corresponding instability conditions using a unified approach. For this purpose, a one-degree-of-freedom system with friction dependent on v_s and T is analysed in the neighbourhood of the steady sliding equilibrium state.

2. Formulation of the governing equation.

2.1. Motion equation.

Consider a body m connected to a base with a spring k and a damper c, as shown in Fig.1. The body is pressed by a force N against a counterbody moving with a velocity v. The frictional interaction between the body and counterbody results in variations in the body displacement $x(t)$ and rise $T(t)$ of the body contact temperature.

Assume that the friction coefficient $\mu(t)$ depends linearly on the sliding velocity $v_\mathrm{s} = v + dx/dt$ and T in the neighbourhood of a steady sliding equilibrium state $(x, dx/dt\,, T) = (-\mu_0 N/k\,, 0, 0)$, i.e.

$$\mu = \mu_0 + a\frac{dx}{dt} + bT \tag{1}$$

where μ_0 is the steady sliding friction coefficient; $a = \partial\mu/\partial v_\mathrm{s}|_{v_\mathrm{s}=v}$ is the μ–v_s slope coefficient; $b = \partial\mu/\partial T|_{T=0}$ is the μ–T slope coefficient. Positive values of a and b correspond to positive μ–v_s and μ–T slopes, respectively. In the preceding and subsequent expressions, it is accepted that the quantities dx/dt and T are small.

With account of Eq.(1), the motion equation is formulated in the following form:

$$m\frac{d^2x}{dt^2} + (c + aN)\frac{dx}{dt} + k\left(x + \frac{\mu_0 N}{k}\right) = -bNT, \qquad t > 0 \tag{2}$$

To solve Eq.(2), it is necessary to describe the behaviour of T.

2.2. Contact temperature equation.

Variations in v_s and T lead to the rise $q(t)$ of the friction heat flux into the body:

$$\begin{aligned} q &= \frac{\alpha N\mu v_\mathrm{s}}{S} - \frac{\alpha N\mu_0 v}{S} = \frac{\alpha N(\mu_0 + av)}{S}\frac{dx}{dt} + \frac{\alpha bNv}{S}T + \frac{\alpha aN}{S}\left(\frac{dx}{dt}\right)^2 + \frac{\alpha bN}{S}\frac{dx}{dt}T \\ &\approx \frac{\alpha N(\mu_0 + av)}{S}\frac{dx}{dt} + \frac{\alpha bNv}{S}T \end{aligned} \tag{3}$$

where S is the contact area; α is the heat-partition coefficient [22]. In Eq.(3), the terms of the second-order smallness are neglected.

According to heat conduction theory [23], the heat source of Eq.(3) results in the surface temperature rise

$$T = \frac{\sqrt{\varkappa}}{\sqrt{\pi}K}\int_0^t \frac{q|_\varsigma\, d\varsigma}{\sqrt{t-\varsigma}} = \frac{\alpha N\sqrt{\varkappa}}{\sqrt{\pi}KS}\int_0^t \left((\mu_0 + av)\frac{dx}{dt} + bvT\right)\bigg|_\varsigma \frac{d\varsigma}{\sqrt{t-\varsigma}} \tag{4}$$

where ς is the integration variable; K is the thermal conductivity coefficient; $\varkappa$ is the thermal diffusivity coefficient. Application of the Laplace integral transform $\mathcal{L}$ to Eq.(4) along with the convolution theorem gives

$$\mathcal{L}[T(t)] = \frac{\alpha N(\mu_0 + av)\sqrt{\varkappa}}{KS}\frac{\mathcal{L}[dx/dt]}{\sqrt{s}-\eta} \tag{5}$$

where s is the transform parameter and

$$\eta = \frac{\alpha N v\sqrt{\varkappa}}{KS}b$$

The inverse transform [23]

$$\mathcal{L}^{-1}\left[\frac{1}{\sqrt{s}-\eta}\right] = \frac{1}{\sqrt{\pi t}} + \eta\exp(\eta^2 t)\left(1+\mathrm{erf}\left(\eta\sqrt{t}\right)\right) \tag{6}$$

allows to express the original of Eq.(5) as

$$T = \frac{\alpha N(\mu_0 + av)\sqrt{\varkappa}}{KS}\int_0^t \left.\frac{dx}{dt}\right|_{t-\varsigma}\left(\frac{1}{\sqrt{\pi\varsigma}} + \eta\exp(\eta^2\varsigma)\left(1+\mathrm{erf}\left(\eta\sqrt{\varsigma}\right)\right)\right)d\varsigma \tag{7}$$

where $\mathrm{erf}(\cdot)$ is the Gauss error function.

2.3. Dimensionless governing equation and initial conditions.

Substitute Eq.(7) into Eq.(2) and obtain the motion equation in the integro-differential form

$$\begin{aligned} m\frac{d^2x}{dt^2} + (c+aN)\frac{dx}{dt} + k\left(x+\frac{\mu_0 N}{k}\right) \\ = -\frac{N\eta(\mu_0+av)}{v}\int_0^t \left.\frac{dx}{dt}\right|_{t-\varsigma}\left(\frac{1}{\sqrt{\pi\varsigma}} + \eta\exp(\eta^2\varsigma)\left(1+\mathrm{erf}\left(\eta\sqrt{\varsigma}\right)\right)\right)d\varsigma \end{aligned} \tag{8}$$

After introducing the dimensionless variables

$$\tau = t\sqrt{k/m}, \qquad X = \frac{kx+\mu_0 N}{v\sqrt{km}}$$

and parameters

$$\zeta = \frac{c+aN}{2\sqrt{km}}, \qquad \psi = \eta\sqrt[4]{m/k}, \qquad F = \frac{N(\mu_0+av)}{v\sqrt{km}}$$

one can rewrite Eq.(8) as follows

$$\frac{d^2X}{d\tau^2} + 2\zeta\frac{dX}{d\tau} + X = \Theta = -F\psi\int_0^\tau \left.\frac{dX}{d\tau}\right|_{\tau-\varsigma}\left(\frac{1}{\sqrt{\pi\varsigma}} + \psi\exp(\psi^2\varsigma)\left(1+\mathrm{erf}\left(\psi\sqrt{\varsigma}\right)\right)\right)d\varsigma \tag{9}$$

This equation contains three parameters. The parameters ζ and F are dependent on a, while the parameter ψ is proportional to b. One can distinguish three dimensionless force affecting the body motion: the spring force $(-X)$, the reduced damping force $(-2\zeta\, dX/d\tau)$, and the friction force Θ associated with the contact temperature variation.

To complete the description of the body motion, the initial conditions are specified in the form of a small velocity perturbation ε_0, that is

$$X|_{\tau=0} = 0, \qquad \left.\frac{dX}{d\tau}\right|_{\tau=0} = \varepsilon_0 \tag{10}$$

Note that the principal results presented further in this study are valid for the general case of the initial conditions, i.e. specified displacement and velocity perturbations. The initial displacement is set to zero in order to decrease the number of parameters and thereby simplify analysis.

3. Analytical expression for the body motion.

Find an analytical solution of Eqs.(9), (10). With account of Eq.(6), X can be expressed in the images of the Laplace integral transform:

$$\mathcal{L}[X(\tau)] = \frac{\varepsilon_0\left(\sqrt{s} - \psi\right)}{s^2\sqrt{s} - \psi s^2 + 2\zeta s\sqrt{s} - \psi(2\zeta - F)s + \sqrt{s} - \psi} \tag{11}$$

Expand Eq.(11) into a series and apply the multinomial theorem [24]:

$$\begin{aligned}
\mathcal{L}[X(\tau)] &= \frac{\varepsilon_0}{s^2}\frac{1}{1 + \dfrac{2\zeta s^{-1/2} - \psi(2\zeta - F)s^{-1} + s^{-3/2} - \psi s^{-2}}{s^{1/2} - \psi}} \\
&= \frac{\varepsilon_0}{s^2}\sum_{i=0}^{\infty}\frac{(-1)^i}{(s^{1/2} - \psi)^i}\left(2\zeta s^{-1/2} - \psi(2\zeta - F)s^{-1} + s^{-3/2} - \psi s^{-2}\right)^i \\
&= \frac{\varepsilon_0}{s^2}\sum_{i=0}^{\infty}\frac{(-1)^i}{(s^{1/2} - \psi)^i}\sum_{j=1}^{(3+i)!/(3!i!)}\frac{i!\,(-\psi)^{\kappa_{1ij}}\left(\psi(F - 2\zeta)\right)^{\kappa_{3ij}}(2\zeta)^{\kappa_{4ij}}}{\kappa_{1ij}!\,\kappa_{2ij}!\,\kappa_{3ij}!\,\kappa_{4ij}!}s^{-2\kappa_{1ij}-\frac{3}{2}\kappa_{2ij}-\kappa_{3ij}-\frac{1}{2}\kappa_{4ij}} \\
&= \varepsilon_0\sum_{i=0}^{\infty}i!\sum_{j=1}^{(3+i)!/(3!i!)}\frac{(-1)^{i+\kappa_{1ij}}\,\psi^{\kappa_{1ij}+\kappa_{3ij}}(F - 2\zeta)^{\kappa_{3ij}}(2\zeta)^{\kappa_{4ij}}}{\kappa_{1ij}!\,\kappa_{2ij}!\,\kappa_{3ij}!\,\kappa_{4ij}!}\frac{s^{-(4+4\kappa_{1ij}+3\kappa_{2ij}+2\kappa_{3ij}+\kappa_{4ij})/2}}{(s^{1/2} - \psi)^i}
\end{aligned} \tag{12}$$

where i and j are the summation indices; $\left(\kappa_{1ij}, \kappa_{2ij}, \kappa_{3ij}, \kappa_{4ij}\right)$ is the j-th combination of non-negative integers satisfying the equality $\kappa_{1ij} + \kappa_{2ij} + \kappa_{3ij} + \kappa_{4ij} = i$. For a fixed i, there are $(3 + i)!/(3!\,i!)$ different such combinations, which can be evaluated by known methods of combinatorial mathematics [24].

Based on the properties of the Mittag-Leffler function, a number of useful Laplace transforms were obtained [25], including the following one:

$$\begin{gathered}
\mathcal{L}^{-1}\left[\frac{s^{(1-\beta)/2}}{(s^{1/2} - \psi)^{i+1}}\right] = \frac{\tau^{(\beta+i-2)/2}}{i!}\,E_{1/2,\beta/2}^{(i)}\left(\psi\,\tau^{1/2}\right), \\
E_{1/2,\beta/2}^{(i)}(z) = \frac{d^i}{dz^i}E_{1/2,\beta/2}(z) = \sum_{n=0}^{\infty}\frac{(i + n)!\,z^n}{n!\,\Gamma((\beta + i + n)/2)}
\end{gathered} \tag{13}$$

where $E_{1/2,\beta/2}(\cdot)$ is the Mittag-Leffler function with two parameters $1/2$ and $\beta/2$; $\Gamma(\cdot)$ is the gamma function; n is the summation index.

Application of the transform of Eq.(13) to Eq.(12) allows to reconstruct the original:

$$X = \varepsilon_0 \left(\tau + \sum_{i=1}^{\infty} i \sum_{j=1}^{(3+i)!/(3!i!)} \frac{(-1)^{i+\kappa_{1ij}} (F-2\zeta)^{\kappa_{3ij}} (2\zeta)^{\kappa_{4ij}}}{\kappa_{1ij}!\ \kappa_{2ij}!\ \kappa_{3ij}!\ \kappa_{4ij}!} \right.$$
$$\left. \times \sum_{n=0}^{\infty} \frac{(i+n-1)!\ \psi^{n+\kappa_{1ij}+\kappa_{3ij}}\ \tau^{(2+i+n+4\kappa_{1ij}+3\kappa_{2ij}+2\kappa_{3ij}+\kappa_{4ij})/2}}{n!\,\Gamma\big((4+i+n+4\kappa_{1ij}+3\kappa_{2ij}+2\kappa_{3ij}+\kappa_{4ij})/2\big)} \right) \tag{14}$$

Equation (14) is validated by using a numerical algorithm based on the Runge-Kutta method and product-integration approximation approach [26]. For example, at $\psi = -1$, $\zeta = 0.1$ and $F = 0.4$, when $X \approx \varepsilon_0 \sin\tau$, the finite sum of Eq.(14) with $i = 1,2,\ldots,20$ and $n = 0,1,\ldots,1000$ has a relative error of the order 10^{-6} on the interval $0 < \tau < 10$.

4. Sliding instability analysis.

The motion of the body is analysed separately for positive μ–T slope ($\psi > 0$) and negative μ–T slope ($\psi < 0$).

4.1. Positive μ–T slope.

Introduce $p = \sqrt{s}$ and represent the denominator of the transfer function of Eq.(11) in the form of a polynomial:

$$p^5 - \psi p^4 + 2\zeta p^3 - \psi(2\zeta - F)p^2 + p - \psi = (p-p_1)(p-p_2)(p-p_3)(p-p_4)(p-p_5) \tag{15}$$

where $p_1,\ldots,p_5$ are the polynomial roots which are either purely real or appear in complex conjugate pairs. Equating the constant terms of the left and right sides of Eq.(15) yields

$$p_1 p_2 p_3 p_4 p_5 = \psi > 0 \tag{16}$$

Since the product of a complex number and its conjugate is always positive, it follows from Eq.(16) that among $p_1,\ldots,p_5$ there exists at least one positive root (let it be p_1 for definiteness). Consequently, in general case the transfer function of Eq.(11) has a purely real pole $s = p_1^2 > 0$ and is hence unstable [27], which implies the exponential increase in $|X|$.

Figure 2 illustrates the behaviour of X due to Eq.(14) and the corresponding location of the poles of Eq.(11) in the complex plane $(\mathrm{Re}\, s, \mathrm{Im}\, s)$ at various values of ψ. The poles are denoted by the symbol "×". It is seen that as ψ increases, the body deviates stronger from the equilibrium position. The motion of the body includes an oscillatory term, which can be decaying, periodic, or growing, as shown in Fig.3.

Thus, at positive μ–T slope coefficient

$$b > 0 \tag{17}$$

the sliding instability occurs in the form of deviation. Physically, the deviation is explained by the positive feedback between μ and T represented by Eq.(4). A small increase in T causes a small increase in μ, which, in its turn, results in a further increase in T.

4.2. Negative μ–T slope.

Figure 4 illustrates the body motion at negative values of ψ. At $\psi = -0.45$ the deviation takes place, while at $\psi = -0.05$ the body oscillates with growing amplitude. When $\psi \approx -0.25$, the two real poles turn into a complex conjugate pair, and the deviation transforms into the oscillation. Thereby, at negative friction–temperature slope, the sliding instability can arise in the form of deviation as well as oscillation.

Different cases of the oscillatory motion are shown in Fig.5. At $\psi = -0.5$ the oscillation grows. On the contrary, it decays at $\psi = -0.1$. In the limit case at $\psi \approx -0.29$, the real part of the complex conjugate poles is zero, and the oscillation is periodic.

The limit points are determined, corresponding to the transitions of deviation / oscillation (see Fig.4) and growing oscillation / decaying oscillation (see Fig.5). Based on these points, the limit curves are found, dividing the parameter plane $(\psi, F/\zeta)$ into the regions «Deviation», «Oscillation» (growing oscillation), and «Stable sliding» (decaying oscillation). Figs.6 and 7 show the limit curves for the cases $\zeta > 0$ and $\zeta < 0$. It is evident that the configuration of the regions depends essentially on the sign of ζ.

Of practical interest is the equation of the limit curve between the regions «Stable sliding» and «Oscillation». Try to find it for the case $|\zeta| \ll 1$ when the limit cycle represents an almost harmonic motion [28] defined for definiteness as $X = \varepsilon_0 \sin \tau$. The work of the spring force over one cycle of oscillation is

$$\int_{\tau}^{\tau+2\pi} \left((-X) \frac{dX}{d\tau} \right) \Bigg|_{\varsigma} d\varsigma = 0 \tag{18}$$

The work of the reduced damping force can be represented as follows:

$$\int_{\tau}^{\tau+2\pi} \left(\left(-2\zeta \frac{dX}{d\tau} \right) \frac{dX}{d\tau} \right) \Bigg|_{\varsigma} d\varsigma = -2\pi \varepsilon_0^2 \zeta \tag{19}$$

The friction force Θ is derived in the following manner [23]:

$$\Theta = -\varepsilon_0 F\psi \int_0^{\tau} \cos(\tau-\varsigma)\left(\frac{1}{\sqrt{\pi\varsigma}} + \psi \exp(\psi^2\varsigma)\left(1+\mathrm{erf}\left(\psi\sqrt{\varsigma}\right)\right)\right) d\varsigma = \mathcal{L}^{-1}\left[-\frac{\varepsilon_0 F\psi s}{(s^2+1)\left(\sqrt{s}-\psi\right)}\right]$$

$$= -\frac{\varepsilon_0 F\psi}{1+\psi^4}\Big(\psi^3 \exp(\psi^2\tau)\left(1+\mathrm{erf}\left(\psi\sqrt{\tau}\right)\right)$$

$$+\left(\sqrt{2}\psi^2\,\mathbf{C}\left(\sqrt{2\tau/\pi}\right)+\sqrt{2}\,\mathbf{S}\left(\sqrt{2\tau/\pi}\right)+\psi\right)\sin\tau$$

$$+\left(\sqrt{2}\,\mathbf{C}\left(\sqrt{2\tau/\pi}\right)-\sqrt{2}\,\psi^2\,\mathbf{S}\left(\sqrt{2\tau/\pi}\right)-\psi^3\right)\cos\tau\Big)$$

where $\mathbf{C}(\cdot)$ is Fresnel cosine integral function; $\mathbf{S}(\cdot)$ is Fresnel sine integral function. At $\psi < 0$ and $\tau \to \infty$ we obtain the periodic function

$$\Theta = -\frac{\varepsilon_0 F\psi}{\sqrt{2}(1+\psi^4)}\left(\left(1+\sqrt{2}\psi+\psi^2\right)\sin\tau + \left(1-\psi^2-\sqrt{2}\psi^3\right)\cos\tau\right)$$

With account of the equation above, the work of the friction force over one cycle of oscillation is

$$\int_{\tau}^{\tau+2\pi} \left(\Theta\frac{dX}{d\tau}\right)\bigg|_{\varsigma} d\varsigma = \frac{\pi\varepsilon_0^2 F|\psi|\left(1-\psi^2+\sqrt{2}|\psi|^3\right)}{\sqrt{2}(1+\psi^4)} \tag{20}$$

Due to the energy balance, the work of all the forces acting on the body for one cycle, i.e. the sum of Eqs.(18)–(20), must be equal to zero. This yields the following limit equation:

$$F\frac{|\psi|\left(1-\psi^2+\sqrt{2}|\psi|^3\right)}{\sqrt{2}(1+\psi^4)} = 2\zeta \tag{21}$$

The validity of Eq.(21) is confirmed by its comparison with the limit points obtained from Eq.(14), as shown in Fig.8.

The instability condition is easily obtained from Eq.(21) in the form

$$\left(\frac{\mu_0}{v}+a\right)\frac{|\psi|\left(1-\psi^2+\sqrt{2}|\psi|^3\right)}{\sqrt{2}(1+\psi^4)} > \frac{c}{N}+a \tag{22}$$

According to Eq.(22), the sliding instability occurrence depends on both a and b, i.e. the μ–v_s and μ–T slopes. Since at $\psi < 0$ it is true that

$$0 < \frac{|\psi|\left(1-\psi^2+\sqrt{2}|\psi|^3\right)}{\sqrt{2}(1+\psi^4)} < 1$$

there can be singled out two particular cases: sliding is stable at $\mu_0/v < c/N$ and $c/N + a > 0$; sliding is unstable at $\mu_0/v > c/N$ and $c/N + a < 0$.

5. Discussion.

Maksimov and Rakhmanov [16, 17] considered the friction of a body on a stationary counterbody, caused by constant normal and tangential forces applied to the body. It was assumed that the friction coefficient μ depends linearly on v_s and T, with dimensionless μ–v_s slope coefficient μ_v and μ–T slope coefficient μ_T; the temperature is uniformly distributed in the body and is governed by the frictional heating and convective heat transfer from the body to the environment. By analysing the response of the body on small velocity and temperature perturbations, the instability conditions were derived, containing μ_v, μ_T, and a dimensionless steady sliding velocity u. It was shown that growing oscillations of v_s and T occur at

$$0 < \mu_T u - \mu_v - 1 < 2\sqrt{\mu_v + \mu_T} \tag{23}$$

while v_s and T increase exponentially at

$$\mu_v + \mu_T < 0 \text{ or } 0 < 2\sqrt{\mu_v + \mu_T} < \mu_T u - \mu_v - 1 \tag{24}$$

The instability conditions of Eqs.(17), (22) proposed in this study differ from Eqs.(23), (24) since they allow for non-uniformity of the temperature field in the body as well as the presence of the spring and damper links. Nevertheless, both Eqs.(17), (22) and Eqs.(23), (24) yield the same qualitative conclusions: (1) sliding can be unstable at positive μ–v_s slope, and it can be stable at negative μ–v_s slope; (2) the μ–T slope can result in the sliding instability; (3) the sliding instability can occur in the form of deviation or oscillation. It is noteworthy that the conclusion (1) has experimental confirmations [12, 29].

Nosonovsky and Bhushan [18] applied the thermodynamic approach to analyse the friction system instability. Assuming that μ depends on v_s and T, while the thermal conductivity coefficient K depends on v_s, the local stability condition [19] was expressed in the following form:

$$\left(\frac{\partial \mu}{\partial v_s} v_s + \mu\right)\left(\frac{\partial \mu}{\partial v_s} v_s + \mu - \frac{\mu v_s}{K}\frac{\partial K}{\partial v_s}\right)(\delta v_s)^2 - \frac{2\mu v_s^2}{T}\frac{\partial \mu}{\partial T}(\delta T)^2 \geq 0 \tag{25}$$

where δv_s and δT are the variations of v_s and T, respectively. If the condition of Eq.(25) is violated, the friction system would likely enter the self-organizing regime with reduced friction and wear rate. One of possible consequences of the self-organization is the sliding instability [18]. It was concluded that the violation of the condition of Eq.(25) takes place either if the μ–T slope is positive ($\partial\mu/\partial T > 0$), which is equivalent to the fulfilment of the instability condition of Eq.(17), or if, in the first term, the parentheses have different signs.

Mortazavi et al. [20] investigated the influence of positive μ–T slope on the sliding instability for a friction body represented as a slab with a length L and a thickness w. It was assumed that the temperature is uniformly distributed in the body in the directions of its width and thickness; μ increases linearly with T. By specifying a non-uniform temperature distribution at the initial moment and simulating the heat conduction along the body length, it was found that the condition

$$\frac{\partial \mu}{\partial T} > \frac{\pi^2 w K}{Q_0 L^2} \tag{26}$$

leads to the unstable sliding with μ and T increasing exponentially. Here Q_0 is the initial heat generation rate per unit area. It is evident that the assumption of uniform temperature distribution along the body thickness is true for small values of w. When $w \to 0$, Eq.(26) transforms into the instability condition of Eq.(17).

Nosko et al. [21] studied the influence of negative μ–T slope on the occurrence and characteristics of stick–slip oscillations in a one-degree-of-freedom system with friction. The temperature in the body was described with account of the heat conduction in the direction perpendicular to the sliding surface and the contact heat transfer between the body and counterbody. The analysis of the energy balance led to the oscillation occurrence condition. For the case when there is no contact heat transfer, this condition degenerates into

$$\frac{\partial\mu}{\partial T} < -\frac{\sqrt{2}cKS\sqrt[4]{k}}{\alpha\mu_0 N^2\sqrt{\varkappa}\sqrt[4]{m}}$$

which coincides with the instability condition of Eq.(22) at $a = 0$ and $|\psi| \ll 1$.

Sextro [30] analysed the energy balance of stationary friction. Assuming that T is proportional to q with a coefficient k_q, the dependency $\mu = \mu_0 - k_T T$ was shown to be equivalent to the dependency

$$\mu = \frac{\mu_0}{1 + k_q k_T p v_s/\mu_0} \approx \mu_0 - k_q k_T p v_s$$

i.e. the model with negative μ–T slope (with the coefficient $b = -k_T$) is almost identical to the model with negative μ–v_s slope (with the coefficient $a = -k_q k_T p$). Here p is the contact pressure. If we assume a sharp negative μ–T slope, i.e. $\psi \to -\infty$, the transfer function of Eq.(11) takes the form

$$\mathcal{L}[X(\tau)] = \frac{\varepsilon_0}{s^2 + (2\zeta - F)s + 1}$$

which is equivalent to the motion equation

$$m\frac{d^2x}{dt^2} + \left(c - \frac{\mu_0 N}{v}\right)\frac{dx}{dt} + k\left(x + \frac{\mu_0 N}{k}\right) = 0, \qquad x|_{t=0} = -\frac{\mu_0 N}{k}, \qquad \left.\frac{dx}{dt}\right|_{t=0} = \varepsilon_0 v \tag{27}$$

The comparison of Eq.(27) with Eq.(2) at $b = 0$ shows that the sharp negative μ–T slope produces the same motion as does the μ–v_s slope with $a = -\mu_0/v$. This result is qualitatively similar to that obtained by Sextro [30].

6. Conclusions.

An analytical expression of Eq.(14) is derived for the motion of a body of a one-degree-of-freedom system with velocity- and temperature-dependent friction about the steady sliding equilibrium position. Based on this expression, the instability conditions of Eq.(17) and Eq.(22) are obtained, which allow to predict the sliding instability of two types — deviation of the body from the equilibrium position and oscillation. The instability conditions are in a good agreement with existing theoretical concepts.

1. Positive friction–temperature slope results in the deviation of the body from the equilibrium position. The deviation increases with the friction–temperature slope coefficient and can be accompanied by growing oscillation.
2. At negative friction–temperature slope, both types of the sliding instability can arise depending on the friction–velocity and friction–temperature slopes. The instability condition of Eq.(22) and the limit curves of Figs.6, 7 allow to predict the instability occurrence.
3. The motion of the body at sharp negative friction–temperature slope almost coincides with the motion at negative friction–velocity slope with the coefficient equal in modulus to the ratio of the friction coefficient to the steady sliding velocity.

References

[1] Ibrahim, R.A.: Friction-induced vibration, chatter, squeal, and chaos. Part 2: Dynamics and modeling. Appl. Mech. Rev. 47 (7), 227−253 (1994)

[2] Kajdanowsky, N.L., Khaikin, S.E.: Mechanical relaxation oscillations. Zhurnal Teknischeskoi Fiziki 3 (1), 91–109 (1933) (in Russian)

[3] Mills, H.R.: Brake squeak. Institution of Automobile Engineers. Report No. 9000 B (1938)

[4] Bowden, F.P., Leben, L.: The nature of sliding and the analysis of friction. P. R. Soc. Lond. A 169, 371–391 (1939)

[5] Ishlinsky, A.Y., Kragelsky, I.V.: On jumps under friction. Zhurnal Teknischeskoi Fiziki 14 (4–5), 276–283 (1944) (in Russian)

[6] Spurr, R.T.: A theory of brake squeal. P. I. Mech. Eng. Automobile Division 15 (1), 33–52 (1961)

[7] Barber, J.R.: Thermoelastic instabilities in the sliding of conforming bodies. P. R. Soc. Lond. A 312, 381–394 (1969)

[8] Awrejcewicz, J., Pyryev, Yu.: Thermo-mechanical model of frictional self-excited vibrations. Int. J. Mech. Sci. 47, 1393–1408 (2005)

[9] Adams, G.G.: Self-excited oscillations of two elastic half-spaces sliding with a constant coefficient of friction. J. Appl. Mech. 62, 867–872 (1995)

[10] Day, A.: Braking of road vehicles. Butterworth-Heinemann (2014)

[11] Kokonin, S.S., Kramarenko, E.I., Matveenko, A.M.: Basis of projecting aircraft wheels and braking systems. Moscow Aviation Institute Publisher, Moscow (2007) (in Russian)

[12] Kensley, R.S., Iwasa, Y.: Frictional properties of metal insulator surfaces at cryogenic temperatures. Cryogenics 20, 25–36 (1980)

[13] Yen, B.K., Ishihara, T.: On temperature-dependent tribological regimes and oxidation of carbon–carbon composites up to 1800 °C. Wear 196, 254–262 (1996)

[14] Venkataraman, B., Sundararajan, G.: The influence of sample geometry on the friction behaviour of carbon–carbon composites. Acta Mater. 50, 1153–1163 (2002)

[15] Nosko, O., Nagamine, T., Mori, H., Sato, Y.: Friction-induced oscillations of a non-asbestos organic pin sliding on a steel disc. Acta Mechanica et Automatica 9 (2), 84−88 (2015)

[16] Maksimov, I.L., Rakhmanov, A.L.: The thermofrictional oscillations under sliding. Phys. Lett. A 121 (8), 399–402 (1987)

[17] Maksimov, I.L.: Thermal instability of sliding and oscillations due to frictional heating effect. J. Tribol. 110, 69–72 (1988)

[18] Nosonovsky, M., Bhushan, B.: Thermodynamics of surface degradation, self-organization and self-healing for biomimetic surfaces. Phil. Trans. R. Soc. A 367, 1607–1627 (2009)

[19] Fox-Rabinovich, G., Veldhuis, S.C., Kovalev, A.I., Wainstein, D.L., Gershman, I.S., Korshunov, S., Shuster, L.S., Endrino, J.L.: Features of self-organization in ion modified nanocrystalline plasma vapor deposited AlTiN coatings under severe tribological conditions. J. Appl. Phys. 102, 074305 (2007)

[20] Mortazavi, V., Wang, C., Nosonovsky, M.: Stability of frictional sliding with the coefficient of friction depended on the temperature. J. Tribol. 134 (4), 041601 (2012)

[21] Nosko, O., Nagamine, T., Mori, H., Sato, Y.: Theoretical study of thermofrictional oscillations due to negative friction–temperature characteristic. Tribol. Int. 61, 235–243 (2013)

[22] Blok, H.: Theoretical study of temperature rise at surfaces of actual contact under oiliness lubricating conditions. General Discussion on Lubrication. Vol. 2. Institution of Mechanical Engineers, London, 222–235 (1937)

[23] Carslaw, H.S., Jaeger, J.C.: Conduction of heat in solids. 2nd ed. Oxford University Press, London (1959)

[24] Feller, W.: An introduction to probability theory and its applications. Vol. 1. 3d ed. Wiley, New York (1968)

[25] Podlubny, I.: Fractional differential equations. Academic Press, San Diego (1999)

[26] Beard, R.E.: Some notes on approximate product-integration. Journal of the Institute of Actuaries 73, 356–416 (1947)

[27] Bonnet, C., Partington, J.R.: Analysis of fractional delay systems of retarded and neutral type. Automatica 38, 1133–1138 (2002)

[28] Andronov, A.A., Vitt, A.A., Khaikin, S.E.: Theory of oscillations. Adison Wesley, New York (1966)

[29] Chen, G.X., Zhou, Z.R.: Correlation of a negative friction–velocity slope with squeal generation under reciprocating sliding conditions. Wear 255, 376–384 (2003)

[30] Sextro, W.: Dynamical contact problems with friction: models, methods, experiments and applications. 2nd ed. Springer, Berlin (2007)

Figures

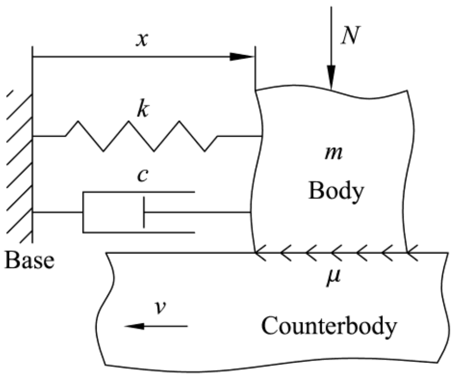

Fig.1 One-degree-of-freedom system with friction

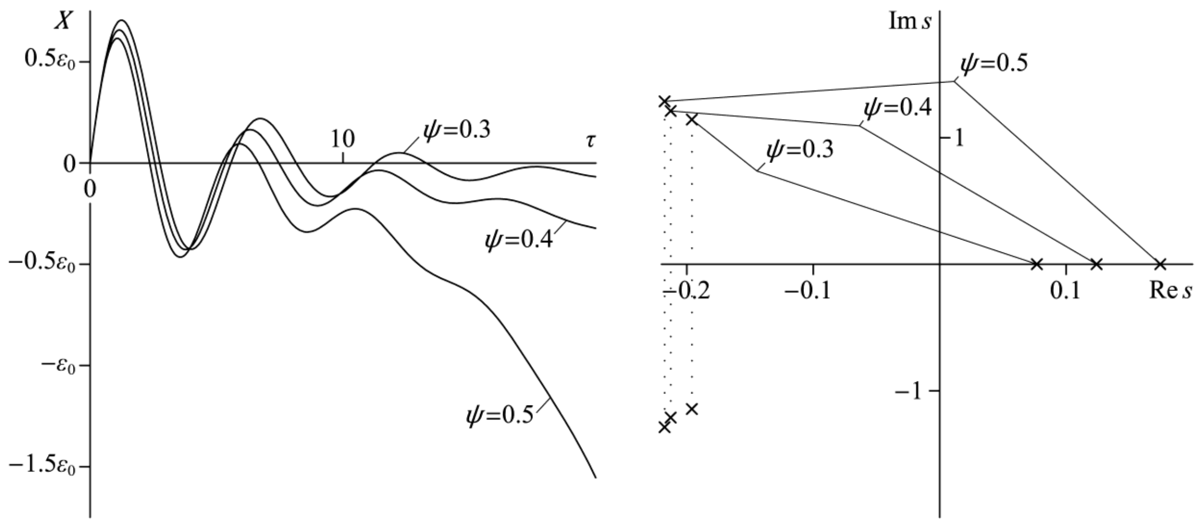

Fig.2 Deviation increasing with ψ at $\zeta = 0.1$ and $F = 1$

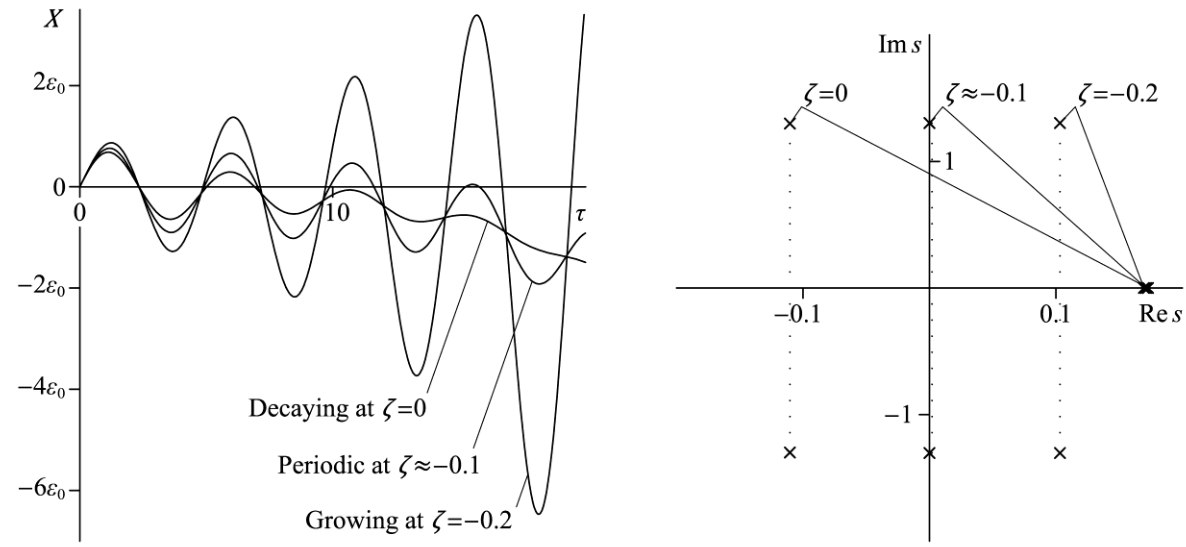

Fig.3 Oscillatory term at $\psi = 0.5$ and $F = 1$

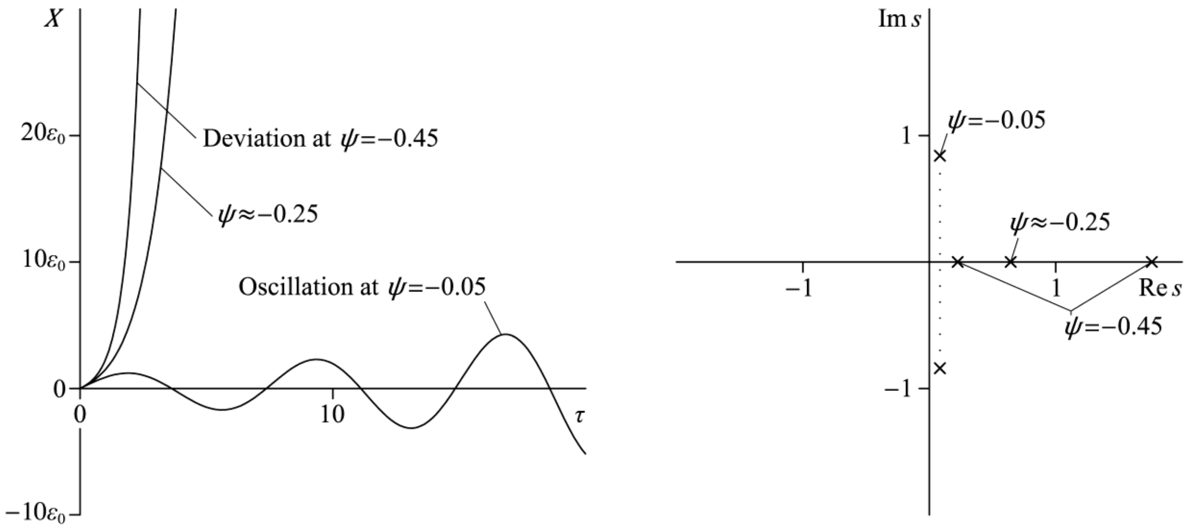

Fig.4 Transition between the deviation and oscillation at $\zeta = 0.1$ and $F = 10$

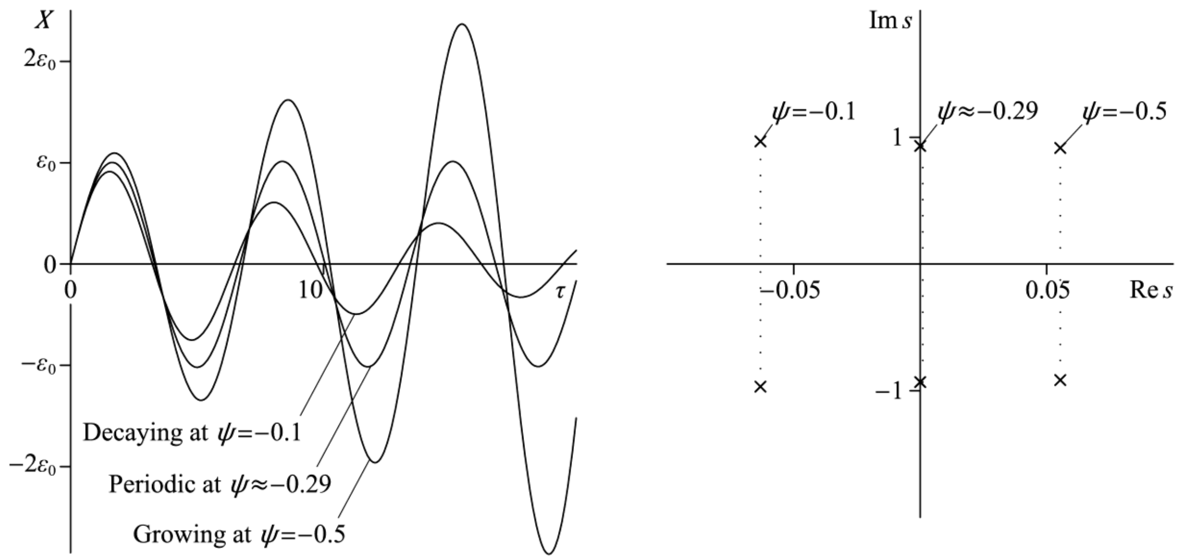

Fig.5 Transition between the growing oscillation and decaying oscillation at $\zeta = 0.1$ and $F = 1$

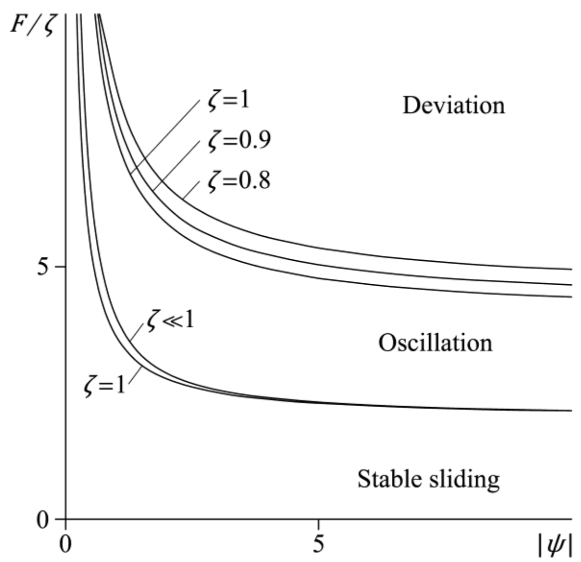

Fig.6 Character of sliding at $\psi < 0$ and $\zeta > 0$

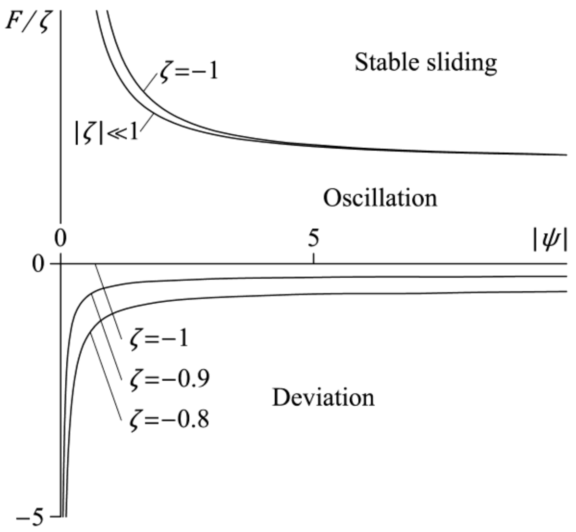

Fig.7 Character of sliding at $\psi < 0$ and $\zeta < 0$

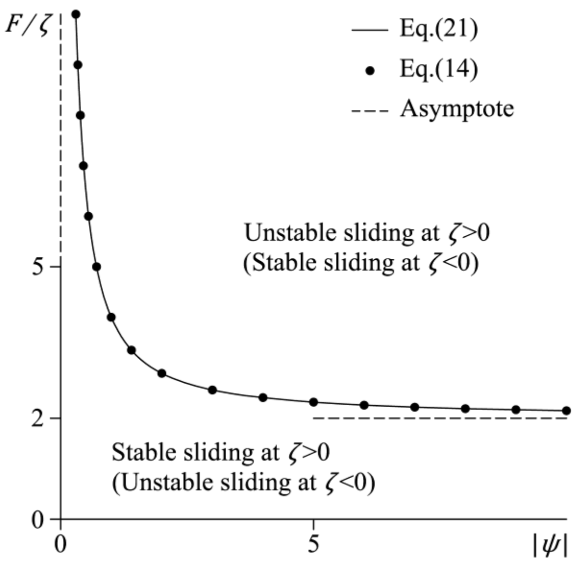

Fig.8 Limit curve of the sliding instability at $\psi < 0$ and $|\zeta| \ll 1$