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## APPLICATION OF MAZUR–ORLICZ’S THEOREM IN AMISE CALCULATION

*Abstract.* An approximation error and an asymptotic formula are given for shift invariant operators of polynomial order  $\varrho$ . Density estimators based on shift invariant operators are introduced and AMISE is calculated.

**1. Asymptotic formulas.** We assume that  $F, G : \mathbb{R}^d \rightarrow \mathbb{R}$  are functions such that there are constants  $C > 0$  and  $0 < q < 1$  such that for all  $x \in \mathbb{R}^d$ ,

$$(1) \quad |F(x)| < Cq^{|x|} \quad \& \quad |G(x)| < Cq^{|x|},$$

where  $|x|^2 = x \cdot x$  and  $x \cdot x$  is the scalar product in  $\mathbb{R}^d$ . Consider the operator given by

$$(2) \quad Qf(x) = \int_{\mathbb{R}^d} K(x, y)f(y)dy,$$

where

$$(3) \quad K(x, y) = \sum_{\alpha \in \mathbb{Z}^d} F(y - \alpha)G(x - \alpha).$$

For  $h > 0$ , define

$$(4) \quad Q_h = \sigma_h \circ Q \circ \sigma_{1/h},$$

where

$$\sigma_h f(x) = f(x/h).$$

We call the operators with kernel of type (3) *shift invariant*. Examples of such operators are:

- spline operators: the Ciesielski–Durrmeyer operator (see [C]), a quasi-projection (see [Dz1]), an orthogonal projection (see [BD2], [BHR]),

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- an orthogonal projection based on multiresolution approximation [M], operators based on shift invariant spaces (see [JZ] and [BDR]; in particular shift invariant spaces constructed by a function which satisfies the Strang–Fix conditions, see [SF]).

Let  $W_p^r$  be a Sobolev space (for details see [M]). Let  $C_0^r$  be the space of  $r$ -differentiable functions with compact support. Set

$$|f|_{r,p} = \sum_{|\beta|=r} \|D^\beta f\|_p, \quad \|f\|_p = \left( \int_{\mathbb{R}^d} |f|^p \right)^{1/p},$$

$$D^\beta f = \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \dots \partial x_d^{\beta_d}}, \quad \beta = (\beta_1, \dots, \beta_d), \quad |\beta| = \beta_1 + \dots + \beta_d.$$

Assume the operator  $Q$  reproduces all polynomials of degree less than  $r$ , i.e.  $Q(P) = P$  provided  $\deg P < r$ . We then say that  $Q$  has *polynomial order*  $r$ . The following theorem is a generalization of [BHR, Proposition 4, p. 63].

**THEOREM 1.1.** *Let  $1 \leq p < \infty$ . Assume that  $Q$  has polynomial order  $r$ . Then there is a constant  $C(p) > 0$  such that for all  $f \in W_p^r(\mathbb{R}^d)$ ,*

$$(5) \quad \|Q_h f - f\|_p \leq C(p) h^r |f|_{r,p},$$

*Proof.* Since the operators  $Q_h$  are bounded from  $L^p$  to  $L^p$  it is sufficient to prove (5) for  $f \in C_0^r$ . Let  $f \in C_0^r$ . Let  $P_x$  be the Taylor polynomial of  $f$  of degree  $r-1$  at  $x$ . Note that  $f(x) = P_x(x)$  and  $Q_h f(x) - f(x) = Q_h(f - P_x)(x)$ . Now Lemma 1.1 below yields (5) for  $1 \leq p < \infty$ . ■

An easy computation shows the assertion for  $p = \infty$  (see proof of [Dz4, Theorem 9.7]).

In statistics we need an asymptotic formula for the error in shift invariant operators. Such a formula was proved in [BD3], [BD4] for an interpolation operator and an orthogonal projection. Those proofs are based on a generalization of Mazur–Orlicz’s theorem (see [BD3]). This theorem goes back to L. Fejér. Recall that a function  $g$  defined on  $\mathbb{R}^d$  is called  $\mathbb{Z}^d$ -periodic if for all  $x \in \mathbb{R}^d$ ,

$$(6) \quad g(x) = g(x + \alpha) \quad \text{for all } \alpha \in \mathbb{Z}^d.$$

**THEOREM 1.2** (Mazur–Orlicz [MO]). *If for  $j = 1, \dots, m$ ,  $g_j$  are measurable, bounded,  $\mathbb{Z}^d$ -periodic functions and  $f_j$  are measurable functions with*

$$\int_{\mathbb{R}^d} |f_j(x)|^p dx < \infty$$

for some  $1 \leq p < \infty$ , then

$$(7) \quad \int_{\mathbb{R}^d} \left| \sum_{j=1}^m f_j(x) g_j(x/h) \right|^p dx \rightarrow \int_{[0,1]^d} \int_{\mathbb{R}^d} \left| \sum_{j=1}^m f_j(t) g_j(x) \right|^p dt dx \quad \text{as } h \rightarrow 0.$$



Earlier results concerning the asymptotic formula can be found in [C], [Dz1] for spline operators, and in [DU], [BD2] for an orthogonal projection in  $L^2$ . See also [DLP]. Let  $[\beta](x) = x^\beta = x_1^{\beta_1} \dots x_d^{\beta_d}$ . We present a new and simpler proof of the asymptotic formula for the error in shift invariant operators.

**THEOREM 1.3.** *Assume that  $Q$  has maximal polynomial order  $\varrho$ . Let  $1 \leq p < \infty$  and  $f \in W_p^\varrho(\mathbb{R}^d)$ . Then*

$$(8) \quad \lim_{h \rightarrow 0^+} \left\| \frac{Q_h f - f}{h^\varrho} \right\|_p^p = \int_{\mathbb{R}^d} \left( \int_{[0,1]^d} \left| \sum_{|\beta|=\varrho} \frac{1}{\beta!} D^\beta f(t) (Q([\beta])(x) - x^\beta) \right|^p dx \right) dt.$$

*Proof.* It is sufficient to prove (8) for the dense subset  $C_0^{\varrho+1}$  of  $W_p^\varrho(\mathbb{R}^d)$  since

$$\left\| \frac{Q_h f - f}{h^\varrho} \right\|_p \leq C |f|_{\varrho,p}.$$

Fix  $f \in C_0^{\varrho+1}$ . Let  $P_x$  be the Taylor polynomial of degree  $\varrho$  of  $f$  at  $x$ . By the triangle inequality (we take  $F(x) = Q_h(P_x)(x) \neq P_x(x)$ )

$$\left\| \frac{Q_h f - f}{h^\varrho} \right\|_p \leq \left\| \frac{Q_h(f - P_x)}{h^\varrho} \right\|_p + \left\| \frac{Q_h P_x - P_x}{h^\varrho} \right\|_p$$

and

$$\left\| \frac{Q_h P_x - P_x}{h^\varrho} \right\|_p \leq \left\| \frac{Q_h f - f}{h^\varrho} \right\|_p + \left\| \frac{Q_h(f - P_x)}{h^\varrho} \right\|_p.$$

If we prove that there is  $C$  such that for all  $f \in C_0^{\varrho+1}$ ,

$$(9) \quad \|Q_h(f - P_x)\|_p \leq C h^{\varrho+1} |f|_{\varrho+1,p},$$

then the proof of (8) is completed by showing that

$$(10) \quad \begin{aligned} \lim_{h \rightarrow 0^+} \left\| \frac{Q_h P_x - P_x}{h^\varrho} \right\|_p^p &= \int_{\mathbb{R}^d} \left( \int_{[0,1]^d} \left| \sum_{|\beta|=\varrho} \frac{1}{\beta!} D^\beta f(t) (Q([\beta])(x) - x^\beta) \right|^p dx \right) dt. \end{aligned}$$

The technical proof of (9) is postponed to Lemma 1.1. Let

$$P_x = T_x + R_x,$$

where  $T_x$  is homogeneous of degree  $\varrho$  and  $\deg R_x < \varrho$ . Since  $Q(R_x) = R_x$  we have

$$(11) \quad \begin{aligned} \frac{Q_h(P_x)(t) - P_x(t)}{h^\varrho} &= \frac{Q_h(T_x)(t) - T_x(t)}{h^\varrho} = Q(T_x)(t/h) - T_x(t/h) \\ &= \sum_{|\beta|=\varrho} \frac{1}{\beta!} D^\beta f(x) (Q([\beta])(t/h) - (t/h)^\beta). \end{aligned}$$



Consequently, from (11) we get

$$\left\| \frac{Q_h P. - P.}{h^\varrho} \right\|_p^p = \int_{\mathbb{R}^d} \left| \sum_{|\beta|=\varrho} \frac{1}{\beta!} D^\beta f(x) (Q([\beta])(x/h) - (x/h)^\beta) \right|^p dx.$$

An easy calculation shows (cf. [Dz3, Lemma 3.3]) that the functions

$$Q([\beta])(x) - x^\beta = (-1)^{|\beta|} \sum_{\alpha \in \mathbb{Z}^d} \int_{\mathbb{R}^d} (x-y)^\beta F(y-\alpha) dy G(x-\alpha)$$

are  $\mathbb{Z}^d$ -periodic. Now the Mazur–Orlicz Theorem (7) implies (10).

LEMMA 1.1. *Let  $1 \leq p < \infty$ . Let  $P_x$  be the Taylor polynomial of degree  $k-1$  of a function  $f$ . There is  $C$  such that for all  $f \in C_0^k$ ,*

$$(12) \quad \|Q_h(f - P.)\|_p \leq Ch^k |f|_{k,p}.$$

*Proof.* By Taylor's formula,

$$\begin{aligned} \|Q_h(f - P.)\|_p^p &= \int_{\mathbb{R}^d} \left| \sum_{\alpha \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \int_0^1 \sum_{|\beta|=k} \frac{1}{\beta!} D^\beta f(x + s(hy - x))(1-s)^{k-1} ds \right. \\ &\quad \left. \times (hy - x)^\beta F(y - \alpha) dy G(x/h - \alpha) \right|^p dx. \end{aligned}$$

To prove (12), using assumption (1), it is sufficient to estimate

$$\begin{aligned} J_\beta &= \int_{\mathbb{R}^d} \left| \sum_{\alpha \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \int_0^1 |D^\beta f(x + s(hy - x))| ds \right. \\ &\quad \left. \times |hy - x|^k q^{|y-\alpha|} dy q^{|x/h-\alpha|} \right|^p dx. \end{aligned}$$

We apply Jensen's inequality three times:

$$\begin{aligned} \left( \int_0^1 g(s) ds \right)^p &\leq \int_0^1 |g(x)|^p dx, \\ \left( \sum_{\alpha \in \mathbb{Z}^d} |a_\alpha| q^{|x-\alpha|} \right)^p &\leq C_1 \sum_{\alpha \in \mathbb{Z}^d} |a_\alpha|^p q^{|x-\alpha|}, \end{aligned}$$

where  $C_1$  is independent of  $x$ , i.e.  $C_1 = \max_x (\sum_{\alpha \in \mathbb{Z}^d} q^{|x-\alpha|})^{p-1}$ ,

$$\left( \int_{\mathbb{R}^d} |g(y)| q^{|y-\alpha|} dy \right)^p \leq C_2 \int_{\mathbb{R}^d} |g(y)|^p q^{|y-\alpha|} dy,$$

where  $C_2$  is independent of  $\alpha$ . Consequently,

$$\begin{aligned} J_\beta &\leq C \sum_{\alpha \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 |D^\beta f(x + s(hy - x))|^p ds \\ &\quad \times |hy - x|^{pk} q^{|y-\alpha|} dy q^{|x/h-\alpha|} dx. \end{aligned}$$

Letting  $x/h - \alpha = u$  yields

$$J_\beta \leq Ch^d \sum_{\alpha \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \int_0^1 |D^\beta f(hu + h\alpha + s(hy - hu - h\alpha))|^p ds \\ \times |hy - hu - h\alpha|^{pk} q^{|y-\alpha|} dy q^{|u|} du$$

and by obvious changes of variables

$$J_\beta \leq Ch^d \sum_{\alpha \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \int_0^1 |D^\beta f(hu + h\alpha + sh(z - u))|^p ds \\ \times h^{pk} |z - u|^{pk} q^{|z|} dz q^{|u|} du \\ = Ch^{d+pk} \sum_{\alpha \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \int_0^1 |D^\beta f(hu + h\alpha + shv)|^p ds |v|^{pk} q^{|u+v|} dv q^{|u|} du.$$

Let us split the integrals:

$$J_\beta \leq Ch^{d+pk} \sum_{n=1}^\infty \sum_{\alpha \in \mathbb{Z}^d} \int_{n-1 < |u| < n} \int_0^1 |D^\beta f(hu + h\alpha + shv)|^p ds \\ \times |v|^{pk} q^{|u+v|} dv q^{n-1} du \\ \leq Ch^{d+pk} \sum_{n=1}^\infty q^{n-1} \sum_{\alpha \in \mathbb{Z}^d} \int_{n-1 < |u| < n} \sum_{j=1}^\infty q^{j-1} \\ \times \int_{j-1 < |u+v| < j} \int_0^1 |D^\beta f(hu + h\alpha + shv)|^p ds |v|^{pk} dv du.$$

Note that if  $|v + u| < j$  and  $|u| < n$  then

$$|v| < |v + u| + |u| < j + n.$$

Thus

$$J_\beta \leq Ch^{d+pk} \sum_{n=1}^\infty q^{n-1} \sum_{\alpha \in \mathbb{Z}^d} \int_{|u| < n} \sum_{j=1}^\infty q^{j-1} \\ \times \int_{|v| < j+n} \int_0^1 |D^\beta f(hu + h\alpha + shv)|^p ds |v|^{pk} dv du.$$

Changing the order of the integrations we get

$$J_\beta \leq Ch^{d+pk} \sum_{n=1}^\infty q^{n-1} \sum_{j=1}^\infty q^{j-1} \\ \times \int_{|v| < j+n} \int_0^1 \sum_{\alpha \in \mathbb{Z}^d} \int_{|u| < n} |D^\beta f(hu + h\alpha + shv)|^p du ds |v|^{pk} dv.$$

Note that if  $|v| < j + n$  then

$$\begin{aligned} h^d \sum_{\alpha \in \mathbb{Z}^d} \int_{|u| < n} |D^\beta f(hu + h\alpha + shv)|^p du \\ \leq \sum_{\alpha \in \mathbb{Z}^d} \int_{|x-h\alpha| < h(2n+j)} |D^\beta f(x)|^p dx \leq (4n+2j)^d \int_{\mathbb{R}^d} |D^\beta f(x)|^p dx \end{aligned}$$

and moreover

$$\int_{|v| < j+n} |v|^{pk} dv = C(j+n)^{pk+d}.$$

Consequently,

$$\begin{aligned} J_\beta &\leq C \sum_{n=1}^{\infty} q^{n-1} h^{pk} \sum_{j=1}^{\infty} q^{j-1} (4n+2j)^d (j+n)^{pk+d} \int_{\mathbb{R}^d} |D^\beta f(x)|^p dx \\ &\leq Ch^{pk} \int_{\mathbb{R}^d} |D^\beta f|^p. \end{aligned}$$

This finishes the proof of the lemma. ■

Let  $X_1, \dots, X_n$  be a random sample from a distribution with density  $f \in W_2^\varrho$ . We define a density estimator based on the kernel  $K$  by

$$(13) \quad f_{h,n}(x) = \frac{1}{n} \sum_{j=1}^n K_h(x, X_j),$$

where

$$K_h(x, y) = (1/h)^d K(x/h, y/h).$$

Note that

$$(14) \quad Ef_{h,n} = Q_h f.$$

As usual we consider the estimation error given by

$$(15) \quad \text{MISE}(f, h) = E \left[ \int_{\mathbb{R}^d} [f_{h,n} - f]^2 \right].$$

It is known that

$$(16) \quad \text{MISE}(f, h) = E \left[ \int_{\mathbb{R}^d} [f_{h,n} - Q_h f]^2 \right] + \int_{\mathbb{R}^d} [Q_h f - f]^2.$$

The asymptotic formula for the second factor in (16) is given in (8). We prove that

**THEOREM 1.4.** *Assume that  $Q$  has maximal polynomial order  $\varrho > 0$ . If  $nh^d \rightarrow \infty$ ,  $h \rightarrow 0$  then*

$$(17) \quad \lim_{nh^d \rightarrow \infty} nh^d E \left[ \int_{\mathbb{R}^d} [f_{h,n} - Q_h f]^2 \right] = \int_{\mathbb{R}^d} \left[ \int_{[0,1]^d} K^2(x, y) dy \right] dx,$$



where

$$(18) \quad \int_{\mathbb{R}^d} \left[ \int_{[0,1]^d} K^2(x, y) dy \right] dx = \sum_{\alpha \in \mathbb{Z}^d} \eta(\alpha) \xi(\alpha)$$

and

$$\eta = G * \check{G}, \quad \xi = F * \check{F}, \quad \check{G}(x) = G(-x), \quad \check{F}(x) = F(-x).$$

*Proof.* Note that

$$\begin{aligned} E \left[ \int_{\mathbb{R}^d} [f_{h,n} - E f_{h,n}]^2 \right] &= \frac{1}{n^2} \sum_{j=1}^n \int_{\mathbb{R}^d} E [K_h(x, X_j) - EK_h(x, X_j)]^2 dx \\ &= \frac{1}{n^2} \sum_{j=1}^n \int_{\mathbb{R}^d} (E[K_h^2(x, X_j)] - [EK_h(x, X_j)]^2) dx. \end{aligned}$$

If  $h \rightarrow 0$  then by (5),

$$\int_{\mathbb{R}^d} [EK_h(x, X_j)]^2 dx = \int_{\mathbb{R}^d} (Q_h f)^2 \rightarrow \int_{\mathbb{R}^d} f^2.$$

On the other hand

$$\int_{\mathbb{R}^d} EK_h^2(x, X_j) dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_h^2(x, y) f(y) dy dx.$$

From Fubini's theorem

$$\int_{\mathbb{R}^d} EK_h^2(x, X_j) dx = \frac{1}{h^d} \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} K^2(u, y/h) du \right] f(y) dy.$$

Note that for all  $\alpha \in \mathbb{Z}^d$ ,

$$(19) \quad \int_{\mathbb{R}^d} K^2(x, y + \alpha) dx = \int_{\mathbb{R}^d} K^2(x, y) dx.$$

From Mazur–Orlicz's theorem we get

$$\lim_{h \rightarrow 0} \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} K^2(u, y/h) du \right] f(y) dy = \int_{[0,1]^d} \left[ \int_{\mathbb{R}^d} K^2(u, y) du \right] dy \int_{\mathbb{R}^d} f(y) dy.$$

We thus get (17). A simple calculation leads to (18). ■

REMARKS. 1. From (16)–(8) we get

$$\begin{aligned} \text{MISE}(f, h) \sim \text{AMISE} &:= \frac{1}{nh^d} \int_{\mathbb{R}^d} \left[ \int_{[0,1]^d} K^2(x, y) dy \right] dx \\ &+ h^{2\varrho} \int_{\mathbb{R}^d} \left( \int_{[0,1]^d} \left| \sum_{|\beta|=\varrho} \frac{1}{\beta!} D^\beta f(t) Q([\beta])(x) - x^\beta \right|^2 dx \right) dt. \end{aligned}$$



So the best choice of  $h > 0$  which minimizes (16) is

$$h \sim n^{-1/(2q+d)}.$$

2. Using the methods of [Dz2] one can prove the central limit theorem. This theorem generalizes the results for wavelet estimators [DL1]–[DL2] in  $\mathbb{R}^d$  and box spline estimators [Dz2]. These results are motivated by the result for the Rosenblatt–Parzen estimator [H].

### References

- [BD1] M. Beška and K. Dziedziul, *The saturation theorem for interpolation and the Bernstein–Schnabl operator*, Math. Comp. 70 (2001), 705–717.
- [BD2] —, —, *The saturation theorem for orthogonal projection*, in: *Advances in Multivariate Approximation*, W. Haussmann et al. (eds.), Wiley, 1999, 73–83.
- [BD3] —, —, *Asymptotic formula for the error in cardinal interpolation*, Numer. Math. 89 (2001), 445–456.
- [BD4] —, —, *Asymptotic formula for the error in orthogonal projection*, Math. Nachr. 233 (2002), 47–53.
- [BDR] C. de Boor, R. A. DeVore and A. Ron, *Approximation from shift-invariant subspaces of  $L_2(\mathbb{R}^d)$* , Trans. Amer. Math. Soc. 341 (1994), 787–806.
- [BHR] C. de Boor, K. Höllig and S. Riemenschneider, *Box Splines*, Springer, 1993.
- [C] Z. Ciesielski, *Asymptotic nonparametric spline density estimation in several variables*, in: Internat. Ser. Numer. Math. 94, Birkhäuser, Basel, 1990, 25–53.
- [DU] I. Daubechies and M. Unser, *On the approximation power of convolution-based least squares versus interpolation*, IEEE Trans. Signal Process. 45 (1997), 1697–1711.
- [DL1] P. Doukhan and J. R. León, *Quadratic deviation of projection density estimates*, Rebrape 7 (1993), 37–63.
- [DL2] —, —, *Déviations quadratiques d’estimateurs de densité par projections orthogonales*, C. R. Acad. Sci. Paris Sér. I Math. 310 (1990), 425–430.
- [DLP] P. Doukhan, J. Léon et F. Portal, *Une mesure de la déviation quadratique d’estimateurs non paramétriques*, Ann. Inst. H. Poincaré Probab. Statist. 22 (1986), 37–66.
- [Dz1] K. Dziedziul, *The saturation theorem for quasi-projections*, Studia Sci. Math. Hungar. 35 (1999), 99–111.
- [Dz2] —, *Central limit theorem for square error of multivariate nonparametric box spline density estimators*, Appl. Math. (Warsaw) 28 (2001), 437–456.
- [Dz3] —, *Asymptotic formulas in cardinal interpolation and orthogonal projection*, in: *Recent Progress in Multivariate Approximation*, W. Haussmann et al. (eds.), Internat. Ser. Numer. Math. 137, Birkhäuser, Basel, 2001, 139–157.
- [Dz4] —, *Box Splines*, Wyd. Politechniki Gdańskiej, 1997 (in Polish).
- [H] P. Hall, *Central limit theorem for integrated square error of multivariate nonparametric density estimators*, J. Multivariate Anal. 14 (1984), 1–16.
- [JZ] K. Jetter and D.-X. Zhou, *Order of linear approximation from shift-invariant spaces*, Constr. Approx. 11 (1995), 423–438.
- [MO] S. Mazur et W. Orlicz, *Sur quelques propriétés de fonctions périodiques*, Studia Math. 9 (1940), 1–16.
- [M] Y. Meyer, *Wavelets and Operators*, Cambridge Univ. Press, Cambridge, 1992.





- [SF] G. Strang and G. Fix, *A Fourier analysis of the finite element variational method*, in: *Constructive Aspects of Functional Analysis*, G. Geymonat (ed.), Ed. Cremonese, Roma, 1973, 793–840.

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