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COMMENTS ON EXISTING ANALYTICAL SOLUTIONS TO THE WAVE-INDUCED CYCLIC RESPONSE OF A POROUS SEABED OF INFINITE THICKNESS

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ABSTRACT

This paper deals with the wave-induced cyclic response of a porous seabed (by means of oscillating parameters: pore-fluid pressure, soil displacement components, effective normal stress and shear stress components) due to a surface sinusoidal water-wave propagating over a seabed of infinite thickness. The main existing analytical solutions to the governing problem are critically discussed, pointing out their meaningful errors and doubtful items. A phase-lag phenomena is particularly studied as an immanent part of any complex-valued analytical solutions having a cyclic nature.

Keywords: porous seabed, infinite seabed thickness, cyclic response, soil saturation, sinusoidal progressive wave, phase-lag phenomenon, analytical solutions

INTRODUCTION

The wave-induced response of a porous seabed loaded by a progressive sinusoidal surface water-wave is still an interesting subject in many coastal engineering problems. In order to solve them, it is necessary to treat the problem either numerically, especially when a certain engineering structure is involved, or analytically by using one of the existing theories and their solutions to the wave-induced cyclic seabed response, particularly when a pure case of the seabed without presence of any structure founded on or embedded in seabed sediments is concerned. These analytical solutions can be further used in many scientific and engineering analyses of the wave-induced seabed instability due to wave-induced momentary liquefaction or/and wave-induced residual liquefaction of the upper part of seabed as a result of continuous build-up of the wave-induced pore-fluid pressure within seabed sediments. But above all, anyone needs to be reminded of the fact that analytical solutions serve very often as an important reference for validation of appropriate numerical solutions.

Assuming only the pore-fluid to be compressible, Moshagen and Tørum [12] presented an analytical solution for the pore-fluid pressure obtained for both infinite and finite thicknesses of the seabed. Following research works pertained to a more advanced case where both phases of the seabed (i.e. pore-fluid and soil skeleton) are considered to be compressible and, thereby, the relative compressibility of the two-phase seabed medium started to become of great importance from the practical point of view. This type of analytical solutions is based on Biot's consolidation theory, Hooke's law (the soil has linear, reversible, isotropic and non-retarded mechanical properties) and Darcy's law for the pore-fluid flow through a porous medium. The obtained solutions are given in terms of six (in case of the two-dimensional space) complex-valued wave-induced and cyclically varying parameters (pore-fluid pressure, two soil displacement components, two soil effective normal stress components and one shear stress component), or at least some of them, assuming simultaneously plain strain and partly saturated soil conditions.

Basically, an infinite thickness of the seabed was studied by Yamamoto et al. [19], Madsen [6] and Okusa [13]. Yamamoto et al. [19] assumed a hydraulically isotropic seabed and presented the final solution only with respect to the wave-induced pore-fluid pressure and soil displacement components. However, Yamamoto et al. [19] presented additionally two simplified approximate solutions for: (a) soils completely saturated with seawater and for most soils except for dense sand, and (b) partially saturated dense sand and sandstones. Only case (a) was associated with a set of final equations obtained for all six wave-induced parameters. Madsen [6], treating the seabed as a hydraulically anisotropic medium, derived the governing partial differential equation of the 6^{th} order and obtained the final solution in terms of all the above mentioned wave-induced parameters. Madsen [6] considered a special simplified case of a fully saturated, isotropic, dense soil and presented appropriate equations only for the pore-fluid pressure and soil stress components. Okusa [13], similarly to Yamamoto et al. [19], assumed the seabed to act as a hydraulically isotropic medium. After solving the governing partial differential equation of the 4^{th} order, Okusa [13] obtained equations for the wave-induced pore-fluid pressure, effective normal stress and shear stress components within the soil skeleton. The general solution (i.e. before applying boundary conditions) was presented in two forms: exact and simplified (the approximation was obtained after identification of negligibly small terms). However, the particular solution (i.e. after applying boundary conditions), presented by Okusa [13], was based only on his simplified general solution. Similarly to his forerunners, Okusa [13] considered a special case of fully saturated soil conditions and presented adequate final equations. Yamamoto et al. [19] and Okusa [13] were kind to discuss the question of phase-lag phenomenon, presenting vertical distributions of this parameter with respect to pore-fluid pressure oscillations. Yamamoto et al. [19] made a comparison with some experimental data and values calculated after Moshagen and Tørum [12], whereas Okusa [13] illustrated graphically two different computational cases and presented, as a bonus, an equation for the phase-lag of pore-fluid pressure oscillations derived from his simplified approximate solution.

In the next step, the "infinite thickness" two-phase compressible seabed model was adopted and extended into a more general "finite thickness" model together with its analytical solutions derived by many researchers, among others: Richwien and Magda [15], Magda [7], Hsu and Jeng [2], Jeng and Hsu [5] and Jeng [3, 4]. Very often, based on their "finite thickness" analytical solutions, the authors formulated also, as a special simpler case, equations reflecting the conditions of infinite thickness of a porous and elastically deformable seabed. Besides that, it is also worth noting that Mei and Foda [9] elaborated a very sophisticated "boundary layer theory". Its easily applicable analytical solution was found to be a useful tool in many seabed response analyses, where, among others, the problem of extrication of large objects from the seabed is particularly studied [1]. This very interesting engineering challenge is still of great importance, as documented in recent works by Michalski [10, 11].

A solid mechanics sign convention for strains and stresses was usually applied in the above mentioned analytical solutions. Only Okusa [13] hold entirely with a soil mechanics sign convention, whereas Madsen [6] presented a kind of "hybrid method". The consistency of mathematical formulations and analytical solutions to the governing problem, based on different sign conventions for strains and stresses, was thoroughly studied by Magda [8].

Using the above mentioned first group of "infinite thickness" analytical solutions in practice, some important drawbacks have been found by the Author of the present paper. Therefore, after a brief description of mathematical models of the wave-induced cyclic seabed response, a critical assessment of some selected "infinite thickness" analytical solutions will be presented in the following, pointing out their weaknesses and mistakes.

All of the errors in the analytical methods under consideration have been detected personally by the Author of the present paper. The entire mathematical procedures, associated with the analytical solutions presented originally by Moshagen and Tørum [12], Yamamoto et al. [19] and Okusa [13], have been repeated from soup to nuts by the Author. The results of all derivative procedures performed by the Author have been compared with the published matter and the differences have been indicated and depicted in details. Additional computations have been executed using the questionable solutions and the corrected equations. Moreover, the computational results obtained from the corrected equations have been collated with appropriate results computed according to the originally perfect analytical solution published by Madsen [6]. All the computations have been performed by the Author of the present paper using his own computer programs written in Fortran.

A CRITICAL REVIEW OF THE EXISTING ANALYTICAL SOLUTIONS

Basic definition sketch of the two-dimensional governing problem is illustrated in Fig. 1. A porous (permeable) and elastically deformable seabed is loaded by a progressive sinusoidal surface water-wave travelling above it. This causes wave-induced cyclic variations of six seabed response parameters, i.e.: pore-fluid pressure, two soil displacement components, two effective normal stress components and one shear stress component.

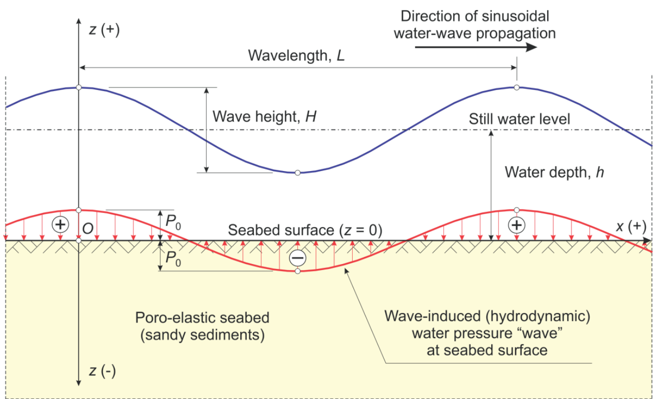

Fig. 1. Definition sketch for analysis of the wave-induced cyclic response of a poro-elastic sandy seabed of infinite thickness

A porous seabed is considered as a two-phase medium, consisting of the soil skeleton and the pore-fluid. Taking into account solutions most interesting from the practical point of view, at least one of the two component phases must be assumed to be compressible. Therefore, the so-called potential model, developed by Putnam [14], where both phases are treated as incompressible media and the problem is governed by the Laplace equation, is out of scope of the present paper. And thus, the following two models of the wave-induced seabed response are applicable, namely:

- diffusion model (governed by the continuity equation in the form of Fick's second law diffusion partial differential equation; only the pore-fluid is assumed to be compressible and the soil skeleton does not obey elastic deformations),
- storage model (governed by the coupled equations of static force and moment equilibrium together with the continuity equation in the form of storage partial differential equation proposed by Verruijt [18]; both the pore-fluid and the soil skeleton are treated as compressible media).

Different analytical solutions to the governing problem, according to the above mentioned mathematical models of the wave-induced seabed response, were treated analytically and numerically by many researchers. However, this paper deals only with some milestone analytical solutions listed in Tab. 1. The present selection of the analytical solutions published in the scientific literature was a consequence of their high citation level. And thus, according to the "Google Scholar" web search engine data from the 14th of March, 2023, the papers by: Moshagen and Tørum [12], Yamamoto et al. [19], Madsen [6], Okusa [13], Hsu and Jeng [2], Jeng and Hsu [5], Jeng [3, 4] are associated with the following number of citations: 133, 877, 714, 393, 390, 162, 208 and 66, respectively. Additionally, it has to be emphasized that the analytical solutions by Yamamoto et al. [19], Madsen [6] and Okusa [13] are used frequently by other researchers, e.g. Sumer [16], Sumer and Fredsøe [17], Jeng [3, 4], in their works and many comparative analyses.

Tab. 1. Chronological list of some milestone theories and their analytical solutions to the wave-induced cyclic response of a porous seabed of infinite thickness

Author/Authors (Year of publication)	Soil skeleton				Pore-fluid
	Compressibility		Hydraulic anisotropy		Compressibility
	No	Yes	No	Yes	Yes
Moshagen and Tørum [12]	x			x	x
Yamamoto et al. [19]		x	x		x
Madsen [6]		x		x	x
Okusa [13]		x	x		x
Hsu and Jeng [2]		x		x	x
Jeng and Hsu [5]		x		x	x
Jeng [3, 4]		x		x	x

It is very characteristic that all the wave-induced seabed response analytical solutions were achieved in the form of complex functions.

SOLUTION BY MOSHAGEN AND TØRUM [12]

An analytical solution to the diffusion problem, describing the wave-induced pore-fluid pressure response in a rigid and porous seabed under the assumption of pore-fluid compressibility, was obtained by Moshagen and Tørum [12] who, using suitable boundary conditions, presented two types of their analytical solution. The first one, more general, is the solution for a finite thickness of a porous seabed layer – the so-called "finite thickness solution". Afterwards, applying $d \to \infty$ (where d denotes the thickness of a porous seabed layer), a special case thereof was also obtained as the "infinite thickness solution"

$$\tilde{p} = P_0 \exp\left(\mu \sqrt{\frac{K_x}{K_z}} z\right) \exp[i(kx - \omega t)] \quad (1)$$

where: $\tilde{p}$ = wave-induced pore-fluid pressure (complex-valued) [kPa], P_0 = amplitude of the hydrodynamic pressure at the seabed surface ($z = 0$) [kPa], μ = parameter (complex-valued) [1/m], K_x and K_z = coefficients of soil permeability in horizontal and vertical directions, respectively [m/s], k = wave number ($k = 2\pi/L$) [1/m], L = wavelength [m], ω = wave angular frequency ($\omega = 2\pi/T$) [rad/s], T = wave period [s], t = time [s], x and z = horizontal and vertical coordinates of the two-dimensional Cartesian coordinate system Oxz, respectively [m], i = imaginary unit ($i = \sqrt{-1}$).

The complex-valued parameter μ can be presented using the following well-known trigonometric form of a complex number

$$\mu = |\mu|(\cos\varphi + i \sin\varphi) \quad (2)$$

(where: $|\mu|$ = absolute value (or modulus or magnitude) of μ [1/m], $\varphi \equiv \arg(\mu)$ = the argument (or phase) of μ [rad].

Moshagen and Tørum [12] gave the following formulas for the absolute value and the argument of complex-valued parameter μ, respectively (see the original Eqs. (15) and (16) in [12], p. 53):

$$|\mu| = \left[k^4 + \left(\frac{\omega n \beta \gamma_w}{K_x}\right)^2\right]^{1/4} \quad (3a)$$

$$\varphi \equiv \arg(\mu) = \frac{1}{2}\arctan\left(\frac{\omega n \beta \gamma_w}{K_x k^2}\right) \quad (3b)$$

where, additionally: n = porosity of soil [–], β = compressibility of pore-fluid [m^2/kN], γ_w = unit weight of pore-fluid (seawater) [kN/m^3].

When using Eq. (1) in an analysis of the hydrodynamic uplift force acting on a submarine pipeline buried in seabed sediments, the Author had noticed some unexpected problems with the phase-lag of wave-induced pore-fluid pressure oscillations. This was the reason why the Author went through the entire derivation procedure in order to find the reason thereof. The comparison of the Author's derivation with the matter printed in [12] has led to the conclusion that the analytical solution by Moshagen and Tørum [12] is burdened with an error which can be easily detected and proved.

And thus, by introducing the "infinite thickness solution" (Eq. (1)) into the governing partial differential equation of the diffusion type (see the original Eq. (8) in [12], p. 51) and performing some additional mathematical operations, one should be able to reach the following expression

$$\mu^2 = k^2 - i\,\frac{\omega n \beta \gamma_w}{K_x} \qquad (4)$$

Next, by raising both sides of Eq. (2) to the power of 2, and keeping in mind the double-angle formulas, one has

$$\mu^2 = |\mu|^2 \cos 2\,\varphi + i|\mu|^2 \sin 2\,\varphi \qquad (5)$$

And now, by analogy between Eq. (4) and Eq. (5), and performing some simple mathematical operations, it becomes possible to prove that the absolute value of μ is described by exactly the same equation as given in [12] (see Eq. (3a)) but the equation for the argument of μ should be rather given as follows

$$\varphi \equiv \arg\,(\mu) = \frac{1}{2}\arctan\left(-\frac{\omega n \beta \gamma_w}{K_x k^2}\right) = \qquad (6a)$$

$$= -\frac{1}{2}\arctan\left(\frac{\omega n \beta \gamma_w}{K_x k^2}\right) \qquad (6b)$$

After comparing Eqs. (3b) and (6b) it is worth noting that the solution for the argument $\phi \equiv \arg(\mu)$ of complex-valued parameter μ is wrongly given in the original Eq. (16) by Moshagen and Tørum [12], p. 53, where the argument of arcus tangent is positively signed, whereas a correct form of the equation should contain a negative value of the argument of inverse trigonometric function, as presented in the above derived Eq. (6a). As a consequence of this fundamental error, sinusoidal oscillations of the pore-fluid pressure in the seabed precede sinusoidal oscillations of the surface water-wave and the hydrodynamic pressure "wave" at the seabed surface ($z = 0$), which stays in contradiction to the phase-lag definition in case of the progressive surface water-wave moving above the seabed. As indicated by many other analytical solutions, e.g. [6, 13, 19], the pore-fluid pressure oscillations within the seabed (at least in its uppermost zone) must always follow the hydrodynamic pressure "wave" irrespectively of the direction of the surface water-wave propagation, as illustrated in Fig. 2.

Apart from Eq. (2), the complex-valued parameter μ can be also represented by the most general defining form

$$\mu = \Re\{\mu\} + i\Im\{\mu\} \qquad (7)$$

where: $\Re\{\mu\}$ and $\Im\{\mu\}$ = real and imaginary parts of μ, respectively [1/m]. After introducing this notation to Eq. (1), assuming hydraulically isotropic soil conditions ($K_x = K_z$), and performing some simple mathematical manipulations, the "infinite thickness solution" by Moshagen and Tørum [12] can be presented in the following form

$$\tilde{p} = P_0 \exp(\Re\{\mu\}z)\,\exp[i(kx - \omega t + \Im\{\mu\}z)] \qquad (8a)$$

the real part of which can be written as (see also the original Eq. (17) in [12], p.53)

$$\Re\{\tilde{p}\} = P_0 \exp(\Re\{\mu\}z)\cos(kx - \omega t + \Im\{\mu\}z) \qquad (8b)$$

On the other hand, still assuming the same traditional direction of surface water-wave propagation as done by Moshagen and Tørum [12] (see Figs. 1 and 2(a)), the wave-induced pore-fluid pressure oscillations within a porous seabed can be represented by the following most general equations:

$$\tilde{p} = P(z)\cos\big[kx - (\omega t - \delta_p)\big] = \qquad (9a)$$

$$= P(z)\cos\big(kx - \omega t + \delta_p\big) \qquad (9b)$$

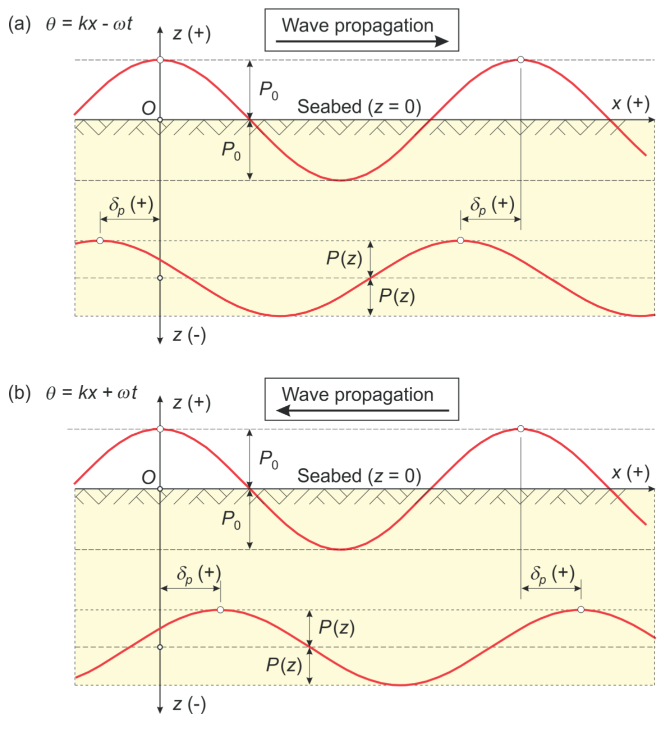

Fig. 2. Graphical presentation of the phase-lag, δ_p, in pore-fluid pressure "travelling" sinusoidal oscillations within the seabed for two opposite directions of surface water-wave propagation

where: $\tilde{p}$ = wave-induced momentary pore-fluid pressure (real-valued) [kPa], $P(z)$ = amplitude of the wave-induced cyclic pore-fluid pressure oscillations [kPa], δ_p = phase-lag (or phase-shift or phase-delay) of the wave-induced cyclic pore-fluid pressure oscillations [rad]. In general, the phase-lag, δ_p, is defined to be positively signed when a "wave" of any oscillating parameter (here the pore-fluid pressure) within the seabed follow the surface water-wave and, as a consequence, the hydrodynamic pore-fluid pressure "wave" at the seabed surface ($z = 0$), as presented in Fig. 2(a).

Because the phase-lag phenomenon is very often misunderstood, a general definition of the phase-lag in pore-fluid pressure sinusoidal oscillations, considered for the case of two opposing directions of surface water-wave propagation, is clearly illustrated in Fig. 2 in a two-dimensional Cartesian coordinate system Oxz. By analogy with Eqs. (9a) and (9b), the application of opposite direction of the surface water-wave propagation (i.e. according to the negative Ox-axis direction) would require the following general equation (see Fig. 2(b)):

$$\tilde{p} = P(z)\cos\left[kx + \left(\omega t - \delta_p\right)\right] = \quad \textbf{(10a)}$$

$$= P(z)\cos\left(kx + \omega t - \delta_p\right) \quad \textbf{(10b)}$$

A short comparison of Eqs. (8b) and (9b) leads to the conclusion that the "infinite thickness solution", obtained by Moshagen and Tørum [12], is associated by the phase-lag of wave-induced pore-fluid pressure increasing linearly with depth ($\delta_p = \Im\{\mu\}z$) which seems to be a rather rough approximation of the real behaviour of wave-induced pore-fluid pressure cyclic oscillations within the seabed.

In order to illustrate the meaning of the error detected, comparative computations have been performed, using the following set of input-data chosen arbitrary in the present paper: wave period $T = 9.78$ s, water depth $h = 10$ m, wavelength $L = 90.04$ m (sinusoidal wave theory), coefficient of soil permeability $K_x = K_z = 10^{-4}$ m/s (fine sand), compressibility of pore-water (seawater) $\beta_w = 5.0\times10^{-7}$ m²/kN, degree of soil saturation $S_r = 1.0$ (fully saturated soil conditions; $\beta = \beta_w$), unit weight of seawater $\gamma_w = 10.06$ kN/m³ (for mean World Ocean salinity equal to 35‰) and porosity of soil $n = 0.4$. The results of computations of the phase-lag of wave-induced pore-fluid pressure are illustrated in Fig. 3.

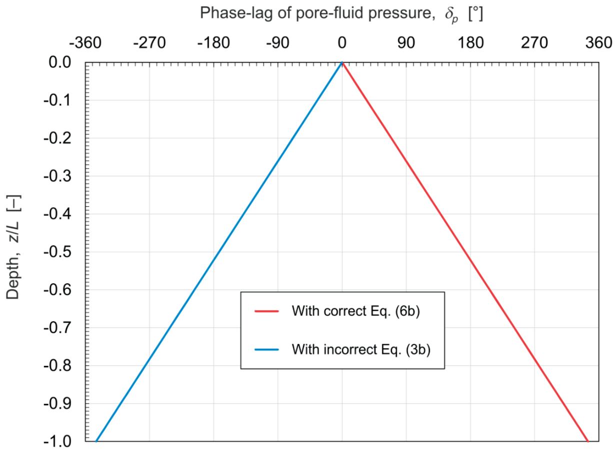

Fig. 3. Vertical distributions of the phase-lag of wave-induced pore-fluid pressure oscillations computed with the incorrect Eq. (3b) by Moshagen and Tørum [12] and the Author's correct Eq. (6b)

Using the erroneous Eq. (3b), one obtains:

$$\mu = (9{,}6631601\text{E}{-}02,\ -6{,}6852028\text{E}{-}02)\ \ [\text{m}^{-1}] \quad \textbf{(11a)}$$

$$\Im\{\mu\} = -6.685\times10^{-2}\ \ \text{m}^{-1} \quad \textbf{(11b)}$$

whereas, taking the correct Eq. (6b), one should calculate:

$$\mu = (9{,}6631601\text{E}{-}02,\ 6{,}6852028\text{E}{-}02)\ \ [\text{m}^{-1}] \quad \textbf{(12a)}$$

$$\Im\{\mu\} = 6.685\times10^{-2}\ \ \text{m}^{-1} \quad \textbf{(12b)}$$

The phase-lag values obtained using the correct Eq. (6b) are exactly additive inverses of the appropriate results calculated using the incorrect Eq. (3b) (see Fig. 3).

SOLUTION BY YAMAMOTO ET AL. [19]

Yamamoto et al. [19] were the first researchers (paper published on July 12, 1978) who presented a theory of the wave-induced cyclic seabed response together with an "infinite thickness" analytical solution, assuming compressibility of both of the two-phase seabed components (i.e. soil skeleton and pore-fluid; see Tab. 1). The solution was presented in the form of complex-valued wave-induced pore-fluid pressure and soil displacement components. Assuming a special simplified case of fully saturated and dense soils, Yamamoto et al. [19] showed also the final equations obtained this time with respect to all six wave-induced parameters.

A sign convention applied by Yamamoto et al. [19] for the wave-induced soil stress components, $\tilde{\sigma}_x'$, $\tilde{\sigma}_z'$ and $\tilde{\tau}_{xz}$ [the original Eqs. (2.7)-(2.9) in [19], p. 196, respectively], is typical for solid mechanics which is opposite to traditionally applied soil mechanics sign convention [8]. Fortunately, Yamamoto et al. [19] were consequent and applied the same solid mechanics sign convention also in case of two equations of equilibrium and the storage equation (the original Eqs. (2.4)-(2.5) and Eq. (2.1) in [19], p. 195, respectively).

Exclusively from all the Authors considered in the present paper, only Yamamoto et al. [19] took the direction of positive *Oz*-axis vertically downwards from the seabed surface. One has to be aware of some consequences thereof when comparing to the other wave-induced seabed response theories and their solutions. Firstly, vertical coordinates of points within the seabed introduced into a computational procedure obviously must not be negative ($z \geq 0$). Secondly, one has also to remember that such assumption will influence the sign of the wave-induced vertical displacement of soil skeleton as well as effective vertical normal stress and the shear stress components within the soil matrix, as discussed by Magda [8].

A boundary condition for the hydrodynamic bottom pressure oscillations is usually taken in the following form

$$\tilde{p} = P_0 \exp[i(kx - \omega t)] \tag{13}$$

where the minus sign in the term $-\omega t$ denotes the water-wave movement from left to right (i.e. along the positive *Ox*-axis direction). Yamamoto et al. [19] decided to introduce an opposite direction of water-wave propagation (i.e. adequately to the negative *Ox*-axis direction); this condition is represented by the positive sign in the term $+\omega t$ within the surface water-wave phase-angle expression $\Theta = kx + \omega t$. If one does not take it into account, the sign of phase-lag can be erroneously read out. The sequence of the wave-induced oscillating soil shear stress can also be changed. For instance, assuming $t = T/4$, the wave-induced shear stress is usually positive for the right-side directed water-wave (of course, after assuming the solid mechanics sign convention and $z \geq 0$ in the seabed) but the computations according to [19] will bring a negative value of the soil shear stress component, as discussed in [8].

Performing an extended comparative analyses of different analytical solutions published in the literature, the Author has found unexpected differences between the solutions by Yamamoto et al. [19] and Madsen [6] (the quality of Madsen's [6] solution had been already proved by the Author through repeating the entire Madsen's [6] analytical derivation procedure). This was the reason why the Author decided to get down to reproducing the whole Yamamoto's [19] analytical derivation procedure which finally has led to the conclusion that the general solution (i.e. before applying the boundary conditions) for the vertical displacement of soil skeleton is wrong, as given by (see the original Eq. (3.8b) in [19], p. 198; please note that the original notation is kept)

$$W = i\left[a_2 + \frac{a_4}{\lambda}\frac{1+(3-4\nu)m}{1+m}\right]\exp(-\lambda z) + \\ + i a_4 z \exp(-\lambda z) + i\frac{\lambda}{\lambda'} a_6 \exp(-\lambda' z) \tag{14}$$

and the correct equation should take the following form

$$W = i\left[a_2 + \frac{a_4}{\lambda}\frac{1+(3-4\nu)m}{1+m}\right]\exp(-\lambda z) + \\ + i a_4 z \exp(-\lambda z) + i\frac{\lambda'}{\lambda} a_6 \exp(-\lambda' z) \tag{15}$$

The above two equations differ in the form of their third terms which are correlated by the following expression

$$\frac{\lambda'}{\lambda} = \frac{\lambda}{\lambda'}\left(1 - \frac{\omega'}{i\lambda^2}\right) \tag{16}$$

After applying the boundary conditions to the general solution, Yamamoto et al. [19] presented a particular solution for both components of the soil skeleton displacement and the pore-fluid pressure. Unfortunately, based on the above mentioned Author's derivation procedure applied to Yamamoto's [19] way of solution, it has been found that the particular solution for the vertical displacement of soil skeleton is wrong as well, as given consequently by (see the original Eq. (3.13b) in [19], p. 198; please note that the original notation is kept)

$$w = \left\{\left[1 + m\frac{1+(1-2\nu)(-\lambda''+i\omega'')}{-\lambda''+i(1+m)\omega''}\right]\exp(-\lambda z) + \\ - \left[1 - \frac{m\lambda''}{-\lambda''+i(1+m)\omega''}\right]\lambda z\exp(-\lambda z) + \\ - \frac{m(1+\lambda'')}{-\lambda''+i(1+m)\omega''}\exp(-\lambda' z)\right\}\frac{p_0}{2\lambda G}\exp[i(kx+\omega t)] \tag{17}$$

whereas the correct equation should be presented as follows

$$w = \left\{\left[1 + m\frac{1+(1-2\nu)(-\lambda''+i\omega'')}{-\lambda''+i(1+m)\omega''}\right]\exp(-\lambda z) + \\ + \left[1 - \frac{m\lambda''}{-\lambda''+i(1+m)\omega''}\right]\lambda z\exp(-\lambda z) + \\ - \frac{m(1+\lambda'')}{-\lambda''+i(1+m)\omega''}\exp(-\lambda' z)\right\}\frac{p_0}{2\lambda G}\exp[i(kx+\omega t)] \tag{18}$$

The difference between Eqs. (17) and (18) can be seen by comparing their second terms in curly brackets; the second term in the wrong Yamamoto's [19] Eq. (17) is negatively signed whereas the second term in the correct Eq. (18), derived solely by the Author, must be signed positively.

The above mentioned two errors could not be discovered by Yamamoto et al. [19] probably because they presented results of computations only with respect to the pore-fluid pressure (both amplitude and phase-lag of pore-fluid pressure oscillations). However, if one wants to derive the wave-induced effective normal stress components, $\tilde{\sigma}'_x$ and $\tilde{\sigma}'_z$, and the wave-induced shear stress component, $\tilde{\tau}_{xz}$, based on the wrong particular solution published by Yamamoto et al. [19], the results obtained will be consequently burdened with an error.

In order to illustrate the meaning of the second error discovered, illustrative and comparative computations were performed, using the formerly presented set of input-data together with the following additional parameters and their values: shear modulus of soil $G = 10^4$ kPa, degree of soil saturation $S_r = 1.0, 0.999, 0.99, 0.9$, and Poisson's ratio of soil $\nu = 0.33$. Vertical distributions of the relative (and dimensionless) amplitude of wave-induced vertical displacement of soil, $\bar{u}_z = \tilde{u}_z/[P_0/(2Gk)] = 2Gk\tilde{u}_z/P_0$, are shown in Fig. 4 for different soil saturation conditions modelled by the degree of soil saturation. The pore-fluid compressibility was computed using the following formula proposed by Verruijt [18]

$$\beta = \beta_w + \frac{1 - S_r}{P_h} \quad \text{for} \quad S_r \geq 0.85 \tag{19}$$

where: β = compressibility of pore-fluid [m²/kN], β_w = compressibility of pure pore-water (seawater without air-bubles content) [m²/kN], S_r = degree of soil saturation [–], $P_h = p_{at} + p_h$ = absolute hydrostatic pressure at the computational point in the seabed (usually at the seabed surface) [kPa], p_{at} = atmospheric pressure (p_{at} = 101,325 kPa), p_h = hydrostatic pressure at the computational point in the seabed (usually at the seabed surface for for which $p_h = \gamma_w h$) [kPa], γ_w = unit weight of seawater [kN/m³], h = water depth [m].

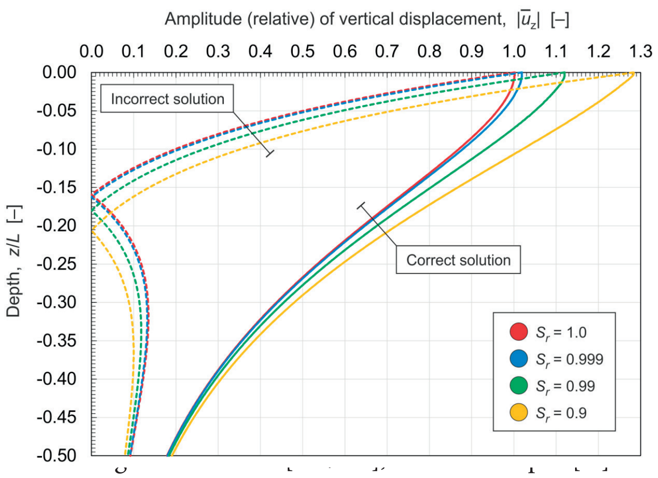

Fig. 4. Vertical distributions of the amplitude of wave-induced vertical displacement of soil skeleton; solid lines — Author's corrected solution given in Eq. (18), dashed lines — incorrect original solution by Yamamoto et al. [19] given in Eq. (17)

The difference between the wrong solution by Yamamoto et al. [19] (see Eq. (17)) and the correct solution obtained by the Author of the present paper (see Eq. (18)) is obvious and very meaningful. The quality of the Author's computations has been proved by performing additional computations based on Madsen's [6] analytical solution where a full agreement has been achieved.

SOLUTION BY MADSEN [6]

Madsen [6], as the second scientist (paper published in December 1978), presented a theory of the wave-induced cyclic seabed response and derived his own "infinite thickness" analytical solution, assuming the two-phase seabed medium to be compressible (see Tab. 1). Madsen [6] was also the first who published resultant equations of the particular solution for a full set of six wave-induced parameters: pore-fluid pressure, two soil displacement components, two effective normal stress components and one shear stress component.

According to the best knowledge of the Author of present paper, Madsen's [6] theory and his analytical solutions for the above mentioned wave-induced parameters are perfect (clearly presented and completely free of errors) in relation with all other similar solutions considered and discussed in the present paper. Although the final equations are perfect, it is interesting to emphasize that Madsen [6] used a kind of "hybrid method" when formulating the governing problem; the solid mechanics sign notation for strains was used whereas the stress-strain relationships were obtained by artificial "attaching" a negative sign (–) to the right-hand sides of the stress-strain equations, allowing thereby switching into the soil mechanics sign convention, as described in [3, 8].

SOLUTION BY OKUSA [13]

Okusa [13], followed the work by Yamamoto et al. [19] and Madsen [6] and, assuming hydraulically isotropic seabed consisting of two compressible phases (see Tab. 1), presented his own "infinite thickness" analytical solution obtained only for the wave-induced pore-fluid pressure and effective normal and shear stress components.

Jeng [3, 4] noted that the considerations presented by Okusa [13] had been based on plane stress conditions (see in [3], p. 11, and repeatedly in [4], p. 8). It has to be stressed that this statement is completely wrong; a closer study of the paper by Okusa [13] reveals that the volumetric strain of soil skeleton, appearing in the equation of mass conservation of fluid (so-called the storage equation), is given under plain strain conditions (see the original Eq. (2) in [13], p. 519); exactly the same situation is with the compatibility equation (see the original Eq. (8) in [13], p. 520) which reflects the plain strain conditions evidently.

The equations for the real-valued wave-induced pore-fluid pressure and the phase-lag of pore-fluid pressure oscillations (see the original Eq. (48) in [13], p. 525), obtained by Okusa [13] as a simplified approximate solution (due to abandoning the negligibly small terms), are as follows (please note that the original notation is kept):

$$U = \left\{ \left[\left(1 - B_1'\right) \exp(\kappa_1 z) \sin(\kappa_2 z) \right]^2 + \right.$$
$$\left. + \left[B_1' \exp(az) + \left(1 - B_1'\right) \exp(\kappa_1 z) \cos(\kappa_2 z) \right]^2 \right\}^{\frac{1}{2}} \times \tag{20a}$$
$$\times \cos\left(ax - \omega t - \delta\right)$$

$$\tan\delta = \frac{(1 - B_1') \exp(\kappa_1 z) \sin(\kappa_2 z)}{B_1' \exp(az) + (1 - B_1') \exp(\kappa_1 z) \cos(\kappa_2 z)} \tag{20b}$$

Based on the computational analysis performed by the Author, it must be stressed that the momentary pore-fluid pressure results, obtained by using Eq. (20a) together with Eq. (20b), are correct. However, these equations separately must be evaluated as incorrect from the formal analytical point of view. The term cos $(ax - \omega t - \delta)$ informs that the pore-fluid pressure oscillations within the seabed precede the hydrodynamic pressure oscillations at the seabed surface ($z = 0$) which, of course, contradicts with formerly described phase-lag definition (see Eq. (9b)). The momentary pore-fluid pressure, computed from the combination of Eqs. (20a) and (20b), are coincidentally correct just because Eq. (20b) gives the values which are opposite to those which are really expected from the derivation procedure performed properly.

The equation for the momentary pore-fluid pressure, written in terms of a complex function, has the following form (see the original Eqs. (23) and (42) in [13], pp. 521 and 523, respectively; please note that the original notation is kept) [13]

$$U = B_1' \exp(az) \exp[i(ax + \omega t)] + \\ +(1 - B_1') \exp(\kappa_1 z) \exp[i(ax - \omega t - \kappa_2 z)] \tag{21}$$

Based on Eq. (21), adopting the form of Eq. (9b), using the fundamental equations for the absolute value and the argument of a complex number, remembering that the cosine function is an even function and the sine function is an odd function, and assuming a convenient value of the water-wave phase angle $\Theta = kx - \omega t = 0$, the following should be easily derived:

$$U = \left\{ \left[\left(1 - B_1'\right) \exp(\kappa_1 z) \sin(\kappa_2 z) \right]^2 + \\ + \left[B_1' \exp(az) + \left(1 - B_1'\right) \exp(\kappa_1 z) \cos(\kappa_2 z) \right]^2 \right\}^{\frac{1}{2}} \times \\ \times \ \cos\,(ax - \omega t + \delta) \tag{22a}$$

$$\tan \delta = -\frac{(1 - B_1') \exp(\kappa_1 z) \sin(\kappa_2 z)}{B_1' \exp(az) + (1 - B_1') \exp(\kappa_1 z) \cos(\kappa_2 z)} \tag{22b}$$

If positive values of the phase-lag of wave-induced pore-fluid pressure oscillations should classically denote a certain delay of the pore-fluid pressure oscillations with respect to the surface water-wave oscillations (or the hydrodynamic pressure oscillations at the seabed surface; see Fig. 2) — and this is the case confirmed by graphical presentation in the original Figs. 6 and 7 in [13], pp. 526–527 — the phase-angle term, responsible for the cyclic character of the phenomena under consideration, should be rather written as $(ax - \omega t + \delta_p)$ (please note that $\delta_p \equiv \delta$), as presented in Eqs. (9b) and (22a), and not in the form of $(ax - \omega t - \delta_p)$ used by Okusa [13] (see Eq. (20a)).

In order to perform illustrative computations of the wave-induced pore-fluid pressure, the following set of input-data was chosen after Okusa's [13] paper: wave period $T = 15$ s, water depth $h = 20$ m, wavelength $L = 197.53$ m (sinusoidal wave theory), volume compressibility of soil $\alpha = 9.18 \times 10^{-4}$ m²/kN, porosity of soil $n = 0.5$, Poisson's ratio of soil $\nu = 0.3$ and Skempton's pore-fluid pressure coefficient $B = 1.0, 0.9, 0.8, 0.7, 0.6, 0.5$. The values of other required parameters: K, β_w, γ_w and p_{at} were assumed as in the previously presented computational examples. A family of vertical distributions of the relative (and dimensionless) amplitude of wave-induced pore-fluid pressure, $\bar{p} = \tilde{p}/P_0$, is shown in Fig. 5 for several different soil saturation conditions modelled by Skempton's coefficient, B.

A comparison analysis has revealed that Okusa's [13] results, obtained from his simplified approximate solution and presented in a graphical form, are lacking in accuracy. The original Fig. 4 in [13], p. 524, and also the original Fig. 5 in [13], p. 525 (obtained for another set of input data), do not show a very characteristic disturbance of monotonicity of the curves in the uppermost zone of the seabed, as indicated in Fig. 5(a) of the present analysis. As far as the distribution of phase-lag of the wave-induced pore-fluid pressure oscillations is concerned (see Fig. 5(b)), maximum values are underestimated by factor one-third in Okusa's [13] illustrations, presented in the original Fig. 6, p. 526, and also in the original Fig. 7, p. 527 (obtained for another set of input data). It seems that these meaningful discrepancies can be explained by insufficient hardware computational capabilities existing almost four decades ago. As before, the quality of the Author's computations has been proved by additional computations based on Madsen's [6] analytical solution, achieving a highly satisfactory agreement.

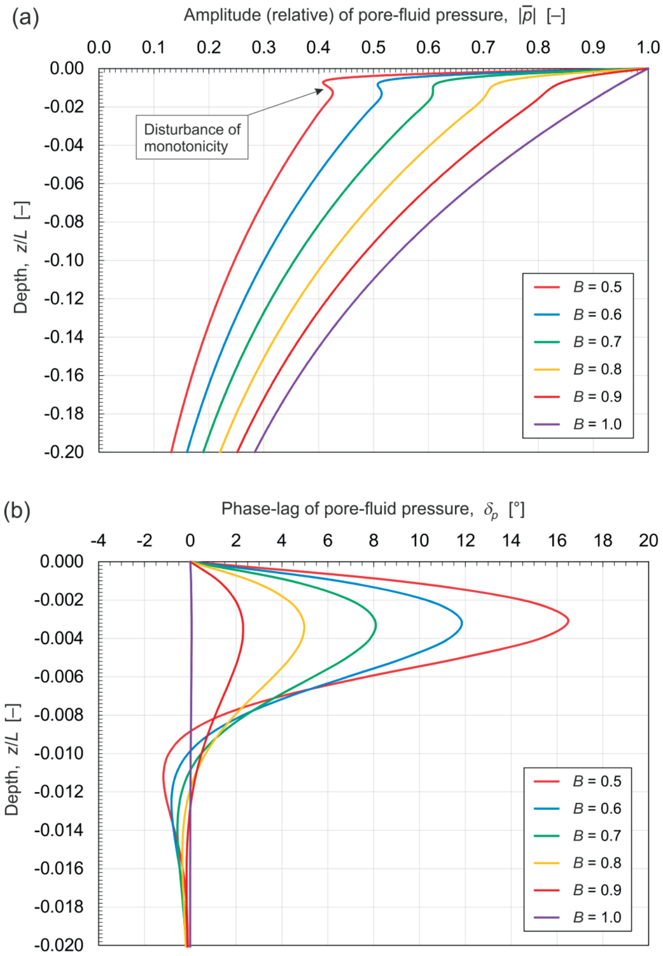

Fig. 5. Vertical distributions of the amplitude and the phase-lag of wave-induced pore-fluid pressure oscillations computed using Okusa's [13] exact and approximate forms of his analytical solution

SOLUTIONS BY HSU AND JENG [2], JENG AND HSU [5], JENG [3, 4]

Hsu and Jeng [2] and Jeng and Hsu [5] presented the three-dimensional governing partial differential equations (the static force and moment equilibrium equations together with the storage equation), clearly indicating the use of the solid mechanics sign convention for stresses throughout their papers; this was also certified by the stress block in the original Fig. 2 in [2], p. 789. The form of equations for the wave-induced effective normal stress and shear stress components also indicate the use of the solid mechanics sign convention. Surprisingly, Jeng and Hsu [5], p. 430, wrote: "A positive sign is used in the present paper, as in equations (7)–(12), i.e. compressive stresses are defined as positive". Nothing could be further from the truth. The form of equilibrium equations and equations for the wave-induced stress components, given by Jeng and Hsu [5], indicates clearly that the solid mechanics sign convention was used by them. Therefore, the solution obtained by Jeng and Hsu [5] for a fully saturated and isotropic soil of infinite thickness should follow strictly the solution presented by Yamamoto et al. [19], of course after transformation of the latter to the positive z-axis directed upwards and setting the water-wave propagation direction to be consistent with the positive Ox-axis direction. This finding makes a clear contradiction between what Jeng and Hsu [5] wrote in the text (please recall the above citation) and what they presented in the equations of their original paper.

Jeng [3] used exactly the same assumptions as Hsu and Jeng [2] and Jeng and Hsu [5]. However, among seven basic assumptions indicated in his book there is no any information regarding the sign convention applied. Again, it can only be deducted from the form of the equilibrium equations (see the original Eqs. (3.10)–(3.12) in [3], pp. 37–38) and the stress block in the soil element, presented in the original Fig. 3.2 in [3], p. 38, that the solid mechanics sign convention is used throughout Chapter 3 of the book by Jeng [3]. Jeng's [3] solution, obtained for fully saturated and hydraulically isotropic seabed of infinite thickness, after transforming it into the two-dimensional case and real-valued functions, and adopting the notation used in the present paper, can be presented as follows:

$$\overline{\sigma}'_x \stackrel{\text{def}}{=} \frac{\widetilde{\sigma}'_x}{P_0} = -kz\exp(kz)\cos(kx-\omega t) \quad \text{(23a)}$$

$$\overline{\sigma}'_z \stackrel{\text{def}}{=} \frac{\widetilde{\sigma}'_z}{P_0} = -kz\exp(kz)\cos(kx-\omega t) \quad \text{(23b)}$$

$$\overline{\tau}_{xz} \stackrel{\text{def}}{=} \frac{\widetilde{\tau}_{xz}}{P_0} = -kz\exp(kz)\sin(kx-\omega t) \quad \text{(23c)}$$

It can be easily recognised that the above solution differs from the solution given in the original Eqs. (47), (49) and (50), pp. 432–433, published formerly by Jeng and Hsu [5]. The wave-induced effective vertical normal stress, $\overline{\sigma}'_z$, which should obviously be compressive under the wave crest (e.g. for the surface water-wave phase-angle $\Theta = kx - \omega t = 0$), this time will always have non-negative values for $z \le 0$, as it is always in the case of soil mechanics sign convention. But this is not the sign convention used by Jeng [3]. The solution to the wave-induced effective vertical normal stress in the soil, given in Eq. (23b), must be evaluated as a wrong one. The signs of other stresses, $\overline{\sigma}'_x$ and $\overline{\tau}_{xz}$, described by Eqs. (23a) and (23c), are in line with the solid mechanics sign convention. Such mixture of two different sign conventions in one set of solution equations is unacceptable, leading very often to many misunderstandings and mistakes, and is not recommended for any practical use.

It has also been found that the original Eqs. (2.46)–(2.48) in [4], p. 45, describing the soil stress components in terms of complex functions, are erroneous because: there is a conflict of units in Eq. (2.46), and the common coefficient C_1^{∞}, appearing in the original Eqs. (2.46)–(2.48) and given in the original Eq. (2.51) in [4], p. 46, does not become equal to unity as it should be when fully saturated soil conditions (practically denoting incompressibility of the pore-fluid) are assumed.

CONCLUSIONS

A closer look has been taken at some selected analytical solutions of the wave-induced cyclic response of a porous seabed of infinite thickness to the sinusoidal surface water-wave loading. A thorough analysis has indicated the following items:

- a perfectly correct analytical solution was published by Madsen [6] and obtained for the entire set of six wave-induced seabed response parameters (pressure, stress and displacement components),
- some minor corrections are required in the analytical solutions delivered by Moshagen and Tørum [12] and Okusa [13] as far as the equations for the phase-lag of wave-induced pore-fluid pressure oscillations are concerned,
- unfortunately, there are two meaningful errors in the analytical solution given by Yamamoto et al. [19]: one in the general solution (i.e. before applying boundary conditions) and one in the particular solution (i.e. after applying boundary conditions) for the vertical component of soil displacement, which makes the Reader impossible to obtain, by means of further differentiations, correct forms of the linked equations for the soil stress components,
- unexpected errors in the analytical "infinite thickness solutions" can be caused by problems with a correct identification or with a lack of consequences in using only one and the same sign convention for stresses in the soil matrix, as found in [3]; the other errors detected especially in [4], are quite inexplicable.

The correct forms of the erroneous equations has been derived personally by the Author and they are included in the present paper for comparison purposes.

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