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Buckling of Frame Braced by Linear Elastic Springs

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In the design codes and specifications, simplified formulae or diagrams are given for determining the buckling lengths of frame columns based on the ruling criterion of considering frames as sway or non sway. Due to the fact that, the code formulae utilize only local stiffness distributions, these formulae may yield in certain cases rather erroneous results. In most code formulas a case of weakly braced frames is usually not considered. In this paper the classical Winter model, developed originally for columns is applied for frame structures and compared with the results of parametric study of frame with bracing. Sensitivity analysis of critical loads of frame due to bracing stiffness variations is carried out and the method for calculation of the threshold bracing stiffness condition for frames is proposed.

Keywords: Frame, buckling, bracing, effective length

1. Introduction

Determination of the buckling (effective) lengths of frame columns is one of the most important phases of frame design. The effective length of frame columns has great influence in the design of cross–section profiles. Even small changes in effective length may cause significant changes in bearing coefficient of the cross–section. Various braces may reduce the frame columns effective length. In many practical frame design problems the buckling length is not calculated but it is assumed by the designer. In design codes [1–2] the effective lengths of frames are based on the sway classification into to groups: sway and non–sway frames. In the design codes only simplified formulae and diagrams are given for determining the buckling lengths of frame columns. In many braced framed structures, where the lateral stiffness of the bracing system is less than condition for non-sway frame (weakly braced frame), the effect of the bracing stiffness on the lateral stability of the frame is entirely neglected.

Only very limited research work on the buckling of braced multistorey frames may be found in the literature. In the research [3] it was shown that simplified formulas for determining buckling length of frame columns may yield erroneous results, especially for irregular frames. Application of code formulas on several numerical examples have shown that the erroneous results may be encountered for both sway and non–sway buckling modes. This problem occurs mainly because, only local stiffness distributions are considered in the code formulae, while the general behavior of the frame is not taken into account.

The research [4] was focused on the determination of simplified formulas of the buckling loads of dual structural systems where frames are braced by vertical columns. In this research approximated formula of the threshold rigidity for the vertical bracing column sufficient to make the frames buckle in a non–sway mode was proposed.

In this paper a study of the buckling of braced frames is presented. The frame structures with braces modeled by elastic springs are considered. A relationship between the frame critical load, effective length of the frame columns and the bracing rigidity are established. In order to obtain a safe lower limit of the buckling load of the braced frame in function of the bracing stiffness, the classical Winter model [5–7] is developed. The results are compared with the parametrical study of braced frames.

The sensitivity analysis method [8–9] is used to establish the variations of a few lowest buckling loads due to the bracing stiffness variations. The changes of buckling modes with the increase of bracing stiffness are analyzed. In worked numerical example functions describing the influence of location of a unit stiffness brace on the first variation of critical loads of the frame are found. The linear approximations of the exact relation of the buckling loads due to the variations of the bracing stiffness are determined.

The threshold rigidity of bracing is also under consideration. The threshold rigidity that is defined as bracing stiffness which is sufficient to make the frames buckle in a non–sway mode is found. A method based on the sensitivity analysis for the estimation of the threshold bracing stiffness for full bracing of the frames is proposed. The threshold bracing stiffness determined by the proposed method is compared with the stiffness found by means of parametrical study of the frames. The advantage of the proposed method is that the maximal magnitude of the frame first buckling load in the function of the bracing stiffness may be determined for the unbraced frame. Another advantage of the proposed method is that the threshold bracing condition can be found in a few approximation steps and labour–consuming parametrical stability analysis for the frame with various bracing stiffness is not necessary. It is worth noting that application of sensitivity analysis is present in many applications [11] but its application in the analysis of threshold condition of full bracing of frames is not present in literature. The above described method was efficiently used to predict the threshold bracing stiffness for the full bracing condition in the case of out–of–plane buckling of a truss [10].

2. Sensitivity analysis of critical forces due to bracing stiffness

The equilibrium equation for finite-dimensional structural system can be written as:

$$(\mathbf{K}-P\mathbf{K}_G)\,\mathbf{z}=\mathbf{0} \tag{1}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{K}_G$ is geometrical matrix, and $\mathbf{z}$ denotes the nodal displacement vector. Equation for first variation of critical load with respect to variation of vector of design variables u was derived in work [9] in the following form:

$$\delta P_{cr} = P_{cr},_{\mathbf{u}}\,\delta\mathbf{u} = \mathbf{z}^T\,(\mathbf{K},_{\mathbf{u}} - P\mathbf{K}_G,_{\mathbf{u}})\,\mathbf{z}\delta\mathbf{u} = \Lambda_{P_{cr},\mathbf{u}}\delta\mathbf{u} \tag{2}$$

Where the vector $\Lambda_{P_{cr},\mathbf{u}}$ describes the influence of the unit change of the design variable, for example bracing stiffness, on the buckling load. It allows to determinate parts of construction where possible applying of bracing may cause the largest variation of buckling load, and allows to calculate approximate buckling load due to bracing stiffness variation.

3. Parametric study of a frame

3.1. *Model description*

As the parametric study consider the two storey frame presented in Fig. 1. All beams and columns have constant cross–sections. The frame columns are loaded at their top by equal forces P. The storey height is h, and the beam span is l. The frame is supported by horizontal linear springs at each floor level. It is assumed that the stiffness of bracing at each floor is constant and that the bracing force–displacement characteristics is linear.

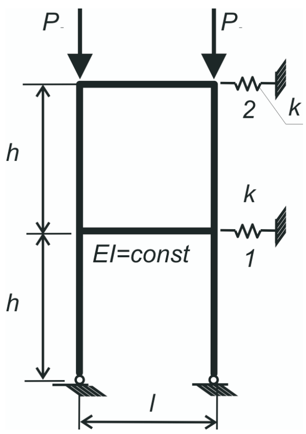

Figure 1 Frame with horizontal bracing

3.2. *Results of numerical simulation*

3.2.1. *Determination of bracing requirements for frame according Winter method*

At first the frame is modified to obtain a Winter – type model. In the beam and column joints a fictitious hinges are introduced (Fig.2). The purpose of development of Winter–type frame model is to calculate a safe lower limit of necessary stiffness of braces that allow to obtain maximal possible critical load of the frame. In the case of columns it was presented that the relation between critical force and bracing stiffness obtained for that model is lower than for similar column without hinges so the model allows to obtain a save lower limit of relation between critical load and bracing stiffness of columns [5–7]. The fictitious hinges at column and beam joints introduced in the Winter–type model of the frame allow to consider that the frame beams and columns are rigid. The total potential energy for Winter–type model of the frame consist of increase in the strain energy stored in the elastic springs and decrease in the potential energy of the external forces P:

$$V = \frac{1}{2}k\delta_1^2 + \frac{1}{2}k\delta_2^2 - 2P\left(2h - h \times \cos(\theta_1 - h\cos\theta_2)\right) \tag{3}$$

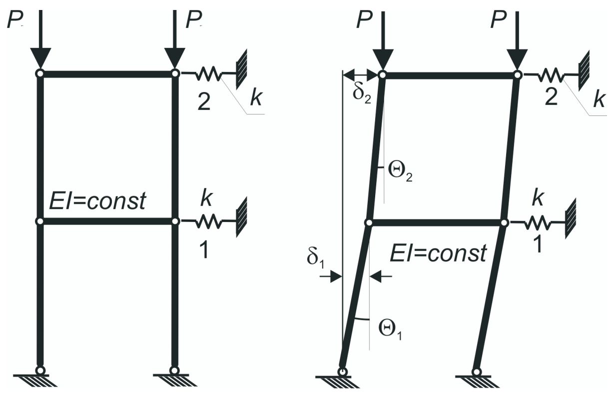

Figure 2 Winter–type model of frame with bracing and laterally distributed position of the frame

At the equilibrium position variation of total potential energy vanish, thus:

$$\begin{bmatrix} \frac{\partial V}{\partial \delta_1} \\ \frac{\partial V}{\partial \delta_2} \end{bmatrix} = \begin{bmatrix} k - \frac{4P}{h} & \frac{2P}{h} \\ \frac{2P}{h} & k - \frac{2P}{h} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4}$$

The determinant of above matrix is equal zero:

$$\det \begin{bmatrix} k - \frac{4P}{h} & \frac{2P}{h} \\ \frac{2P}{h} & k - \frac{2P}{h} \end{bmatrix} = 0 \tag{5}$$

We arrive at two critical loads that correspond to the buckling modes of the Winter–type frame. The buckling modes are shown in Fig. 3.

$$\begin{aligned} &P_{cr1} = 0.19098 \times hk \\ &P_{cr2} = 1.30902 \times hk \\ &P_{cr1} \le P_{cr0} \qquad P_{cr2} \le P_{cr0} = \frac{\pi^2 EI}{h^2} \end{aligned} \tag{6}$$

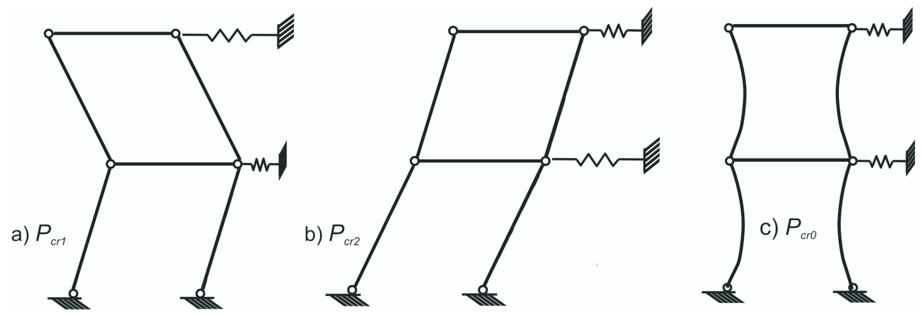

Figure 3 Buckling modes corresponding to calculated buckling loads and to the maximal buckling load for Winter–type model of frame

The Winter poly–line that describes the relation between buckling load and the bracing stiffness is constructed by means of construction lines. Assuming that the brace stiffness increases we obtain the increase of the critical forces given by Eq. (6). When that forces are equal to the buckling force for the simply supported column of length h, then the buckling mode changes to the mode shown at the Fig. 3c and the critical load is constant with further increase of the bracing stiffness. This idea allows one to find the endpoints of constructions lines of the Winter poly–line. The coordinates of the end points of the lines are (5.236, 1) and (0.764, 1) (see Fig. 4). The starting points of constructions line are set out as first two critical loads of analyzed frame without bracings and without hinges: (0.1845, 0), (0.6111, 0). The poly–line calculated for Winter–type model of frame is compared with the exact relationship between critical force and bracing stiffness parameter (Fig. 4.). Buckling loads are related to critical force of simply supported column of length $h(P_{cr0})$. The above presented analysis allows to conclude that for bracing stiffness parameter α between 0.4–1.2 the buckling load predicted by means of Winter method is greater than calculated for frame model without fictitious hinges, so for this stiffness of bracing Winter method doesn't provide a safe lower limit of critical load of frame.

3.2.2. *Sensitivity analysis of buckling load of a frame braced by elastic springs*

The sensitivity analysis of the buckling load due to the bracing stiffness variation is also carried out. According to Eq. (2) the first variation of critical load of flexural buckling due to the variation of the stiffness of braces is found.

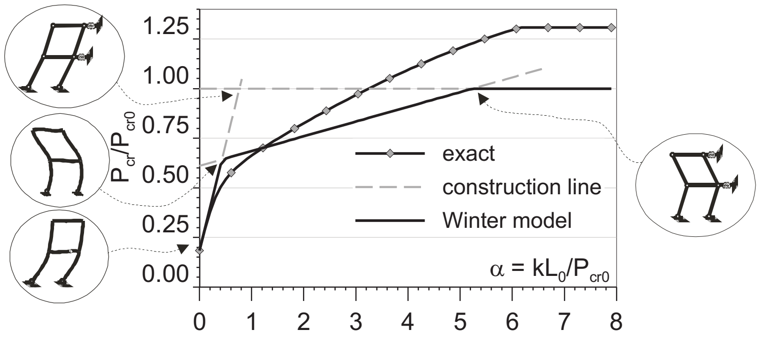

Figure 4 Relationship of relative critical load due to bracing stiffness parameter α

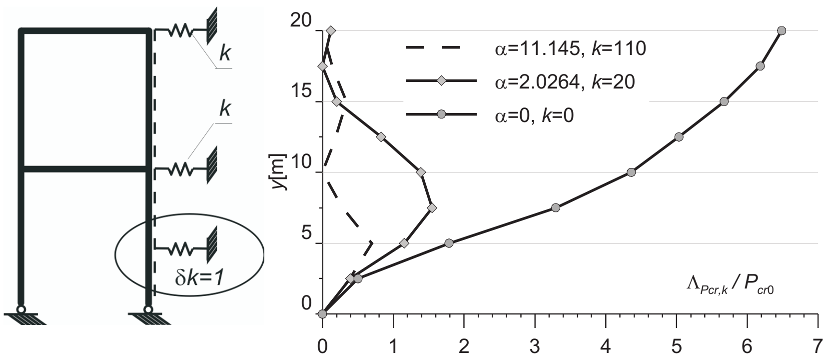

Figure 5 Influence lines of the relative variation of critical load of flexural buckling due to the location of new unit stiffness bracing for various initial bracing parameter α

The analysis was performed by means of commercial structure analysis program [12] that allows to find buckling mode and spreadsheet program [13]. The design sensitivity analysis allows to predict the buckling load variation due to location of a new unit stiffness brace along the columns length. The influence lines of the variation of the flexural buckling load due to the location of unit stiffness brace for different initial stiffness of braces in the frame are presented in (Fig.5). The influence lines are related to the critical load of simply supported column of length h, that is 98.7 kN (EI/h^2=10). It is worth noting that the lines magnitude depends on the initial bracing stiffness. In the case of frame without bracing the influence line has maximal value at the top of the frame so location of new brace near the top of the frame is most effective in increasing of buckling load. Then the same analysis is carried out for the case of frame with bracing of stiffness k=20 kN/m (α=2.0264). It is found that in this case most effective increase of buckling load may be obtained after location of brace at the coordinate y=7.5 m measured from the bottom of the frame. The third influence line is found for the frame with bracing of stiffness k=110 kN/m (α=11.145). One can conclude that increasing stiffness of bracing at the joints between beams and columns doesn't cause increase of buckling load. These line shows that further increase of buckling load would be attained when additional bracing was located in the middle of unbraced part of the columns. The parametrical analysis of relation between the first three critical buckling loads and the coefficient of bracing stiffness is presented in Fig. 6.

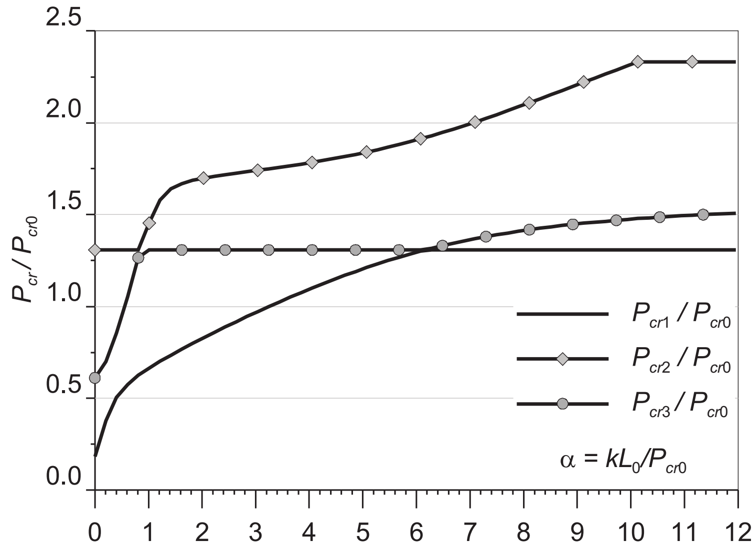

Figure 6 Relationship of relative critical loads (1–3) to coefficient of the bracing rigidity α

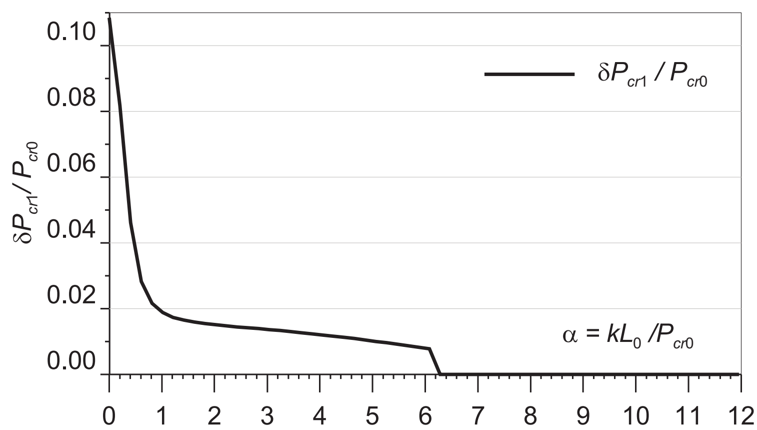

Figure 7 Relative variation of the first buckling load vs. bracing stiffness parameter α

The first variation of the second and third buckling load due to the stiffness variation of the brace located in the frame joints is also carried out. The relation between the first variation of buckling loads and the bracing stiffness are presented in Figs 7–9 for the first, second and third buckling load respectively. It is interesting to note that when the rigidity of the bracing is low (α <0.82) the third buckling load is not sensitive to the increase in the bracing stiffness, than the second buckling load becomes insensitive to the increase in the bracing stiffness at bracing stiffness parameter between $0.82 < \alpha < 6.2$, and finally when initial stiffness of bracing is greater than k > 62 kN/m ($\alpha > 6.28$), the first buckling load becomes insensitive to the chances in bracing stiffness. This stiffness is the threshold value of stiffness required for non–sway buckling mode of the frame.

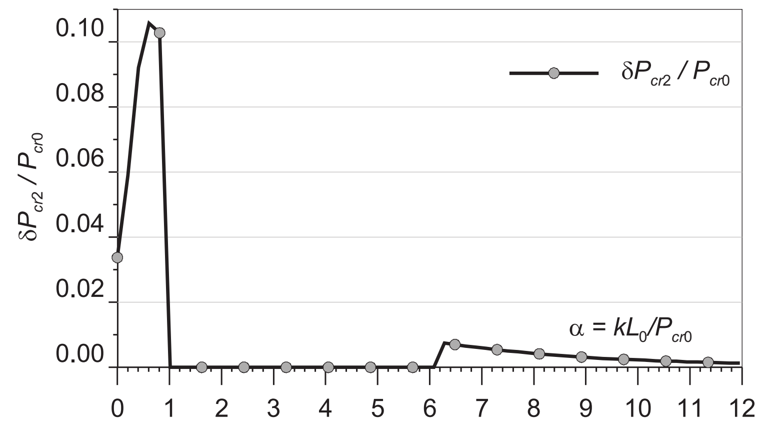

Figure 8 Relative variation of the second buckling load vs. bracing stiffness parameter α

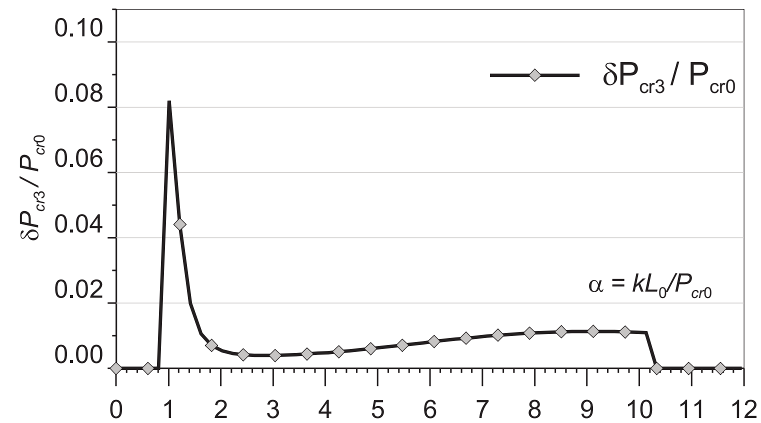

Figure 9 Relative variation of the third buckling load vs. bracing stiffness parameter α

An interesting observation is, that increasing the bracing stiffness causes an increase in the first buckling load, but the maximal critical force that may be reached is equal to that of critical buckling loads of higher order of initially an unbraced frame, that is not sensitive to the changes in bracing stiffness. So when the maximum of the first buckling load need to be determined it can be obtained by means of sensitivity analysis as that buckling load for unbraced frame that is insensitive to the location of the new unit stiffness brace. The level of the critical buckling load, that is not sensitive to the bracing stiffness variations is constant. It is also interesting to explain the insensitivities of buckling loads due to the initial stiffness of frame bracing. These relation between buckling loads and bracing stiffness may be explained as changes of the overall mode of frame buckling. At the low bracing stiffness the first mode of the frame buckling is sway mode. The third mode of buckling is non-sway buckling mode. Then with increase of the bracing stiffness non–sway buckling mode corresponds to the second buckling load and finely to the first buckling load when the buckling load for sway buckling is greater than that for the non-sway buckling mode (Fig. 10).

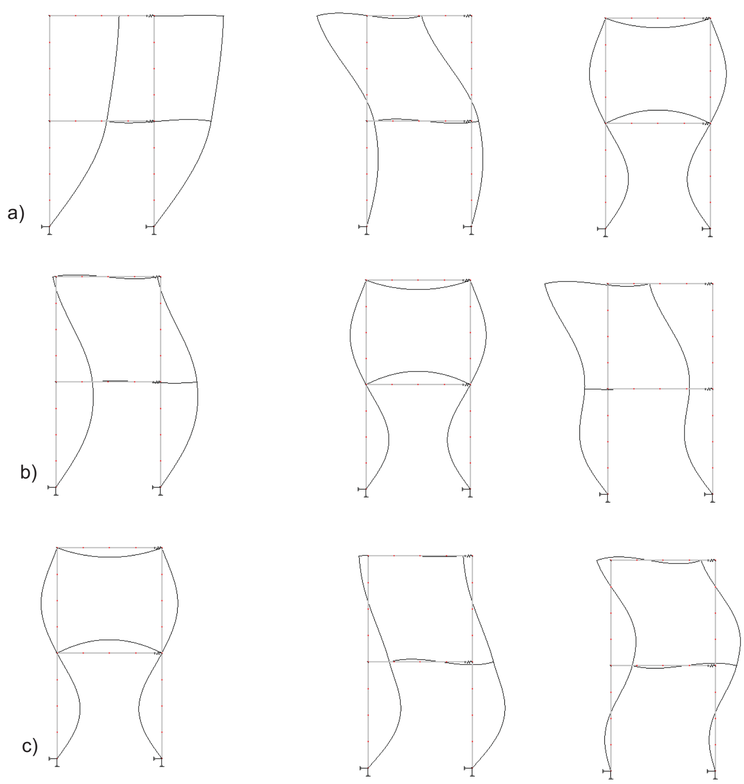

Figure 10 The three first buckling modes for a) α=0, b) α=1.1, c) α=6.5

3.2.3. *Threshold bracing condition for a frame braced by elastic springs*

Sensitivity analysis may be helpful in calculation of the full bracing condition that is defined as a threshold bracing stiffness that allow to obtain maximal critical load of the frame. These condition may be also interpreted as non-sway condition of the frame. In order to calculate the threshold bracing stiffness the following method is introduced. At the beginning of the analysis, the frame without bracing is considered. The first variation of the first few buckling loads should be calculated. Two important results should be found out from the sensitivity analysis. The first information is which buckling load is insensitive to the changes of bracing stiffness. That load value is the maximal magnitude of the first buckling load that may be reached due to an increase in bracing stiffness. The second result is the first variation of the first buckling load due to a variation of bracing stiffness. Then a linear approximation of the exact relation between the buckling load and bracing stiffness k can be found by means of following relation:

$$P_{cr1} = P_{cr1,0} + \frac{\partial P_{cr1}}{\partial k} \delta k \tag{7}$$

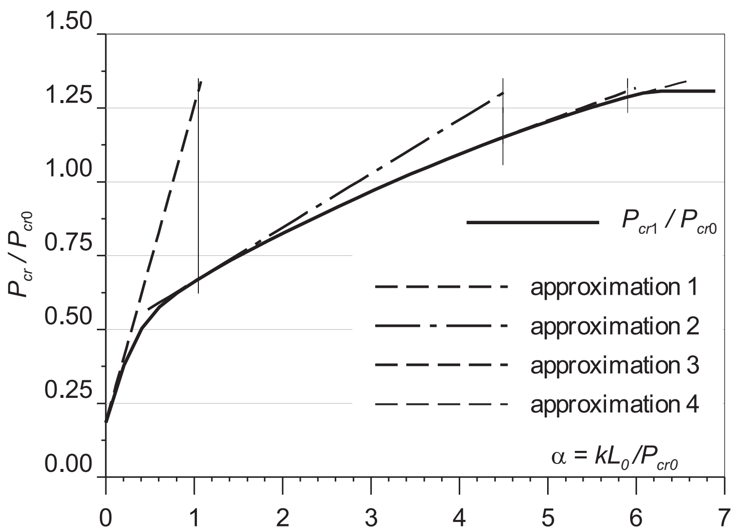

Figure 11 Relative first buckling load vs. relative bracing stiffness and its approximations constructed to find the threshold bracing stiffness condition

The first increment of bracing stiffness can be found after assuming that the required value of the first buckling load is equal to the magnitude of the higher buckling loads for an unbraced frame that is not sensitive to the bracing variation. Then the first buckling load and its first variation for new bracing stiffness need to be found and next increment of the bracing stiffness can be calculated. The calculation may be repeated until required accuracy is reached. In that way the threshold value of bracing stiffness for full bracing condition of the examined frame was determined. The approximation procedure is graphically illustrated in the Fig.11. The calculation results are presented in Table 1. The coefficient of bracing stiffness required for full bracing condition is α=6.161 at bracing stiffness k=60.803 kN/m.

Table 1 Frame with horizontal bracing

k	P_{cr1}	P_{cr3}	δP_{cr1}	δk
0.000	18.213	129.03	10.703	10.354
10.354	66.275	129.03	1.825	34.394
44.354	113.64	129.03	1.106	13.910
58.264	127.03	129.03	0.813	2.460
60.724	128.97	129.03	0.759	0.079
60.803	129.03	129.03	0.758	

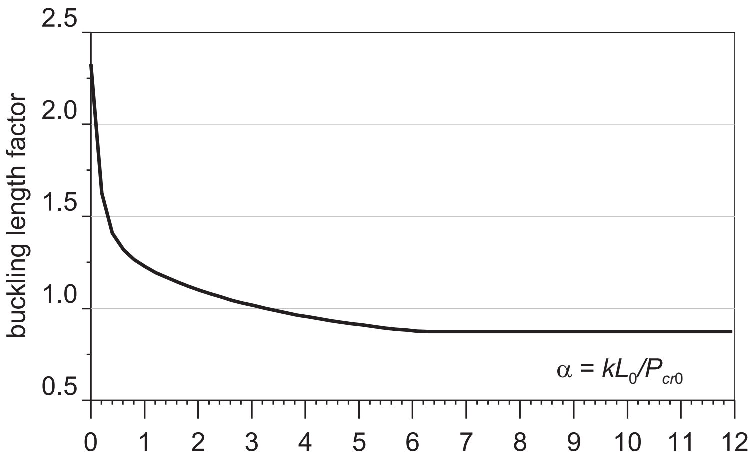

Figure 12 The effective length factor vs. bracing stiffness parameter

3.2.4. *Buckling length factor*

When the buckling load P_{cr} is determined, the buckling length of an individual column can be computed. The buckling length related to the storey heigh of the frame columns due to the bracing stiffness was also investigated (Fig. 12). The effective length factor for non–sway frame is 0.875 and in the case of sway frame is 2.328. According the design codes for weakly braced frames the effective length is calculated as for sway frames. This approach gives a save value of the buckling length but is not precise and may cause not economical design.

4. Conclusions

The effect of bracing stiffness on the critical buckling load of regular frame was investigated. The results of the numerical analysis and sensitivity analysis allow one to draw some conclusions regarding the effect of bracing stiffness on the critical buckling load.

- The critical buckling load of the frame depends on the stiffness and position of braces.
- Winter type model of the frame with fictitious hinges doesn't provide a safe lower limit of critical load of the frame for the whole range of bracing stiffness.
- The sensitivity analysis allows one to obtain the influence lines of the buckling load variation due to location of a new unit stiffness brace.
- The relationship between the buckling load and the bracing stiffness allows to conclude that it is possible to account for positive effect of bracing also for weakly braced frames, classified according code requirements as sway structures.
- Buckling length factor for columns of weakly braced frames is lower than for sway structures and this effect is neglected in simplified code formulas.
- The threshold bracing stiffness of the regular frame can be determined by means of the sensitivity analysis. The higher-order critical load, calculated for the frame without bracing, that is insensitive to the change in bracing stiffness is the maximum of the first buckling load that may be reached due to the increase in bracing stiffness. At the threshold bracing stiffness the braced frames buckle in a non-sway mode.
- In the calculation of the threshold bracing stiffness the effect of various imperfections must be considered before it can be used in practice because bracing carries also the horizontal loading that induce stresses both in the bracing member and in the frame. This load may cause decrease of the stiffness of the bracing member. When the bracing column is loaded by vertical forces, part of its lateral stiffness must be used to prevent buckling by the axial force, so the stiffness used to strengthen the lateral stability of the frame decreases. Above mentioned effects should be taken into account and verified before the proposed method for calculation of the threshold bracing condition could be used in practice.

References

[1] PN-90/B-03200 Steel structures. Design rules (Polish standard), *PKN*, Warszawa, **1990**.

[2] Eurocode 3: Design of Steel Structures. Part1.1: General rules and rules for buildings, ENV 1993–1–1, *CEN*, Brussels, **1992**.

[3] **Özmen, G. and Girgin, K.**: Buckling lengths of unbraced multi–storey frame columns, *Struct. Eng. and Mechs*, 19(1), 55–71, **2005**.

[4] **Tong, G.S. and Ji, Y.**: Buckling of frames braced by flexural bracing, *J. Construct. Steel Res.*, 63, 229–236, **2007**.

[5] **Winter, G.**: Lateral bracing of columns and beams, *Proc. ASCE*, 84, 1561–1–1561–22, **1958**.

[6] **Yura, J.A.**: Winter's bracing approach revised, *Eng. Struct.*, 18(10), 821–825, **1996**.

[7] **Trahair, N.**:Flexural–Torsional Buckling of Structures, *E&FN Spon*, London, **1993**.

[8] **Haug, E.J., Choi, K.K., and Komkov, V.**: Design sensitivity analysis of structural system, *Academic Press*, Orlando, **1986**.

[9] **Iwicki, P.**: Sensitivity analysis of critical loads of I–columns, *Arch. Civil Eng.*, 53(4), 591–606, **2007**.

[10] **Szymczak, C.**: Sensitivity analysis of thin–walled members, problems and application, *Thin–Walled Structures*, 41, 271–290, **2003**.

[11] **Iwicki, P.**: Sensitivity analysis of critical forces of trusses with side bracing, *J. Construct. Steel Res.*, **66**, 923–930, **2010**.

[12] ROBOT STRUCTURAL ANALYSIS PROFESSIONAL User Manual, Version 2010, Autodesk Inc, **2009**.

[13] EXCEL User Manual, Microsoft Corporation, **2007**.