Correlation experiments in nonlinear quantum mechanics

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Abstract

We show how one can compute multiple-time multi-particle correlation functions in nonlinear quantum mechanics in a way which guarantees locality of the formalism.

I. INTRODUCTION

Exactly linear dynamics of states is a rarity in physics. Linear theories are in general approximations to nonlinear ones. The exception is quantum mechanics. Hence the question: Can one construct a consistent nonlinear theory which contains quantum mechanics as a special case?

Nonlinear extensions of quantum mechanics are not obvious. Nonlinear Schrödinger and von Neumann equations can be justified but do not seem to be allowed in the usual interpretation. A probability interpretation of nonlinear operators is not clear. Of particular interest are difficulties with multi-particle entangled states [1–11]. Standard textbook calculations of correlation experiments based on the projection postulate lead to nonlocal effects.

In this paper [12] we show how to compute conditional and joint probabilities in a way which eliminates unphysical influences between *separated* systems, and which coincides

with the usual prescription if the dynamics is linear. We do not exclude the possibility that the scheme we propose should be modified if there exist causal influences between the measurements. Suggestions concerning a generalization of our proposal can be found in [13].

We discuss a two particle system in the tensor space $H_1 \otimes H_1$ of one particle Hilbert spaces. We analyse correlation experiments: Measure on particle #1 at time t_1 an observable X_1 with two possible outcomes ($+$ or $-$) through a projection operator E_1, and on particle #2 at time t_2 an observable X_2 through a projection operator E_2. A nonlinear evolution of the pair of spin-1/2 particles is constructed via the Polchinski extension from nonlinear one-particle equations. As opposed to the original Polchinski formulation [3] we do not resort to the Many Worlds Interpretation. It is shown that different results are found depending on whether the two particles are viewed as closed or open systems. In the latter case there are no nonlocal effects. In the linear case the two alternative approaches give the same result.

The material is arranged as follows.

In section II and III we compare different methods of computing two-particle correlations in linear quantum mechanics. Two approaches are used.

(a) The two particles are treated as a closed system. The Hamiltonian is not time dependent. Two-time probabilities are calculated via projections-at-a-distance, as used by Gisin [2] and Mielnik [11].

(b) The two particles are treated as an open system. The environment contains the measuring devices acting at two different times. The Hamiltonian is time dependent, the two different times appearing as parameters. Two-time probabilities are calculated without reference to projections at-a-distance.

Section IV gives a short review of nonlinear evolution equations and Polchinski's multi-particle extension is introduced. Section V contains the central result of this paper: The open-system generalization of Polchinski's extension. It is shown that nonlocal effects do not occur in two-time measurements if the open-system formalism is employed. The results are illustrated in Section VI by explicit solutions for a two-particle entangled state. In Section VII we show how to modify in nonlinear quantum mechanics the projection-at-a-

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distance approach in order to eliminate the nonlocal effects. In Section VIII we discuss problems of probability reduction in systems which are causally related (preparation at-a-distance, teleportation, and Russian roulette with a cheating player). Some technical points are briefly explained in the Appendix.

II. CORRELATION EXPERIMENTS IN LINEAR QUANTUM MECHANICS — HEISENBERG PICTURE

For the description of the correlation experiment we start with a two-particle entangled state $|\Psi_0\rangle$ prepared at time $t = 0$. The two particles evolve independently by unitary operators $V_1(t) = e^{-iH_1 t}$ and $V_2(t) = e^{-iH_2 t}$. At times t_1 and t_2 one performs measurements of two quantities ("yes-no observables") represented by projectors E_1 and E_2 on particles #1 and #2, respectively.

We can now view the two particles as a closed or as an open system in which the measuring devices are a part of the environment. For the closed system the time dependence of the E_k is

$$E_k(t_k) = V_k(t_k)^\dagger E_k V_k(t_k) = e^{iH_k t_k} E_k e^{-iH_k t_k} \tag{1}$$

with time independent Hamiltonians.

Directly measurable probabilities are:

- Probability of the result "yes" for E_1 on particle #1

$$P[E_1(t_1)] = \langle \Psi_0 | E_1(t_1) \otimes I_2 | \Psi_0 \rangle, \tag{2}$$

- Probability of the result "yes" for E_2 on particle #2

$$P[E_2(t_2)] = \langle \Psi_0 | I_1 \otimes E_2(t_2) | \Psi_0 \rangle, \tag{3}$$

- Joint probability of results "yes" for both particles

$$P[E_1(t_1) \cap E_2(t_2)] = \langle \Psi_0 | E_1(t_1) \otimes E_2(t_2) | \Psi_0 \rangle. \tag{4}$$

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The conditional probability of the result "yes" for E_2 on particle #2 under the condition that "yes" is found for E_1 on particle #1

$$P[E_2(t_2)|E_1(t_1)] = \frac{P[E_1(t_1) \cap E_2(t_2)]}{P[E_1(t_1)]}, \tag{5}$$

is calculated from the joint probability and the probability of the condition.

In such an experiment the behaviour of particle #1 for times later than t_1 is irrelevant. The measurement at t_1 is a *destructive* measurement of the property represented by $E_1(t_1)$.

In the Heisenberg picture one expects that any operator, also $E_1 \otimes E_2$, has a unitary time dependence

$$E_1(t) \otimes E_2(t) = U(t)^\dagger E_1 \otimes E_2 U(t) \tag{6}$$

with some generator H. For the open system we construct a Hamiltonian in which the parameters t_1 and t_2 are encoded. Such an operator is interesting from the fundamental point of view and is also essential for later applications. The following time dependent Hamiltonian has the required properties

$$H_{t_1,t_2}(t) = \theta(t - t_1)H_1 \otimes I_2 + \theta(t - t_2)I_1 \otimes H_2. \tag{7}$$

were $\theta(x)$ is the step function equal 1 for $x < 0$ and 0 otherwise (note that $\theta(x) = \Theta(-x)$ where Θ is the Heaviside function). t_k are parameters indicating the times when interaction with the detectors takes place. The evolution of the projectors is

$$E_1(t) \otimes E_2(t) = e^{i \int_0^t H_{t_1,t_2}(t')dt'} E_1 \otimes E_2 e^{-i \int_0^t H_{t_1,t_2}(t')dt'}.$$

In particular,

$$E_1(t) \otimes E_2(t) = E_1(t_1) \otimes E_2(t_2) \quad \text{if } t_1, t_2 \leq t. \tag{8}$$

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III. CORRELATION EXPERIMENTS IN LINEAR QUANTUM MECHANICS — SCHRÖDINGER PICTURE

How to do the same calculation in the Schrödinger picture? This is a relevant question in our context, since in nonlinear quantum mechanics the Heisenberg picture in the usual sense does not exist.

As in Sec. II there are at least two posibilities which are based on a "projection at-a-distance" (closed system), and the time dependent Hamiltonian (open system).

A. Projection-at-a-distance approach

The dynamics of the state is

$$|\Psi(t)\rangle = V_1(t) \otimes V_2(t)|\Psi_0\rangle. \tag{9}$$

The calculation of the probabilities can be done with the following algorithm.

- Evolve the two-particle state until $t = t_1$ by means of (9).

- At $t = t_1$ project with $E_1 \otimes I_2$ and normalize

$$|\Psi(t_1)\rangle \mapsto \frac{E_1 \otimes I_2|\Psi(t_1)\rangle}{\| E_1 \otimes I_2|\Psi(t_1)\rangle \|} =: |\tilde{\Psi}(t_1)\rangle. \tag{10}$$

The projector E_1 represents the proposition which gave the result "yes" in the measurement performed on particle #1.

- Evolve the resulting state for $t_1 < t < t_2$ starting at t_1 with the initial condition (10) by means of $I_1 \otimes V_2(t - t_1)$, i.e.

$$I_1 \otimes V_2(t - t_1)|\tilde{\Psi}(t_1)\rangle. \tag{11}$$

- Calculate at $t = t_2$ the average of $I_1 \otimes E_2$ in the state (11)

$$\frac{\langle\Psi(t_1)|E_1 \otimes V_2(t_2 - t_1)^\dagger E_2 V_2(t_2 - t_1)|\Psi(t_1)\rangle}{\langle\Psi(t_1)|E_1 \otimes I_2|\Psi(t_1)\rangle}. \tag{12}$$

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This is the conditional probability of the result "yes" for the second particle under the condition that the appropriate measurement gave "yes" for the first particle.

- The interpretation of the denominator in (12) shows that the joint probability is given by the numerator of (12),

$$\langle\Psi_0|V_1(t_1)^\dagger E_1 V_1(t_1) \otimes V_2(t_2)^\dagger E_2 V_2(t_2)|\Psi_0\rangle. \tag{13}$$

which is the formula we wanted to derive. Here the conditional probability and the probability of the condition imply the joint probability.

B. Open-system approach

There exists a simpler and more straightforward method of computing the correlation function if we use the time dependent Hamiltonian (7). Solving the SE with (7) we find

$$|\Psi_{t_1,t_2}(t)\rangle = e^{-iH_1\otimes I_2 \int_0^t \theta(\tau-t_1)d\tau - iI_1\otimes H_2 \int_0^t \theta(\tau-t_2)d\tau}|\Psi_0\rangle. \tag{14}$$

The joint probability (4) is, like in the Heisenberg picture, directly available,

$$P[E_1(t_1)\cap E_2(t_2)] = \langle\Psi_{t_1,t_2}(t)|E_1 \otimes E_2|\Psi_{t_1,t_2}(t)\rangle, \tag{15}$$

with $t_1, t_2 \le t$.

IV. NONLINEAR HAMILTONIAN EVOLUTIONS

We restrict the nonlinear one-particle Schrödinger equations, for simplicity, to the classical Hamiltonian class, i.e. to those which can be written as

$$i\dot{\psi}_A(x) = \{\psi_A(x), \mathcal{H}\} = \frac{\delta\mathcal{H}}{\delta\bar{\psi}^A(x)}. \tag{16}$$

Linear Schrödinger-type equations are in this class; furthermore also some nonlinear Schrödinger equations (NLSE) can be formulated in this way ("$|\psi(x)|^2$ NLSE" [14], the

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Białynicki-Birula–Mycielski NLSE [15], certain family of Doebner–Goldin NLSE [16], and the equations discussed by Weinberg [17]). Weinberg's NLSE simultaneously belong to a family of generalized SE defined in an analogous way on projective spaces and Kähler manifolds [18–22].

As mentioned in the introduction an extension of the dynamics from one to many particles can be constructed, in the tensor product space, in different ways. If one wants a local two-particle NLSE (for example, such that a potential applied to one of the particles does not influence the other one) the extensions are restricted. Of particular interest in this context is the sub-class of one-particle NLSE with Hamiltonian functions satisfying the Polchinski condition [3]:

$$\mathcal{H}(\psi, \bar{\psi}) = \mathcal{H}(\rho)|_{\rho=|\psi\rangle\langle\psi|}. \tag{17}$$

For example, in a two-dimensional Hilbert space $|\psi\rangle = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ represents a spin-1/2 system. The Hamiltonian function

$$\begin{aligned} \mathcal{H}(\psi, \bar{\psi}) = \mathcal{H}(\psi_+, \psi_-, \bar{\psi}_+, \bar{\psi}_-) &= (\psi_+\bar{\psi}_- + \psi_-\bar{\psi}_+)^2 \\ &= \langle\psi|\sigma_x|\psi\rangle^2 = (\operatorname{Tr}|\psi\rangle\langle\psi|\sigma_x)^2 \\ &= (\operatorname{Tr}\rho\sigma_x)^2|_{\rho=|\psi\rangle\langle\psi|} =: \mathcal{H}(\rho)|_{\rho=|\psi\rangle\langle\psi|} \end{aligned} \tag{18}$$

satisfies the Polchinski condition, whereas

$$\mathcal{H}(\psi, \bar{\psi}) = (\psi_+\psi_- + \bar{\psi}_-\bar{\psi}_+)^2 \tag{19}$$

does not: (19) is not invariant under $|\psi\rangle \mapsto e^{i\alpha}|\psi\rangle$.

In linear quantum mechanics Hamiltonian functions can be written as

$$\begin{aligned} \mathcal{H}(\psi, \bar{\psi}) = \langle\psi|H|\psi\rangle &= \operatorname{Tr}(|\psi\rangle\langle\psi|H) \\ &= \operatorname{Tr}\rho H|_{\rho=|\psi\rangle\langle\psi|} =: \mathcal{H}(\rho)|_{\rho=|\psi\rangle\langle\psi|} \end{aligned} \tag{20}$$

and, hence, fulfil the condition. Białynicki-Birula–Mycielski and "$|\psi(x)|^2$" are examples of NLSE satisfying the Polchinski condition. A weakened version of the condition is applicable to all Doebner–Goldin equations [10].

Assume now that we have two particles with Hamiltonian functions fulfilling the above criterion, i.e.

$$\mathcal{H}_1(\psi_1, \bar{\psi}_1) = \mathcal{H}_1(\rho)|_{\rho=|\psi_1\rangle\langle\psi_1|}, \tag{21}$$

$$\mathcal{H}_2(\psi_2, \bar{\psi}_2) = \mathcal{H}_2(\rho)|_{\rho=|\psi_2\rangle\langle\psi_2|}, \tag{22}$$

and a generic *entangled* state [1]

$$|\Psi(t)\rangle = \sum_{k_1 k_2} \Psi(t)_{k_1 k_2} |k_1\rangle |k_2\rangle. \tag{23}$$

States of the one particle subsystems may be represented by reduced density matrices [$\Psi = \Psi(t)$]

$$\rho_1 = \sum_{k_1 l_1 k_2} \bar{\Psi}_{k_1 k_2} \Psi_{l_1 k_2} |k_1\rangle\langle l_1|, \tag{24}$$

$$\rho_2 = \sum_{k_1 k_2 l_2} \bar{\Psi}_{k_1 k_2} \Psi_{k_1 l_2} |k_2\rangle\langle l_2|. \tag{25}$$

Polchinski defined a *two-particle* Hamiltonian function by their sum evaluated at appropriate one-particle states of particles #1 and #2, respectively, i.e. as

$$\mathcal{H}_{1+2}(\Psi, \bar{\Psi}) := \mathcal{H}_1(\rho)|_{\rho_1} + \mathcal{H}_2(\rho)|_{\rho_2}. \tag{26}$$

The corresponding two-particle NLSE has the Hamiltonian form

$$i\dot{\Psi}_{k_1 k_2} = \frac{\partial \mathcal{H}_{1+2}(\Psi, \bar{\Psi})}{\partial \bar{\Psi}_{k_1 k_2}}. \tag{27}$$

In typical situations (see the Appendix) the solution of (27) can be written as

$$|\Psi(t)\rangle = V_1(\Psi_0, t) \otimes V_1(\Psi_0, t)|\Psi_0\rangle \tag{28}$$

$$= V_1(\rho_1(0), t) \otimes V_1(\rho_2(0), t)|\Psi_0\rangle. \tag{29}$$

We can write with its help reduced density matrices of the subsystems. It can be shown at different levels of generality [3,5,8] that the dynamics of a reduced density matrix of one of

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[1]From now on we employ notation more appropriate for systems with discrete degrees of freedom. This is motivated by finite-dimensional examples we will discuss later.

the subsystems is independent of the choice of Hamiltonian function of the other subsystem (for a simple proof see Appendix). This establishes locality of the extension.

V. CORRELATION EXPERIMENTS IN NONLINEAR QUANTUM MECHANICS — SCHRÖDINGER PICTURE

We mentioned already that in nonlinear quantum mechanics the usual Heisenberg picture may not exist. For a nonlinear evolution of pure one-particle states the Schrödinger picture is automatically given. Hence we describe the correlation experiment in the Schrödinger picture. We have shown that there are two posibilities: the projection-at-a-distance approach and the open-system approach. In the linear case they give the same results (which agree also with those from the Heisenberg picture).

The projection-at-a-distance approach was employed to two-particle systems in nonlinear quantum mechanics by Gisin [2] and recently by Mielnik [11]. The conclusion of these papers was that a nonlocal effect necessarily appears independently of the form of one-particle nonlinearity and the form of two-particle extension. In the next section we show on an explicit example and using the Polchinski extension that the above conclusion is correct if one sticks to this particular representation of the projection postulate. However, the argument does not work if one uses an open-system approach.

To adapt to nonlinear quantum mechanics the open-system approach one has to modify the two-particle extension. We generalize the Polchinski two-particle Hamiltonian function as follows

$$\mathcal{H}_{t_1,t_2}(t,\Psi,\bar{\Psi}) = \theta(t-t_1)\mathcal{H}_1(\rho)|_{\rho_1} + \theta(t-t_2)\mathcal{H}_2(\rho)|_{\rho_2}. \tag{30}$$

The Schrödinger equation for the two particles reads again

$$i\dot{\Psi}_{k_1k_2} = \frac{\partial \mathcal{H}_{t_1t_2}(t,\Psi,\bar{\Psi})}{\partial\bar{\Psi}_{k_1k_2}} \tag{31}$$

$(\Psi_{k_1k_2} = \Psi_{t_1,t_2}(t)_{k_1k_2})$. Solutions of (31) are of the form (cf. Sec. VI and the Appendix)

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$$|\Psi_{t_1,t_2}(t)\rangle = V_1(\rho_1(0), t, t_1) \otimes V_2(\rho_2(0), t, t_2)|\Psi_0\rangle \tag{32}$$

where V_k depend only on (nonlinear and time dependent) Hamiltonians and initial reduced density matrices of k-th particles.

It follows that the reduced density matrices are

$$\rho_k(t) = V_k(\rho_k(0), t, t_k)\rho_k(0)V_k(\rho_k(0), t, t_k)^\dagger, \quad k = 1, 2. \tag{33}$$

As a consequence one cannot influence the dynamics of particle #1 by modifications of potentials, moments of detection, and initial conditions corresponding to particle #2, and vice versa. This establishes locality of the dynamics.

Let us note that the open-system approach is independent of projections at-a-distance and one can directly use the formula from linear quantum mechanics: If $|\Psi_{t_1,t_2}(t)\rangle$ is a solution of (31) then, for $t_1, t_2 \leq t$, the joint probability is

$$P[E_1(t_1) \cap E_2(t_2)] = \langle\Psi_{t_1,t_2}(t)|E_1 \otimes E_2|\Psi_{t_1,t_2}(t)\rangle. \tag{34}$$

To illustrate how this works we consider an explicit example.

VI. EXAMPLE: EVOLUTION OF A PAIR OF SPIN-1/2 PARTICLES

We start with one-particle Hamiltonian functions

$$\mathcal{H}_1(\rho) = A[\operatorname{Tr}(\rho\sigma_z)]^2/2 \tag{35}$$

$$\mathcal{H}_2(\rho) = B[\operatorname{Tr}(\rho\sigma_z)]^2/2 \tag{36}$$

$$\mathcal{H}_1(\psi_1, \bar{\psi}_1) = A[\operatorname{Tr}(|\psi_1\rangle\langle\psi_1|\sigma_z)]^2/2 \tag{37}$$

$$= A\langle\psi_1|\sigma_z|\psi_1\rangle^2/2 \tag{38}$$

$$\mathcal{H}_2(\psi_2, \bar{\psi}_2) = B[\operatorname{Tr}(|\psi_2\rangle\langle\psi_2|\sigma_z)]^2/2 \tag{39}$$

$$= B\langle\psi_2|\sigma_z|\psi_2\rangle^2/2. \tag{40}$$

A and B are real constants and $|\psi_1\rangle$, $|\psi_2\rangle$ are one-particle state-vectors. The corresponding one-particle equations obtained from these Hamiltonian finctions are

$$i|\dot{\psi}_1\rangle = A\langle\psi_1|\sigma_z|\psi_1\rangle\sigma_z|\psi_1\rangle, \tag{41}$$

$$i|\dot{\psi}_2\rangle = B\langle\psi_2|\sigma_z|\psi_2\rangle\sigma_z|\psi_2\rangle. \tag{42}$$

Both nonlinear Hamiltonian operators are of the form $\boldsymbol{b}\cdot\boldsymbol{\sigma}$ where

$$\boldsymbol{b} \sim (0,0,\langle\sigma_z\rangle). \tag{43}$$

This is a mean-field type interaction of a Curie-Weiss type.

The Polchinski two-particle extension is

$$\mathcal{H}_{1+2}(\Psi,\bar{\Psi}) = A[\,\mathrm{Tr}\,(\rho_1\sigma_z)]^2/2 + B[\,\mathrm{Tr}\,(\rho_2\sigma_z)]^2/2 \tag{44}$$

$$= A\langle\Psi|\sigma_z\otimes I|\Psi\rangle^2/2 + B\langle\Psi|I\otimes\sigma_z|\Psi\rangle^2/2 \tag{45}$$

and the two-particle Schrödinger equation derived from this Hamiltonian function is

$$i|\dot{\Psi}\rangle = \Big(A\langle\Psi|\sigma_z\otimes I|\Psi\rangle\sigma_z\otimes I + B\langle\Psi|I\otimes\sigma_z|\Psi\rangle I\otimes\sigma_z\Big)|\Psi\rangle. \tag{46}$$

A. Open-system approach

The generalized Polchinski two-particle Hamiltonian function is

$$\mathcal{H}_{t_1,t_2}(\Psi,\bar{\Psi}) = \theta(t-t_1)A\langle\Psi|\sigma_z\otimes I|\Psi\rangle^2/2 + \theta(t-t_2)B\langle\Psi|I\otimes\sigma_z|\Psi\rangle^2/2 \tag{47}$$

and

$$i|\dot{\Psi}\rangle = \Big(\theta(t-t_1)A\langle\Psi|\sigma_z\otimes I|\Psi\rangle\sigma_z\otimes I + \theta(t-t_2)B\langle\Psi|I\otimes\sigma_z|\Psi\rangle I\otimes\sigma_z\Big)|\Psi\rangle. \tag{48}$$

The general solution of (48) is

$$|\Psi_{t_1,t_2}(t)\rangle = e^{-iA\langle\Psi_0|\sigma_z\otimes I|\Psi_0\rangle\sigma_z\otimes I\int_0^t\theta(\tau-t_1)d\tau - iB\langle\Psi_0|I\otimes\sigma_z|\Psi_0\rangle I\otimes\sigma_z\int_0^t\theta(\tau-t_2)d\tau}|\Psi_0\rangle$$

$$= e^{-iA\langle\sigma_z(0)\rangle_1\sigma_z\kappa(t,t_1)}\otimes e^{-iB\langle\sigma_z(0)\rangle_2\sigma_z\kappa(t,t_2)}|\Psi_0\rangle \tag{49}$$

where $\langle\sigma_z(0)\rangle_k = \mathrm{Tr}\,(\rho_k(0)\sigma_z)$, $\kappa(t,t_k) = \int_0^t\theta(\tau-t_k)d\tau$. The averages in the exponents are evaluated in $|\Psi_0\rangle$. This is a consequence of

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$$\langle\Psi_0|\sigma_z \otimes I|\Psi_0\rangle = \langle\Psi_{t_1,t_2}(t)|\sigma_z \otimes I|\Psi_{t_1,t_2}(t)\rangle \tag{50}$$

$$\langle\Psi_0|I \otimes \sigma_z|\Psi_0\rangle = \langle\Psi_{t_1,t_2}(t)|I \otimes \sigma_z|\Psi_{t_1,t_2}(t)\rangle \tag{51}$$

as one can verify by direct substitution.

(49) describes the entire history of the two particles: From their "birth" at $t = 0$ to their "deaths" at $t = t_1$ and $t = t_2$. The solution of (46) is recovered in the limits $t_1, t_2 \to +\infty$.

Using (49) we can explicitly compute the dynamics of the two subsystems. The reduced density matrices are

$$\rho_1(t) = e^{-iA\langle\sigma_z(0)\rangle_1 \sigma_z \kappa(t,t_1)}\rho_1(0)e^{iA\langle\sigma_z(0)\rangle_1 \sigma_z \kappa(t,t_1)} \tag{52}$$

$$\rho_2(t) = e^{-iB\langle\sigma_z(0)\rangle_2 \sigma_z \kappa(t,t_2)}\rho_2(0)e^{iB\langle\sigma_z(0)\rangle_2 \sigma_z \kappa(t,t_2)}. \tag{53}$$

The form of the above explicit solutions is instructive because of the following properties:

- The subsystems evolve independently of each other.

- The solutions are uniquely determined by the initial condition $|\Psi_0\rangle$ at $t = 0$.

- The evolution operator for the pair is

$$V_1(\Psi_0, t) \otimes V_2(\Psi_0, t) = V_1(\rho_1(0), t) \otimes V_2(\rho_2(0), t) \tag{54}$$

  i.e. is a product of *unitary* operators which depend on $\rho_k(0)$ and not on their decompositions in particular bases.

From the solution (49) one can calculate correlation functions for any observable (see Sec. V).

Operationally there is no ambiguity in the open-system formulation. If one wants to know predictions for an experiment one has to insert the detection times, t_1 and t_2, into (49).

In an actual experiment one deals with N pairs. If we assume for simplicity that for all the pairs the times of flight $\Delta t_k^i = t_k^i - t_0^i$, $k = 1, 2$, $i = 1, \ldots, N$, are the same and equal Δt_k we can compute averages of observables, say, $X_1 \otimes X_2$, by

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$$\langle X_1 \otimes X_2 \rangle_{\Psi, \Delta t_1, \Delta t_2} = \langle \Psi_{\Delta t_1, \Delta t_2}(t) | X_1 \otimes X_2 | \Psi_{\Delta t_1, \Delta t_2}(t) \rangle. \tag{55}$$

Averages of one-system observables, say X_1, are computed in the standard way

$$\langle X_1 \otimes I_2 \rangle_{\Psi, \Delta t_1, \Delta t_2} = \langle \Psi_{\Delta t_1, \Delta t_2}(t) | X_1 \otimes I_2 | \Psi_{\Delta t_1, \Delta t_2}(t) \rangle$$

$$= \mathrm{Tr} \left(e^{iA\langle \sigma_z(0)\rangle_1 \sigma_z \kappa(t, \Delta t_1)} X_1 e^{-iA\langle \sigma_z(0)\rangle_2 \sigma_z \kappa(t, \Delta t_1)} \rho_1(0) \right). \tag{56}$$

The average does not depend on Δt_2. As we have already said this is a consequence of the local properties of the Polchinski extension.

B. Projection-at-a-distance approach

We follow the calculation from Sec. III A step by step. Consider measurements of spin in direction $\boldsymbol{a}_k$, $k = 1, 2$, i.e. the observable is $X_k = \boldsymbol{a}_k \cdot \boldsymbol{\sigma}$ with projectors $E_k = E_k^{\pm} = (I_k \pm X_k)/2$.

- At $t = t_1$ the state is

$$|\Psi(t_1)\rangle = \underbrace{e^{-iA\langle \sigma_z(0)\rangle_1 \sigma_z t_1}}_{V_1(\Psi_0, t_1)} \otimes \underbrace{e^{-iB\langle \sigma_z(0)\rangle_2 \sigma_z t_1}}_{V_2(\Psi_0, t_1)} |\Psi_0\rangle. \tag{57}$$

- At $t = t_1$ project with $E_1^{\pm} \otimes I_2$ and normalize

$$|\Psi(t_1)\rangle \mapsto \frac{E_1^{\pm} \otimes I_2 |\Psi(t_1)\rangle}{\| E_1^{\pm} \otimes I_2 |\Psi(t_1)\rangle \|} =: |\Psi_{\pm}(t_1)\rangle. \tag{58}$$

- Evolve the resulting state for $t_1 < t < t_2$ but starting at t_1 with the initial condition (58)

$$|\Psi_{\pm}(t_2)\rangle = I_1 \otimes \underbrace{e^{-iB\langle \Psi_{\pm}(t_1)|I_1 \otimes \sigma_z |\Psi_{\pm}(t_1)\rangle \sigma_z (t_2 - t_1)}}_{V_2 \left(\Psi_{\pm}(t_1), t_2 - t_1 \right)} |\Psi_{\pm}(t_1)\rangle. \tag{59}$$

- Compute the conditional probability

$$\langle \Psi_{\pm}(t_2) | I_1 \otimes E_2^s | \Psi_{\pm}(t_2) \rangle$$
$$= \frac{\langle \Psi(t_1) | E_1^{\pm} \otimes e^{iB\langle \Psi_{\pm}(t_1)|I_1 \otimes \sigma_z |\Psi_{\pm}(t_1)\rangle \sigma_z (t_2 - t_1)} E_2^s e^{-iB\langle \Psi_{\pm}(t_1)|I_1 \otimes \sigma_z |\Psi_{\pm}(t_1)\rangle \sigma_z (t_2 - t_1)} |\Psi(t_1)\rangle}{\langle \Psi(t_1) | E_1^{\pm} \otimes I_2 | \Psi(t_1)\rangle} \tag{60}$$

where E_2^s is E_2^+ or E_2^-.

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The joint probability

$$\langle \tilde{\Psi}^{\pm}_{t_1,t_2}(t_2)|E_1^{\pm} \otimes E_2^s|\tilde{\Psi}^{\pm}_{t_1,t_2}(t_2)\rangle, \tag{61}$$

where

$$|\tilde{\Psi}^{\pm}_{t_1,t_2}(t_2)\rangle = e^{-iA\langle\Psi_0|\sigma_z\otimes I|\Psi_0\rangle\sigma_z t_1} \otimes e^{-iB\langle\Psi_\pm(t_1)|I\otimes\sigma_z|\Psi_\pm(t_1)\rangle\sigma_z(t_2-t_1)}e^{-iB\langle\Psi_0|I\otimes\sigma_z|\Psi_0\rangle\sigma_z t_1}|\Psi_0\rangle, \tag{62}$$

can be calculated from (60).

Just for comparison let us note that the open-system calculation produces at this point joint probability of the form

$$\langle \Psi_{t_1,t_2}(t_2)|E_1^{\pm} \otimes E_2^s|\Psi_{t_1,t_2}(t_2)\rangle. \tag{63}$$

Now we can pinpoint the difference between the two approaches. The frequencies of spin rotation are different. In the projection-at-a-distance approach we have

$$B\langle\Psi_\pm(t_1)|I_1 \otimes \sigma_z|\Psi_\pm(t_1)\rangle$$

and in the open-system approach

$$B\langle\Psi_0|I \otimes \sigma_z|\Psi_0\rangle.$$

They depend on the projected state taken at t_1 and the initial state at $t = 0$, respectively.

C. Numerical example

For a numerical illustration of previous considerations we take a convenient initial state

$$|\Psi_0\rangle = \frac{1}{3}|1\rangle|2\rangle - \frac{2\sqrt{2}}{3}|2\rangle|1\rangle \tag{64}$$

where

$$|1\rangle = \begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8) \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} -\sin(\pi/8) \\ \cos(\pi/8) \end{pmatrix}. \tag{65}$$

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The parameters in Hamiltonians are $A = 8$, $B = 1/2$, and the detection times are $t_1 = 3.5$ and $t_2 = 8$ (all in dimensionless units). Figs. 1 and 2 show averages of $\sigma_x \otimes \sigma_x$ (solid), $\sigma_x \otimes I_2$ (dashed), and $I_1 \otimes \sigma_x$ (dotted) calculated by means of the two approaches.

In Fig. 1 we used the open-system approach. The dotted line representing the average of $I_1 \otimes \sigma_x$ does not "notice" the measurement performed on particle #1. In Fig. 2 the projection at-a-distance was employed. One can observe a slight change in the doted curve at $t = t_1$. This is the nonlocal effect of the type described by Gisin [2]. Until $t = t_1$ the evolution is described in the open-system way. One can see from the figures that projection-at-a-distance reasoning leads even in this case to the nonlocal influence between the two particles.

VII. NONLINEAR GENERALIZATION OF PROJECTION AT-A-DISTANCE

As the final step of our analysis we show that there exists a generalization of the projection-at-a-distance algorithm leading to results equivalent to those from the open-system approach. The algorithm is applicable if there is no causal relation between the correlated measurements. Modifications are needed if causal realtions do occur (see the next section). The modified algorithm follows steps analogous to those from Sec. VI:

- Evolve the two-particle state until $t = t_1$ by means of the evolution generated by (30). The solution has the form (see Appendix)

$$|\Psi_{t_1,t_2}(t)\rangle = V_1(\Psi_0, t) \otimes V_2(\Psi_0, t)|\Psi_0\rangle. \tag{66}$$

- At $t = t_1$ project and again normalize

$$|\Psi_{t_1,t_2}(t_1)\rangle \mapsto \frac{E_1 \otimes I_2|\Psi_{t_1,t_2}(t_1)\rangle}{\| E_1 \otimes I_2|\Psi_{t_1,t_2}(t_1)\rangle \|}. \tag{67}$$

- Evolve this state by $I_1 \otimes V_2(\Psi_0, t - t_1)$, i.e.

$$\frac{E_1 \otimes V_2(\Psi_0, t_2 - t_1)|\Psi_{t_1,t_2}(t_1)\rangle}{\| E_1 \otimes I_2|\Psi_{t_1,t_2}(t_1)\rangle \|} \tag{68}$$

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- Calculate at $t = t_2$ the average of $I_1 \otimes E_2$ in the state (68)

$$\frac{\langle \Psi_{t_1,t_2}(t_1)|E_1 \otimes V_2(\Psi_0, t_2 - t_1)^\dagger E_2 V_2(\Psi_0, t_2 - t_1)|\Psi_{t_1,t_2}(t_1)\rangle}{\langle \Psi_{t_1,t_2}(t_1)|E_1 \otimes I_2|\Psi_{t_1,t_2}(t_1)\rangle}. \tag{69}$$

The denominator in (69) is the probability of the condition. Therefore the joint probability is given by the numerator of (69). Using (66) we obtain (34).

As we can see there is only one modification with respect to the derivation which led in the example to the nonlocal effect: Instead of

$$V_2(\Psi_\pm(t_1), t_2 - t_1) \tag{70}$$

the following expression appears

$$V_2(\Psi_0, t_2 - t_1) = V_2(\rho_2(0), t_2 - t_1), \tag{71}$$

where $|\Psi_0\rangle$ and $\rho_2(0)$ are the *initial* conditions for the pair and the second particle, respectively.

VIII. REMARKS ON CAUSALLY RELATED CORRELATION EXPERIMENTS

Our discussion was purposefully restricted to measurements which are spacelike separated. However, a dual problem remains: What about measurements which are *not* spacelike separated, a situation one encounters in preparation at-a-distance?

A. Preparation at-a-distance and teleportation

Preparation at-a-distance is a procedure which produces a state of a physical system #1 on the basis of destructive measurements performed on a correlated system #2. Active quantum teleportation is a particular case of this procedure. The procedure is often referred to as a *non-destructive* measurement.

Assume, for example, that we have to produce "an ensemble of white stones" which are selected at random from a box containing black and white pebbles. How do we do this? We

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take a randomly chosen stone and "look at it". If the stone is white, we keep it. Otherwise we thow it away.

The experimental setup involves two steps. In the first step we scatter some light on the stones and our eyes perform *destructive* measurements of the scattered photons. The second step involves a *local action* (keeping or removing the stone) performed on the ensemble of black and white pebbles.

The second step is as necessary for the preparation as the first one, and is performed in the *future light-cone* of the detection event. In (active) quantum teleportation an analogue of the second step is typically referred to as a "classical communication channel supplemented by local operations". We cannot prepare in such a way an ensemble of white stones (or spins "up") in a region of space-time which is spacelike separated from the detection area. For the same reason teleportation cannot be faster than light.

In the next section we discuss a probabilistic game which in many respects is analogous to the nonlinear EPR problem. The example shows that in correlation experiments involving a nonlinear dynamics one has to take into account propagation of information between correlated subsystems.

B. Russian roulette with a cheating player

The nonlinear EPR problem is not, in its essence, a problem of quantum mechanics. It is a general difficulty present in all nonlinear systems whose dynamics depends on probability and which involve reduction of probability via correlations. The Russian roulette with a cheating player is an example of a situation where the required properties occur.

The Russian roulette is a game whose simplest version is the following. There are two players, Anna and Boris, a gun with two chambers, and one bullet. The players do not know which chamber is loaded. They put certain amounts of money into the pool and Boris begins the game: He points the gun at himself and pulls the trigger. If he is unlucky then Anna wins and collects all the money. However, if the bullet was in the other chamber, the

next move belongs to Anna...

There are two variants of continuation.

(a) Anna is not informed about the result of Boris' trial before she pulls the trigger.

(b) She knows what happened to Boris.

The first case involves separated events. The nontrivial formal element of the game is the behaviour of Anna in the second case. If Boris had a bad day she can safely pull the trigger and wins. In the opposite case she knows this time the gun will fire and it makes no sense to continue, so she tries to cheat.

How to formally model the game? Thinking of a *real-life* version of the duel it is rather clear that the change of behaviour of a player is due to his (her) lack of knowledge about the actual location of the cartridge in the gun. Real versions of the game involved six players and a six-chamber gun. As the game continues the probability that the next player will get killed increases if the players are not allowed to randomly spin the chambers after each trial. It seems that in a formal model of the game we should assume that the behaviour of the cheating player (who nevertheless tries to spin the chamber) is probability dependent (the greater the probability of getting killed the greater the motivation to circumvent the rules). If we agree on this viewpoint the roulette becomes an interesting playground for testing the concepts of probability reduction in systems whose dynamics is *probability dependent*.

More instructive is the version of the game with a six-chamber gun and three bullets which are placed in such a way that between two loaded chambers there is an empty one. Each time one pulls the trigger the chamber shifts by one place. If Boris was lucky then the loaded chamber is in place and she should cheat (rotate it by one position). However, if Boris was not lucky she will shoot herself if she cheats. Cheating and non cheating are here statistically equivalent: One half of the ensemble of Annas will not survive the game if they are not informed, independently of whether they cheat or not.

We can say that the dynamics of Anna is independent of probability (i.e. linear) if she is not informed. Let us note that this is exactly analogous to the example we give in Sec. VI: The nonlinearity vanishes if the average involves the entire density matrix of the subsystem.

The behaviour of Anna changes at the moment she gets the information and not at the moment Boris makes his "measurement". This can be verified statistically since now the entire ensemble of cheating Annas will survive. This variant is analogous to the EPR problem as discussed by Gisin in [2]: The nonlinearity reacts to the density matrix which involves reduced probabilities. Still, the reduction of Anna's probability is not instantaneous. How to describe the reduction is a completely different issue. An interesting discussion of a similar problem can be found in a recent paper by Kent [23].

Let us finally note that the link of the game to the nonlinear EPR problem becomes even more evident if one assumes that Anna makes her decision on the basis of an *incomplete* information. Then in order to survive she estimates the probability that the information she obtained is reliable and her behaviour is explicitly probability dependent.

IX. SUMMARY

Among other obstructions for the formulation of a physically motivated and mathematically decent nonlinear extension of quantum mechanics one encounters the following problem: How to build from a one-particle system a time evolution of a multi-particle one, and how to compute correlation experiments in this system. There is an additional condition: We want a local theory. Hence we use the Polchinski multi-particle extension which is sufficient for a local description of equal-time correlation experiments. To include multiple-time and spacelike-separated correlation experiments we generalize the Polchinski formalism by treating the system as an open one with detectors in the role of an environment. Now multi-particle Hamiltonians are time dependent and parametrized by the detection times. On this basis we derive a generalization of the projection-at-a-distance algorithm which is appropriate for nonlinear correlation experiments with spacelike separated events. The modified algorithm predicts the same probabilities as the open-system generalization of the Polchinski approach and the nonlocal effects are eliminated. We also give a new argument against an instantaneous reduction of probability in correlation experiments.

X. APPENDIX: SOLUTIONS OF (31) AND LOCALITY

Take Hamiltonian functions

$$\mathcal{H}_1(\rho_1) = \mathcal{H}_1(\{\rho_{1a_1b_1}\}) = \mathcal{H}_1(\{\sum_{c_2} \Psi_{a_1c_2}\bar{\Psi}_{b_1c_2}\}) \tag{72}$$

$$\mathcal{H}_2(\rho_2) = \mathcal{H}_1(\{\rho_{2a_2b_2}\}) = \mathcal{H}_1(\{\sum_{c_1} \Psi_{c_1a_2}\bar{\Psi}_{c_1b_2}\}). \tag{73}$$

Define the following Hamiltonian operators

$$H_1(\rho_1)_{b_1a_1} = \frac{\partial \mathcal{H}_1}{\partial \rho_{1a_1b_1}} \tag{74}$$

$$H_2(\rho_2)_{b_2a_2} = \frac{\partial \mathcal{H}_2}{\partial \rho_{2a_2b_2}}. \tag{75}$$

The operators are Hermitian since ρ_1 and ρ_2 are Hermitian. Let $\mathcal{H}_{t_1,t_2}$ be given by (30). Using the chain rule one can show that

$$i|\dot{\Psi}\rangle = \sum_{k_1k_2} \frac{\partial \mathcal{H}_{t_1,t_2}}{\partial \bar{\Psi}_{k_1k_2}}|k_1\rangle|k_2\rangle$$
$$= \left(\theta(t-t_1)H_1(\rho_1) \otimes I_2 + \theta(t-t_2)I_1 \otimes H_2(\rho_2)|\Psi\rangle. \tag{76}$$

If the Cauchy problem for (76) is well posed, its solution $|\Psi(t)\rangle$ is uniquely determined by the initial condition $|\Psi_0\rangle$ at $t = 0$. Assume $|\Psi(t)\rangle = |\Psi[\Psi_0, t]\rangle$ is known. Substituting the solution into (76) and denoting

$$\tilde{H}_k(\Psi_0, t) = \theta(t - t_k) H_k(\rho_k(\Psi[\Psi_0, t], \bar{\Psi}[\Psi_0, t])) \tag{77}$$

$$= \theta(t - t_k) H_k(\rho_k(t)) \tag{78}$$

$$= \tilde{H}_k(\rho_k(0), t) \tag{79}$$

we can see that $|\Psi(t)\rangle$ is a solution of

$$i|\dot{\Psi}_{t_1, t_2}(t)\rangle = \left(\tilde{H}_1(\Psi_0, t) \otimes I_2 + I_1 \otimes \tilde{H}_2(\Psi_0, t)\right)|\Psi_{t_1, t_2}(t)\rangle. \tag{80}$$

For a fixed initial value $|\Psi_{t_1, t_2}(0)\rangle = |\Psi_0\rangle$ this is a *linear* Schrödinger equation with time-dependent Hamiltonian (the dependence on the set of parameters defining the initial condition is nonlinear). Using results from linear quantum mechanics we conclude that there exist unitary operators $V_k(\Psi_0, t) = V_k(\rho_k(0), t)$ such that

$$|\Psi(t)\rangle = V_1(\Psi_0, t) \otimes V_2(\Psi_0, t)|\Psi_0\rangle \tag{81}$$

$$= V_1(\rho_1(0), t) \otimes V_2(\rho_2(0), t)|\Psi_0\rangle. \tag{82}$$

To each $|\Psi_0\rangle$ there corresponds an orbit of the dynamics. The difference with respect to linear quantum mechanics is that on different orbits we have different unitary evolutions.

The reduced density matrices evolve by

$$\rho_k(t) = V_k(\rho_k(0), t)\rho_k(0)V_k(\rho_k(0), t)^\dagger. \tag{83}$$

The behaviour of the subsystems is determined entirely by local Hamiltonians and local initial conditions for states. This establishes locality.

This would not be the case if $\tilde{H}_k(\Psi_0, t)$ did not depend on one-particle states of the kth particle. This also shows that different local two-particle extensions may be possible if different one-particle representations of states are used.

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REFERENCES

[1] R. Haag and U. Bannier, Comm. Math. Phys. **60** (1978) 1.

[2] N. Gisin, Phys. Lett. A **143**, 1 (1990); Helv. Phys. Acta **62**, 363 (1989).

[3] J. Polchinski, Phys. Rev. Lett. **66**, 397 (1991).

[4] M. Czachor, Found. Phys. Lett. **4**, 351 (1991).

[5] T. F. Jordan, Ann. Phys. **225** (1993) 83.

[6] G. A. Goldin and G. Svetlichny, J. Math. Phys. **35**, 3322 (1994).

[7] W. Lücke, in *Nonlinear, deformed and Irreversible Quantum Systems*, p. 140-154, H.-D. Doebner et al. eds., World Sientific, Singapore, 1995.

[8] M. Czachor, Phys. Lett. A **225** (1997) 1-12.

[9] M. Czachor and M. Kuna, Phys. Rev. A **58** (1998) 128.

[10] M. Czachor, Phys. Rev. A **57** (1998) 4122.

[11] B. Mielnik, Phys. Lett. A **289**, 1 (2001).

[12] The present paper is an extended version of the preprint M. Czachor and H.-D. Doebner, quant-ph/0110008.

[13] S. Gheorghiu-Svirschevski, quant-ph/0207042; cf. also S. Gheorghiu-Svirschevski, quant-ph/0203153.

[14] L. A. Takhtajan, L. D. Fadeev, *Hamiltonian Approach in Theory of Solitons*, Nauka, Moscow (1986)

[15] I. Białynicki-Birula and J. Mycielski, Ann. Phys. **100** (1976) 62.

[16] H.-D. Doebner and G. A. Goldin, Phys. Lett. A **162** (1992) 397; J. Phys. A: Math. Gen. **27** (1994) 1771; Phys. Rev. A **54** (1996) 3764.

[17] S. Weinberg, Ann. Phys. (NY) **194**, 336 (1989).

[18] T. W. Kibble, *Comm. Math. Phys.* **64**, 73 (1978); *Comm. Math. Phys.* **65**, 189 (1979).

[19] R. Cirelli, A. Mania, and L. Pizzocchero, J. Math. Phys. **31** (1990) 2891-2897; J. Math. Phys. **31** (1990) 2898-2903;

[20] A. Ashtekar and T. A. Schilling, in *On Einstein's Path*, A. Harvey, ed., Springer, Berlin, 1998.

[21] D. C. Brody and L. P. Hughston, J. Geom. Phys. **38** (2001) 19-53.

[22] P. Bóna, Acta Phys. Slov. **50** (2000) 1, also available as math-ph/9909022.
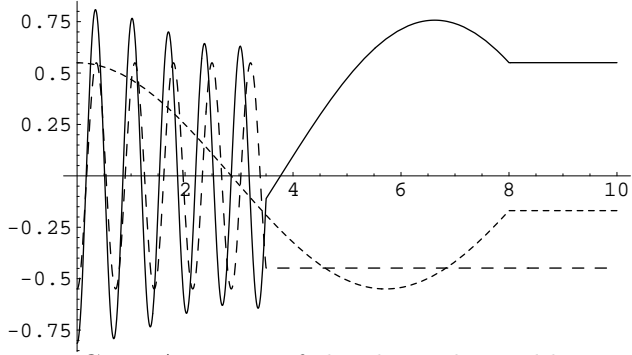
[23] A. Kent, quant-ph/0204106.

FIGURES

FIG. 1. Averages of the three observables in the open-system formulation. The dotted line shows the evolution of observable σ_x associated with particle #2 which is detected at $t = t_2 = 8$. Earlier detection of particle #1 at $t_1 = 3.5$ does not influence particle #2.
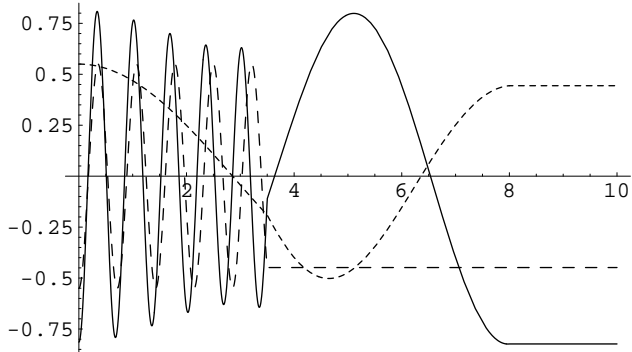

FIG. 2. Averages of the three observables in the standard projection-at-a-distance formulation. Measurement at $t = t_1 = 3.5$ performed on particle #1 nonlocally influences the behaviour of particle #2. As opposed to the plot from Fig. 1 the dotted line is modified at $t = 3.5$. This is Gisin-type nonlocality.