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Combinatorics

Bounds on the vertex–edge domination number of a tree

Bornes sur le nombre de domination sommet–arête d'un arbre

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ABSTRACT

A vertex–edge dominating set of a graph G is a set D of vertices of G such that every edge of G is incident with a vertex of D or a vertex adjacent to a vertex of D. The vertex–edge domination number of a graph G, denoted by $\gamma_{ve}(T)$, is the minimum cardinality of a vertex–edge dominating set of G. We prove that for every tree T of order $n \geqslant 3$ with l leaves and s support vertices, we have $(n-l-s+3)/4 \leqslant \gamma_{ve}(T) \leqslant n/3$, and we characterize the trees attaining each of the bounds.

RÉSUMÉ

Un ensemble sommet–arête dominant d'un graphe G est un ensemble D de sommets de G tel que chaque arête de G soit incidente à un sommet de D ou à un sommet adjacent à un sommet de D. Le nombre de domination sommet–arête d'un graphe G, noté $\gamma_{ve}(T)$, est le cardinal minimum d'un ensemble sommet–arête dominant de G. Nous prouvons que, pour chaque arbre T d'ordre $n \geqslant 3$ avec l feuilles et des sommets s de soutien, que nous avons $(n-l-s+3)/4 \leqslant \gamma_{ve}(T) \leqslant n/3$, et nous caractérisons les arbres atteignant chacune des limites.

1. Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). We denote by P_n the path on n vertices. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that the subtree

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resulting from T by removing the edge vx and which contains the vertex x as a leaf, is a path P_n. By a star we mean a connected graph in which exactly one vertex has degree greater than one.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of G is the minimum cardinality of a dominating set of G. For a comprehensive survey of domination in graphs, see [2,3].

An edge $e \in E(G)$ is vertex–edge dominated by a vertex $v \in V(G)$ if e is incident to v, or e is adjacent to an edge incident to v. A subset $D \subseteq V(G)$ is a vertex–edge dominating set, abbreviated VEDS, of G if every edge of G is vertex–edge dominated by a vertex of D. The vertex–edge domination number of G, denoted by $\gamma_{\mathrm{ve}}(G)$, is the minimum cardinality of a vertex–edge dominating set of G. A vertex–edge dominating set of G of minimum cardinality is called a $\gamma_{\mathrm{ve}}(G)$-set. Vertex–edge domination in graphs was introduced in [7], and further studied in [6].

Chellali and Haynes [1] established the following lower bound on the total domination number of a tree. For every tree T of order n with l leaves, we have $\gamma_t(T) \geqslant (n-l+2)/2$. They also characterized the extremal trees. In [4] a lower bound on the total outer-independent domination number of a tree was given together with the characterization of the extremal trees. Lemańska [5] proved that the domination number of a tree is bounded below by $(n-l+2)/3$.

We prove the following bounds on the vertex–edge domination number of a tree T of order $n \geqslant 3$ with l leaves and s support vertices, $(n-l-s+3)/4 \leqslant \gamma_{\mathrm{ve}}(T) \leqslant n/3$. We also characterize the trees attaining each of the bounds.

2. Results

We begin with the following straightforward observation.

Observation 1. *For every connected graph G of diameter at least two, there exists a $\gamma_{\mathrm{ve}}(G)$-set that contains no leaf.*

First we show that if T is a nontrivial tree of order n with l leaves and s support vertices, then $\gamma_{\mathrm{ve}}(T)$ is bounded below by $(n-l-s+3)/4$. For the purpose of characterizing the trees attaining this bound, we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let T_1 be a path P_5. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of T_k.
- Operation $\mathcal{O}_2$: Attach a path P_2 by joining one of its vertices to a vertex of T_k, which is not a leaf and is adjacent to a support vertex of degree two.
- Operation $\mathcal{O}_3$: Attach a path P_4 by joining one of its leaves to a leaf of T_k adjacent to a weak support vertex.

We now prove that for every tree T of the family $\mathcal{T}$ we have $\gamma_{\mathrm{ve}}(T) = (n-l-s+3)/4$.

Lemma 2. *If $T \in \mathcal{T}$, then $\gamma_{\mathrm{ve}}(T) = (n-l-s+3)/4$.*

Proof. We use the induction on the number k of operations performed to construct the tree T. If $T = T_1 = P_5$, then $(n-l-s+3)/4 = (5-2-2+3)/4 = 1 = \gamma_{\mathrm{ve}}(T)$. Let k be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let n' be the order of the tree T', l' the number of its leaves, and s' the number of support vertices. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. We have $n = n'+1$, $l = l'+1$ and $s = s'$. It is straightforward to see that any $\gamma_{\mathrm{ve}}(T')$-set is a VEDS of the tree T. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T')$. Obviously, $\gamma_{\mathrm{ve}}(T') \leqslant \gamma_{\mathrm{ve}}(T)$. This implies that $\gamma_{\mathrm{ve}}(T) = \gamma_{\mathrm{ve}}(T')$. We now get $\gamma_{\mathrm{ve}}(T) = \gamma_{\mathrm{ve}}(T') = (n'-l'-s'+3)/4 = (n-1-l+1-s+3)/4 = (n-l-s+3)/4$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. We have $n = n'+2$, $l = l'+1$ and $s = s'+1$. It is straightforward to see that any $\gamma_{\mathrm{ve}}(T')$-set is a VEDS of the tree T. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T')$. This implies that $\gamma_{\mathrm{ve}}(T) = \gamma_{\mathrm{ve}}(T')$. We now get $\gamma_{\mathrm{ve}}(T) = \gamma_{\mathrm{ve}}(T') = (n'-l'-s'+3)/4 = (n-2-l+1-s+1+3)/4 = (n-l-s+3)/4$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. We have $n = n'+4$, $l = l'$ and $s = s'$. We denote by x the leaf to which P_4 is attached. Let $v_1v_2v_3v_4$ be the attached path. Let v_1 be joined to x. Let D' be any $\gamma_{\mathrm{ve}}(T')$-set. It is easy to see that $D' \cup \{v_2\}$ is a VEDS of the tree T. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T')+1$. Now let us observe that there exists a $\gamma_{\mathrm{ve}}(T)$-set that does not contain the vertices v_4, v_3, and v_1. Let D be such a set. To dominate the edge v_3v_4, we have $v_2 \in D$. Observe that $D \setminus \{v_2\}$ is a VEDS of the tree T'. Therefore $\gamma_{\mathrm{ve}}(T') \leqslant \gamma_{\mathrm{ve}}(T)-1$. We now conclude that $\gamma_{\mathrm{ve}}(T) = \gamma_{\mathrm{ve}}(T')+1$. We get $\gamma_{\mathrm{ve}}(T) = \gamma_{\mathrm{ve}}(T')+1 = (n'-l'-s'+3)/4+1 = (n-4-l-s+7)/4 = (n-l-s+3)/4$. $\square$

We now give a lower bound on the vertex–edge domination number of a tree together with the characterization of the extremal trees.

Theorem 3. *If T is a nontrivial tree of order n with l leaves and s support vertices, then $\gamma_{\mathrm{ve}}(T) \geqslant (n-l-s+3)/4$ with equality if and only if $T \in \mathcal{T}$.*

Proof. If $\mathrm{diam}(T) = 1$, then $T = P_2$. We have $(n-l-s+3)/4 = (2-2-2+3)/4 < 1 = \gamma_{\mathrm{ve}}(T)$. If $\mathrm{diam}(T) = 2$, then T is a star. We have $l = n-1$ and $s = 1$. Consequently, $(n-l-s+3)/4 = (n-n+1-1+3)/4 = 3/4 < 1 = \gamma_{\mathrm{ve}}(T)$.

Now assume that $\mathrm{diam}(T) \geqslant 3$. Thus the order n of the tree T is at least four. We obtain the result by the induction on the number n. Assume that the theorem is true for every tree T' of order $n' < n$ with l' leaves and s' support vertices.

First assume that some support vertex of T, say x, is strong. Let y be a leaf adjacent to x. Let $T' = T - y$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s$. Obviously, $\gamma_{\mathrm{ve}}(T') \leqslant \gamma_{\mathrm{ve}}(T)$. We get $\gamma_{\mathrm{ve}}(T) \geqslant \gamma_{\mathrm{ve}}(T') \geqslant (n' - l' - s' + 3)/4 = (n - 1 - l + 1 - s + 3)/4 = (n - l - s + 3)/4$. If $\gamma_{\mathrm{ve}}(T) = (n - l - s + 3)/4$, then obviously $\gamma_{\mathrm{ve}}(T') = (n' - l' - s' + 3)/4$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If $\mathrm{diam}(T) \geqslant 4$, then let w be the parent of u. If $\mathrm{diam}(T) \geqslant 5$, then let d be the parent of w. If $\mathrm{diam}(T) \geqslant 6$, then let e be the parent of d. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that some child of u, say x, is a leaf. Let $T' = T - x$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. We get $\gamma_{\mathrm{ve}}(T) \geqslant \gamma_{\mathrm{ve}}(T') \geqslant (n' - l' - s' + 3)/4 = (n - 1 - l + 1 - s + 1 + 3)/4 > (n - l - s + 3)/4$.

Now assume that among the children of u there is a support vertex other than v. Let $T' = T - T_v$. We have $n' = n - 2$, $l' = l - 1$ and $s' = s - 1$. We get $\gamma_{\mathrm{ve}}(T) \geqslant \gamma_{\mathrm{ve}}(T') \geqslant (n' - l' - s' + 3)/4 = (n - 2 - l + 1 - s + 1 + 3)/4 = (n - l - s + 3)/4$. If $\gamma_{\mathrm{ve}}(T) = (n - l - s + 3)/4$, then obviously $\gamma_{\mathrm{ve}}(T) = (n' - l' - s' + 3)/4$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(u) = 2$. Assume that $d_T(w) \geqslant 3$. First assume that some child of w, say x, is a leaf. Let $T' = T - x$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. We get $\gamma_{\mathrm{ve}}(T) \geqslant \gamma_{\mathrm{ve}}(T') \geqslant (n' - l' - s' + 3)/4 = (n - 1 - l + 1 - s + 1 + 3)/4 > (n - l - s + 3)/4$.

Now assume that no child of w is a leaf. Let $T' = T - T_u$. We have $n' = n - 3$, $l' = l - 1$ and $s' = s - 1$. Let us observe that there exists a $\gamma_{\mathrm{ve}}(T)$-set that does not contain the vertices t and v. Let D be such a set. To dominate the edge vt, we have $u \in D$. Let us observe that $D \setminus \{u\}$ is a VEDS of the tree T'. Therefore $\gamma_{\mathrm{ve}}(T') \leqslant \gamma_{\mathrm{ve}}(T) - 1$. We now get $\gamma_{\mathrm{ve}}(T) \geqslant \gamma_{\mathrm{ve}}(T') + 1 \geqslant (n' - l' - s' + 3)/4 + 1 = (n - 3 - l + 1 - s + 1 + 7)/4 > (n - l - s + 3)/4$.

If $d_T(w) = 1$, then $T = P_4$. We have $(n - l - s + 3)/4 = (4 - 2 - 2 + 3)/4 < 1 = \gamma_{\mathrm{ve}}(T)$. Now assume that $d_T(w) = 2$. First assume that $d_T(d) \geqslant 3$. Let $T' = T - T_w$. We have $n' = n - 4$, $l' = l - 1$ and $s' = s - 1$. Let us observe that there exists a $\gamma_{\mathrm{ve}}(T)$-set that does not contain the vertices t, v and w. Let D be such a set. To dominate the edge vt, we have $u \in D$. Observe that $D \setminus \{u\}$ is a VEDS of the tree T'. Therefore $\gamma_{\mathrm{ve}}(T') \leqslant \gamma_{\mathrm{ve}}(T) - 1$. We now get $\gamma_{\mathrm{ve}}(T) \geqslant \gamma_{\mathrm{ve}}(T') + 1 \geqslant (n' - l' - s' + 3)/4 + 1 = (n - 4 - l + 1 - s + 1 + 7)/4 > (n - l - s + 3)/4$.

Now assume that $d_T(d) = 2$. First assume that some child of e is a leaf. Let $T' = T - T_w$. We have $n' = n - 4$, $l' = l$ and $s' = s - 1$. Similarly as in the previous possibility, we conclude that $\gamma_{\mathrm{ve}}(T') \leqslant \gamma_{\mathrm{ve}}(T) - 1$. We get $\gamma_{\mathrm{ve}}(T) \geqslant \gamma_{\mathrm{ve}}(T') + 1 \geqslant (n' - l' - s' + 3)/4 + 1 = (n - 4 - l - s + 1 + 7)/4 > (n - l - s + 3)/4$.

Now assume that no child of e is a leaf. Let $T' = T - T_w$. We have $n' = n - 4$, $l' = l$ and $s' = s$. If $n' = 1$, then $T = P_5 = T_1 \in \mathcal{T}$. Assume that $n' \geqslant 2$. Similarly as earlier, we conclude that $\gamma_{\mathrm{ve}}(T') \leqslant \gamma_{\mathrm{ve}}(T) - 1$. We now get $\gamma_{\mathrm{ve}}(T) \geqslant \gamma_{\mathrm{ve}}(T') + 1 \geqslant (n' - l' - s' + 3)/4 + 1 = (n - 4 - l - s + 7)/4 = (n - l - s + 3)/4$. If $\gamma_{\mathrm{ve}}(T) = (n - l - s + 3)/4$, then obviously $\gamma_{\mathrm{ve}}(T') = (n' - l' - s' + 3)/4$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. $\square$

Next we show that if T is a tree of order $n \geqslant 3$, then $\gamma_{\mathrm{ve}}(T)$ is bounded above by $n/3$. For the purpose of characterizing the trees attaining this bound, we introduce a family $\mathcal{F}$ of trees $T = T_k$ that can be obtained as follows. Let T_1 be a path P_3. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by attaching a path P_3 by joining one of its leaves to a vertex of T_k adjacent to a path P_2 or P_3.

We now prove that for every tree T of the family $\mathcal{F}$ we have $\gamma_{\mathrm{ve}}(T) = n/3$.

Lemma 4. *If $T \in \mathcal{F}$, then $\gamma_{\mathrm{ve}}(T) = n/3$.*

Proof. We use the induction on the number k of operations performed to construct the tree T. If $T = T_1 = P_3$, then $\gamma_{\mathrm{ve}}(T) = 1 = n/3$. Let k be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{F}$ constructed by $k-1$ operations. Let n' be the order of the tree T'. Let $T = T_{k+1}$ be a tree of the family $\mathcal{F}$ constructed by k operations. We have $n = n' + 3$. We denote by x the vertex to which is attached P_3. Let $v_1v_2v_3$ be the attached path. Let v_1 be adjacent to x. Let D' be any $\gamma_{\mathrm{ve}}(T')$-set. It is easy to see that $D' \cup \{v_1\}$ is a VEDS of the tree T. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') + 1$. If x is adjacent to a path P_2, then let us observe that there exists a $\gamma_{\mathrm{ve}}(T)$-set that contains the vertices v_1 and x. Let D be such a set. The set D is minimal, thus $v_2, v_3 \notin D$. It is easy to observe that $D \setminus \{v_1\}$ is a VEDS of the tree T'. If x is adjacent to a path P_3 different from $v_1v_2v_3$, say abc, then let a and x be adjacent. Let us observe that there exists a $\gamma_{\mathrm{ve}}(T)$-set that contains the vertices v_1 and a. Let D be such a set. The set D is minimal, thus $v_2, v_3 \notin D$. Let us observe that $D \setminus \{v_1\}$ is a VEDS of the tree T'. We now conclude that $\gamma_{\mathrm{ve}}(T') \leqslant \gamma_{\mathrm{ve}}(T) - 1$, and consequently, $\gamma_{\mathrm{ve}}(T) = \gamma_{\mathrm{ve}}(T') + 1$. We get $\gamma_{\mathrm{ve}}(T) = \gamma_{\mathrm{ve}}(T') + 1 = n'/3 + 1 = (n - 3)/3 + 1 = n/3$. $\square$

We now give an upper bound on the vertex–edge domination number of a tree together with the characterization of the extremal trees.

Theorem 5. *If T is a tree of order $n \geqslant 3$, then $\gamma_{\mathrm{ve}}(T) \leqslant n/3$ with equality if and only if $T \in \mathcal{F}$.*

Proof. First assume that $\mathrm{diam}(T) = 2$. Thus T is a star. If $T = P_3$, then $T = T_1 \in \mathcal{F}$. If T is a star different from P_3, then we get $n/3 > 1 = \gamma_{\mathrm{ve}}(T)$.

Now assume that $\mathrm{diam}(T) \geqslant 3$. Thus the order n of the tree T is at least four. We obtain the result by the induction on the number n. Assume that the theorem is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y be a leaf adjacent to x. Let $T' = T - y$. It is straightforward to see that any $\gamma_{\mathrm{ve}}(T')$-set is a VEDS of the tree T. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T')$. We now get $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') \leqslant n'/3 < n/3$.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If $\mathrm{diam}(T) \geqslant 4$, then let w be the parent of u. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that some child of u, say x, is a leaf. Let $T' = T - x$. Let D' be a $\gamma_{\mathrm{ve}}(T')$-set that contains no leaf. It is easy to observe that D' is a VEDS of the tree T. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T')$. We now get $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') \leqslant n'/3 < n/3$.

Now assume that among the children of u there is a support vertex other than v. Let $T' = T - T_v$. Let us observe that there exists a $\gamma_{\mathrm{ve}}(T')$-set that contains the vertex u. Let D' be such a set. It is easy to see that D' is a VEDS of the tree T. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T')$. We now get $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') \leqslant n'/3 < n/3$.

Now assume that $d_T(u) = 2$. First assume that w is adjacent to a leaf, say x. Let $T' = T - x$. Let us observe that there exists a $\gamma_{\mathrm{ve}}(T')$-set that contains the vertex u. Let D' be such a set. It is easy to see that D' is a VEDS of the tree T. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T')$. We now get $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') \leqslant n'/3 < n/3$.

Now assume that there is a child of w other than u, say x, such that the distance of w to the most distant vertex of T_x is two or three. It suffices to consider only the possibilities when T_x is a path P_2 or P_3. Let $T' = T - T_u$. We have $n' = n - 3$. Let D' be any $\gamma_{\mathrm{ve}}(T')$-set. It is easy to observe that $D' \cup \{u\}$ is a VEDS of the tree T. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') + 1$. We now get $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') + 1 \leqslant n'/3 + 1 = n/3$. If $\gamma_{\mathrm{ve}}(T) = n/3$, then obviously $\gamma_{\mathrm{ve}}(T') = n'/3$. By the inductive hypothesis we have $T' \in \mathcal{F}$. The tree T can be obtained from T' by attaching a path P_3 by joining one of its leaves to the vertex u. Thus $T \in \mathcal{F}$.

Now assume that $d_T(w) = 2$. Let $T' = T - T_w$. We have $n' = n - 4$. If $n' = 1$, then $T = P_5$. We have $\gamma_{\mathrm{ve}}(P_5) = 1 < 5/3$. If $n' = 2$, then $T = P_6$. The path P_6 can be obtained from two paths P_3 by joining them through leaves. Thus $T \in \mathcal{F}$. Now assume that $n' \geqslant 3$. Let D' be any $\gamma_{\mathrm{ve}}(T')$-set. Let us observe that $D' \cup \{u\}$ is a VEDS of the tree T. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') + 1$. We now get $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') + 1 \leqslant n'/3 + 1 = (n-1)/3 < n/3$. $\square$

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