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Dedicated to the Memory of Professor Zdzisław Kamont

DIFFERENCE FUNCTIONAL INEQUALITIES AND APPLICATIONS

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Abstract. The paper deals with the difference inequalities generated by initial boundary value problems for hyperbolic nonlinear differential functional systems. We apply this result to investigate the stability of constructed difference schemes. The proof of the convergence of the difference method is based on the comparison technique, and the result for difference functional inequalities is used. Numerical examples are presented.

Keywords: initial boundary value problems, difference functional inequalities, difference methods, stability and convergence, interpolating operators, error estimates.

Mathematics Subject Classification: 35R10, 65M12, 65M15.

1. INTRODUCTION

The classical theory of partial differential inequalities has applications in several differential problems. As basic examples we can give: estimates of solutions of partial differential equations, estimates of the domain of the existence of classical or generalized solutions, criteria of uniqueness and continuous dependence. Difference inequalities, or in other words a discrete version of differential inequalities, are frequently used to prove the convergence of the numerical schemes.

The papers [7, 16] initiated the investigation of difference inequalities generated by the first order partial differential equations. The results presented in [7, 16] were extended on functional differential problems in papers [2, 17] and in [12–14] were generalized on differential and differential functional equations of parabolic type. In the mentioned papers explicit difference schemes were considered. We use in the paper general ideas for finite difference equations which can be found in [18, 19].

We formulate our functional differential problem. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X to Y. We denote by $\mathbb{R}^{k\times n}$ the space of real $k \times n$ matrices. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $p = (p_1, \ldots, p_k) \in \mathbb{R}^k$ and for the matrix $U \in \mathbb{R}^{k\times n}$, $U = [u_{ij}]_{i=1,\ldots,k, j=1,\ldots,n}$ we write

$$\|x\| = \sum_{i=1}^{n} |x_i| \quad \text{and} \quad x \diamond y = (x_1 y_1, \ldots, x_n y_n),$$

$$\|p\|_\infty = \max\{|p_i| : 1 \le i \le k\}, \quad \|U\| = \max\Big\{\sum_{j=1}^{n} |u_{ij}| : 1 \le i \le k\Big\}.$$

For each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we write $x = (x', x'')$ where $x' = (x_1, \ldots, x_\kappa)$, $x'' = (x_{\kappa+1}, \ldots, x_n)$, where $0 \le \kappa \le n$ is fixed. If $\kappa = n$ we have $x' = x$, if $\kappa = 0$ then $x'' = x$. Let $\mathbb{R}_+ = [0, \infty)$. Suppose that $a > 0$, $d_0 \in \mathbb{R}_+$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n_+$ and $d = (d_1, \ldots, d_n) \in \mathbb{R}^n_+$, are given. We define the sets

$$E = [0, a] \times [-b', b') \times (-b'', b''], \quad D = [-d_0, 0] \times [0, d'] \times [-d'', 0].$$

Let $c = (c_1, \ldots, c_n) = b + d$ and

$$E_0 = [-d_0, 0] \times [-b', c'] \times [-c'', b''], \quad \partial_0 E = ((0, a] \times [-b', c'] \times [-c'', b'']) \setminus E,$$

$$E^* = E_0 \cup E \cup \partial_0 E.$$

Suppose that $z : E^* \to \mathbb{R}^k$, $z = (z_1, \ldots, z_k)$, and $(t, x) \in E$. We define the function $z_{(t,x)} : D \to \mathbb{R}^k$ as follows

$$z_{(t,x)}(\tau, y) = z(t + \tau, x + y), \ \ (\tau, y) \in D.$$

The function $z_{(t,x)}$ is the restriction of z to $[t - d_0, t] \times [x', x' + d'] \times [x'' - d'', x'']$ and this restriction is shifted to the set D. For a function $w \in C(D, \mathbb{R}^k)$ we define

$$\|w\|_D = \max\{\|w(t, x)\|_\infty : (t, x) \in D\}.$$

Put $\Omega = E \times \mathbb{R}^k \times C(D, \mathbb{R}^k) \times \mathbb{R}^n$ and suppose that

$$f : \Omega \to \mathbb{R}^k, \quad f = (f_1, \ldots, f_k),$$

$$\varphi : E_0 \cup \partial_0 E \to \mathbb{R}^k, \quad \varphi = (\varphi_1, \ldots, \varphi_k)$$

are given functions. We consider the system of differential functional equations

$$\partial_t z_i(t, x) = f_i(t, x, z(t, x), z_{(t,x)}, \partial_x z_i(t, x)), \quad 1 \le i \le k, \tag{1.1}$$

with the initial boundary condition

$$z(t, x) = \varphi(t, x) \ \text{ on } \ E_0 \cup \partial_0 E, \tag{1.2}$$

where $\partial_x z_i = (\partial_{x_1} z_i, \ldots, \partial_{x_n} z_i)$.

Systems of differential equations with deviated variables and differential integral problems can be derived from (1.1) by specializing the operator $f = (f_1, \ldots, f_k)$. Difference methods described in the paper have the potential for application in the numerical solution of the above problems.

A function $v : E^* \to \mathbb{R}^k$ is a classical solution of (1.1), (1.2) if:

(i) $v \in C(E^*, \mathbb{R}^k)$ and v is of class C^1 on E,
(ii) $v = (v_1, \ldots, v_k)$ satisfies the system of equations (1.1) on E and condition (1.2) holds.

We are interested in establishing a method of numerical approximation of classical solutions to problem (1.1), (1.2) with solutions of associated difference schemes and in estimating the difference between these solutions.

We formulate a class of difference schemes for (1.1), (1.2). Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers, respectively. We define a mesh on E^* and D in the following way. Let (h_0, h'), $h' = (h_1, \ldots, h_n)$, stand for steps of the mesh. For $h = (h_0, h')$ and $(r, m) \in \mathbb{Z}^{1+r}$, where $m = (m_1, \ldots, m_n)$, we define nodal points as follows

$$t^{(r)} = rh_0, \;\; x^{(m)} = (x_1^{(m_1)}, \ldots, x_n^{(m_n)}) = m \diamond h'.$$

Let us denote by H the set of all $h = (h_0, h')$ such that there are $K_0 \in \mathbb{Z}$ and $N = (N_1, \ldots, N_n) \in \mathbb{N}^n$ with the properties $K_0 h_0 = d_0$ and $N \diamond h' = d$. Let $K \in \mathbb{N}$ be defined by the relations $Kh_0 \le a < (K+1)h_0$. Write

$$R_h^{1+n} = \{(t^{(r)}, x^{(m)}) : (r, m) \in \mathbb{Z}^{1+n}\}$$

and

$$E_h = E \cap R_h^{1+n}, \;\; E_{h.0} = E_0 \cap R_h^{1+n}, \;\; D_h = D \cap R_h^{1+n},$$

$$\partial_0 E_h = \partial_0 E \cap R_h^{1+n}, \;\; E_h^* = E_{h.0} \cup E_h \cup \partial_0 E_h.$$

Moreover, we put $I = [0, a]$ and

$$I_h = \{t^{(r)} : 0 \le r \le K\}, \;\; I'_h = I_h \setminus \{t^{(K)}\}.$$

Set

$$E'_h = \{(t^{(r)}, x^{(m)}) \in E_h : \; 0 \le r \le K - 1\}$$

and

$$E_{h.r}^* = E_h^* \cap ([-d_0, t^{(r)}] \times \mathbb{R}^n).$$

For functions $z : E_h^* \to \mathbb{R}^k$ and $w : D_h \to \mathbb{R}^k$ we write

$$z^{(r,m)} = z(t^{(r)}, x^{(m)}) \;\text{ on }\; E_h^* \quad \text{and} \quad w^{(r,m)} = w(t^{(r)}, x^{(m)}) \;\text{ on }\; D_h.$$

For the above z and for a point $(t^{(r)}, x^{(m)}) \in E_h$ we define the function $z_{[r,m]} : D_h \to \mathbb{R}^k$ by the formula

$$z_{[r,m]}(\tau, y) = z(t^{(r)} + \tau, x^{(m)} + y), \;\; (\tau, y) \in D_h.$$

We write

$$x^{(m')} = (x_1^{(m_1)}, \ldots, x_\kappa^{(m_\kappa)}), \quad x^{(m'')} = (x_{\kappa+1}^{(m_{\kappa+1})}, \ldots, x_n^{(m_n)}).$$

The function $z_{[r,m]}$ is the restriction of z to the set

$$([t^{(r)} - d_0, t^{(r)}] \times [x^{(m')}, x^{(m')} + d'] \times [x^{(m'')} - d'', x^{(m'')}]) \cap R_h^{1+n}$$

and this restriction is shifted to the set D_h. For a function $w : D_h \to \mathbb{R}^k$ we write

$$\|w\|_{D_h} = \max\{\|w^{(r,m)}\|_\infty : (t^{(r)}, x^{(m)}) \in D_h\}.$$

Let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$, $1 \le j \le n$, where 1 is the j-th coordinate. We consider difference operators δ_0 and $\delta = (\delta_1, \ldots, \delta_n)$ defined in the following way. Suppose that $\omega : E_h^* \to \mathbb{R}$, we put

$$\delta_0 \omega^{(r,m)} = \frac{1}{h_0}\Big(\omega^{(r+1,m)} - \omega^{(r,m)}\Big) \tag{1.3}$$

and

$$\delta_j \omega^{(r,m)} = \frac{1}{h_j}\Big(\omega^{(r,m+e_j)} - \omega^{(r,m)}\Big), \quad 1 \le j \le \kappa, \tag{1.4}$$

$$\delta_j \omega^{(r,m)} = \frac{1}{h_j}\Big(\omega^{(r,m)} - \omega^{(r,m-e_j)}\Big), \quad \kappa + 1 \le j \le n. \tag{1.5}$$

Note that $\delta\omega^{(r,m)}$ is given by (1.5) if $\kappa = 0$ and $\delta\omega^{(r,m)}$ is defined by (1.4) for $\kappa = n$.

Let us denote by $F(X, Y)$ the class of all functions defined on X and taking values in Y, where X and Y are arbitrary sets. Put $\Omega_h = E'_h \times \mathbb{R}^k \times F(D_h, \mathbb{R}^k) \times \mathbb{R}^n$ and suppose that

$$f_h : \Omega_h \to \mathbb{R}^k, \quad f_h = (f_{h.1}, \ldots, f_{h.k}),$$
$$\varphi_h : E_{h.0} \cup \partial_0 E_h \to \mathbb{R}^k, \quad \varphi_h = (\varphi_{h.1}, \ldots, \varphi_{h.k})$$

are given functions. Write

$$\delta_0 z = (\delta_0 z_1, \ldots, \delta_0 z_k),$$
$$F_h[z]^{(r,m)} = (F_{h.1}[z]^{(r,m)}, \ldots, F_{h.k}[z]^{(r,m)})$$

and

$$F_{h.i}[z]^{(r,m)} = f_{h.i}(t^{(r)}, x^{(m)}, z^{(r,m)}, z_{[r,m]}, s_i \delta z_i^{(r,m)} + (1 - s_i)\delta z_i^{(r+1,m)}), \quad 1 \le i \le k,$$

where

$$s_i \delta z_i^{(r,m)} = (s_{i1}\delta_1 z_i^{(r,m)}, \ldots, s_{in}\delta_n z_i^{(r,m)}),$$
$$(1 - s_i)\delta z_i^{(r+1,m)} = ((1 - s_{i1})\delta_1 z_i^{(r+1,m)}, \ldots, (1 - s_{in})\delta_n z_i^{(r+1,m)})$$

and where $0 \le s_{ij} \le 1$, $i = 1, \ldots, k$, $j = 1, \ldots, n$, are given constants. We consider the difference functional system

$$\delta_0 z^{(r,m)} = F_h[z]^{(r,m)} \tag{1.6}$$

with initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \ \text{ on } \ E_{h.0} \cup \partial_0 E_h. \tag{1.7}$$

The above difference method has the following property: each equation in system (1.6) contains the parameter $s_i = (s_{i1}, \ldots, s_{in})$, $1 \le i \le k$. If $s_i = (1, \ldots, 1) \in \mathbb{R}^n$ for $1 \le i \le k$ then (1.6), (1.7) reduces to the explicit difference scheme. It is clear that there exists exactly one solution of problem (1.6), (1.7) in this case. Sufficient conditions for the convergence of the explicit difference methods for first order partial differential equations can be found in the monograph [5] (Chapter V), see also [1, 2].

Note that if $k = 1$ and $s = (s_1, \ldots, s_n) = (0, \ldots, 0) \in \mathbb{R}^n$ then (1.6), (1.7) reduces to the strong implicit difference scheme considered in [6].

Numerical methods for nonlinear parabolic problems were investigated in [8–11]. Difference schemes considered in the above papers depend on two parameters s, $\widetilde{s} \in [0, 1]$. Right hand sides of difference equations corresponding to parabolic equations contain the expressions

$$s\delta z^{(r,m)} + (1 - s)\delta z^{(r+1,m)} \quad \text{and} \quad \widetilde{s}\delta^{(2)} z^{(r,m)} + (1 - \widetilde{s})\delta^{(2)} z^{(r+1,m)},$$

where $\delta = (\delta_1, \ldots, \delta_n)$ and $\delta^{(2)} = [\delta_{ij}]_{i,j=1,\ldots,n}$ are difference operators corresponding to the partial derivatives $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$ and $\partial_{xx} = [\partial_{x_i x_j}]_{i,j=1\ldots,n}$ and z is a scalar unknown function. Our results are motivated by papers [8–11].

In the paper we first investigate difference functional inequalities generated by mixed problems for nonlinear equations. We use this results to prove the theorem on the error estimates of approximate solutions for functional difference schemes. The proof of the convergence is based on the stability of difference equation with initial condition. It is well known that with the adopted proof technique the convergence is equivalent to the stability of difference schemes.

In theorems on the convergence of explicit functional difference methods for (1.1), (1.2) we need assumptions on the mesh which are known as the (CFL) conditions. In our investigations we need the (CFL) condition which depends on the values of parameter s. The conclusion of the theoretical analysis carried out in the paper is that only the strong implicit method is unconditionally stable. The same conclusions can be found in the papers [8–11] for nonlinear parabolic partial differential equations of second order. Nevertheless the numerical experiments show that the (CFL) condition, required in the adopted method of the proof, is only sufficient but not necessary.

The paper is organized as follows. Section 2 deals with a comparison results for the difference functional inequalities. Estimation of the difference between exact and approximate solutions can be found in Section 3. A convergence theorem and an error estimate for the difference methods are presented in Section 4. At the end of the paper we give numerical examples.

2. DIFFERENCE FUNCTIONAL INEQUALITIES

We begin with the maximum principle for implicit difference functional inequalities generated by (1.6), (1.7). Write

$$B = [-b', b') \times (-b'', b''], \quad B^* = [-b', c'] \times [-c'', b'']$$

and

$$R^n_h = \{x^{(m)} : m \in \mathbb{Z}^n\}$$

where $h \in H$. We consider the sets

$$B_h = B \cap R^n_h, \ \ B^*_h = B^* \cap R^n_h$$

and $\partial_0 B_h = B^*_h \setminus B_h$. Define $\theta = (\theta_1, \ldots, \theta_n)$, where

$$\theta_j = 1 \ \text{ for } \ 1 \le j \le \kappa \ \text{ and } \ \theta_j = -1 \ \text{ for } \ \kappa + 1 \le j \le n.$$

Theorem 2.1. *Suppose that $0 \le r \le K - 1$ is fixed and the function $\lambda_h : E_h \times F(D_h, \mathbb{R}^k) \to \mathbb{R}^{k \times n}$, $\lambda_h = [\lambda_{h.ij}]_{i=1,\ldots,k, j=1,\ldots,n}$, is such that for $i = 1, \ldots, k$ we have*

$$\lambda_{h.i}(t, x, w) \diamond \theta \ge 0 \ \ \text{on} \ \ E_h \times F(D_h, \mathbb{R}^k),$$

where $\lambda_{h.i} = (\lambda_{h.i1}, \ldots, \lambda_{h.in})$.

(I) *If $w : E^*_h \to \mathbb{R}^k$, $w = (w_1, \ldots, w_k)$, satisfies the difference inequalities*

$$w_i^{(r+1,m)} \le h_0 \sum_{j=1}^{n} \lambda_{h.ij}(t^{(r)}, x^{(m)}, w_{[r,m]})(1 - s_{ij})\delta_j w_i^{(r+1,m)}, \quad 1 \le i \le k,$$

for $x^{(m)} \in B_h$ and $\mu^{(i)} \in \mathbb{Z}^n$, $\mu^{(i)} = (\mu_1^{(i)}, \ldots, \mu_n^{(i)})$, is such that

$$w_i^{(r+1,\mu^{(i)})} = M^{(i)} \quad \text{for} \quad 1 \le i \le k,$$

where

$$M^{(i)} = \max\{w_i^{(r+1,m)} : x^{(m)} \in B^*_h\} \ \ \text{and} \ \ M^{(i)} > 0, \quad 1 \le i \le k, \tag{2.1}$$

then $x^{(\mu^{(i)})} \in \partial_0 B_h$.

(II) *If $w : E^*_h \to \mathbb{R}^k$, $w = (w_1, \ldots, w_k)$, satisfies the difference inequalities*

$$w_i^{(r+1,m)} \ge h_0 \sum_{j=1}^{n} \lambda_{h.ij}(t^{(r)}, x^{(m)}, w)(1 - s_{ij})\delta_j w_i^{(r+1,m)}, \quad 1 \le i \le k,$$

for $x^{(m)} \in B_h$ and $\tilde{\mu}^{(i)} \in \mathbb{Z}^n$, $\tilde{\mu}^{(i)} = (\tilde{\mu}_1^{(i)}, \ldots, \tilde{\mu}_j^{(i)})$, is such that

$$w_i^{(r+1,\tilde{\mu}^{(i)})} = \tilde{M}^{(i)},$$

where

$$\tilde{M}^{(i)} = \min\{w_i^{(r+1,m)} : x^{(m)} \in B^*_h\} \ \ \text{and} \ \ \tilde{M}^{(i)} < 0, \quad 1 \le i \le k,$$

then $x^{(\tilde{\mu}^{(i)})} \in \partial_0 B_h$.

Proof. Consider the case (I). Suppose that i is fixed, $1 \le i \le k$ and $x^{(\mu^{(i)})} \in B_h$. Then

$$w_i^{(r+1,\mu^{(i)})} \le$$

$$\le h_0 \sum_{j=1}^{\kappa} \frac{1}{h_j} \lambda_{h.ij}(t^{(r)}, x^{(\mu^{(i)})}, w_{[r,\mu^{(i)}]})(1 - s_{ij})\Big(w_i^{(r+1,\mu^{(i)}+e_j)} - w_i^{(r+1,\mu^{(i)})}\Big) +$$

$$+ h_0 \sum_{j=\kappa+1}^{n} \frac{1}{h_j} \lambda_{h.ij}(t^{(r)}, x^{(\mu^{(i)})}, w_{[r,\mu^{(i)}]})(1 - s_{ij})\Big(w_i^{(r+1,\mu^{(i)})} - w_i^{(r+1,\mu^{(i)}-e_j)}\Big).$$

This gives

$$M^{(i)} \Big[1 + h_0 \sum_{j=1}^{n} \frac{1}{h_j} (1 - s_{ij}) |\lambda_{h.ij}(t^{(r)}, x^{(\mu^{(i)})}, w_{[r,\mu^{(i)}]})| \Big] \le$$

$$\le h_0 \sum_{j=1}^{\kappa} \frac{1}{h_j} \lambda_{h.ij}(t^{(r)}, x^{(\mu^{(i)})}, w_{[r,\mu^{(i)}]})(1 - s_{ij}) w_i^{(r+1,\mu^{(i)}+e_j)} -$$

$$- h_0 \sum_{j=\kappa+1}^{n} \frac{1}{h_j} \lambda_{h.ij}(t^{(r)}, x^{(\mu^{(i)})}, w_{[r,\mu^{(i)}]})(1 - s_{ij}) w_i^{(r+1,\mu^{(i)}-e_j)} \le$$

$$\le h_0 M^{(i)} \sum_{j=1}^{n} \frac{1}{h_j} (1 - s_{ij}) |\lambda_{h.ij}(t^{(r)}, x^{(\mu^{(i)})}, w_{[r,\mu^{(i)}]})|.$$

We thus get $M^{(i)} \le 0$ which contradicts (2.1). Then $x^{(\mu^{(i)})} \in \partial_0 B_h$, which is our claim. In a similar way we prove that $x^{(\tilde{\mu}^{(i)})} \in \partial_0 B_h$ for $1 \le i \le k$ in the case of (II). This completes the proof. □

Assumption H[f_h]. Suppose that the function $f_h : \Omega_h \to \mathbb{R}^k$, $f_h = (f_{h.1}, \dots, f_{h.k})$, of the variables (t, x, p, w, q) is such that:

1) f_h is nondecreasing with respect to the functional variable and for each $P = (t, x, p, w, q) \in \Omega_h$ there exist partial derivatives

$$\partial_p f_h(P) = (\partial_{p_1} f_{h.1}(P), \dots, \partial_{p_k} f_{h.k}(P)), \quad \partial_q f_h(P) = [\partial_{q_j} f_{h.i}(P)]_{i=1,\dots,k, j=1,\dots,n}$$

and

$$\partial_{p_i} f_{h.i}(t, x, \cdot, w, q) \in C(\mathbb{R}^k, \mathbb{R}),\ 1 \le i \le k,$$
$$\partial_q f_h(t, x, p, w, \cdot) \in C(\mathbb{R}^n, \mathbb{R}^{k \times n}),$$

2) for each $P \in \Omega_h$ and for $i = 1, \dots, k$ we have

$$\partial_q f_{h.i}(P) \diamond \theta \ge 0$$

and

$$1 + h_0 \partial_{p_i} f_{h.i}(P) - h_0 \sum_{j=1}^{n} \frac{1}{h_j} s_{ij} |\partial_{q_j} f_{h.i}(P)| \ge 0, \tag{2.2}$$

3) the function f_h satisfies the following monotonicity condition with respect to the variable p: for each i, $1 \le i \le k$, if $p \le \bar{p}$, $p = (p_1, \ldots, p_k)$, $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_k)$, and $p_i = \bar{p}_i$, then $f_{h.i}(t, x, p, w, q) \le f_{h.i}(t, x, \bar{p}, w, q)$.

Remark 2.2. The assumption (2.2) is called the Courant-Friedrichs-Levy (CFL) condition for problem (1.6)-(1.7) (see [3, Chapter III] and [5, Chapter V]).

Now we formulate the theorem on functional difference inequalities.

Theorem 2.3. *Suppose that Assumption* H$[f_h]$ *is satisfied and the functions* $u, v : E_h^* \to \mathbb{R}^k$ *are such that the implicit difference inequalities*

$$\delta_0 u^{(r,m)} - F_h[u]^{(r,m)} \le \delta_0 v^{(r,m)} - F_h[v]^{(r,m)}, \quad (t^{(r)}, x^{(m)}) \in E_h', \tag{2.3}$$

and the initial boundary estimates

$$u^{(r,m)} \le v^{(r,m)} \quad \text{on} \quad E_{h.0} \cup \partial_0 E_h, \tag{2.4}$$

are satisfied.

Then

$$u^{(r,m)} \le v^{(r,m)} \quad \text{on} \quad E_h^*. \tag{2.5}$$

Proof. We prove (2.5) by induction on r. It follows from (2.4) that assertion (2.5) is satisfied for $r = 0$ and $(t^{(0)}, x^{(m)}) \in E_h^*$. Suppose that $u_i^{(\tau,m)} \le v_i^{(\tau,m)}$ for $(t^{(\tau)}, x^{(m)}) \in E_{h.r}^*$, where $0 \le r < K$ and for $i = 1, \ldots, k$. Assume now that i is fixed, $1 \le i \le k$. It follows easily that

$$\begin{aligned}
(u_i - v_i)^{(r+1,m)} &\le (u_i - v_i)^{(r,m)} + \\
&+ h_0 \Big[f_{h.i}(t^{(r)}, x^{(m)}, u^{(r,m)}, u_{[r,m]}, s_i \delta u_i^{(r,m)} + (1 - s_i) \delta u_i^{(r+1,m)}) - \\
&- f_{h.i}(t^{(r)}, x^{(m)}, v^{(r,m)}, v_{[r,m]}, s_i \delta v_i^{(r,m)} + (1 - s_i) \delta v_i^{(r+1,m)}) \Big] \le \\
&\le (u_i - v_i)^{(r,m)} \left(1 + h_0 \partial_{p_i} f_{h.i}(P) - h_0 \sum_{j=1}^{n} \frac{1}{h_j} s_{ij} |\partial_{q_j} f_{h.i}(P)| \right) + \\
&+ h_0 \Big[f_{h.i}(t^{(r)}, x^{(m)}, v^{(r,m)}, u_{[r,m]}, s_i \delta v_i^{(r,m)} + (1 - s_i) \delta v_i^{(r+1,m)}) - \\
&- f_{h.i}(t^{(r)}, x^{(m)}, v^{(r,m)}, v_{[r,m]}, s_i \delta v_i^{(r,m)} + (1 - s_i) \delta v_i^{(r+1,m)}) \Big] + \\
&+ h_0 \sum_{j=1}^{\kappa} \frac{1}{h_j} s_{ij} \partial_{q_j} f_{h.i}(P) (u_i - v_i)^{(r,m+e_j)} - h_0 \sum_{j=\kappa+1}^{n} \frac{1}{h_j} s_{ij} \partial_{q_j} f_{h.i}(P) (u_i - v_i)^{(r,m-e_j)} + \\
&+ h_0 \sum_{j=1}^{n} \partial_{q_j} f_{h.i}(P) (1 - s_{ij}) \delta_j (u_i - v_i)^{(r+1,m)},
\end{aligned}$$

where $x^{(m)} \in B_h$ and $P \in \Omega_h$ is an intermediate point. We thus get

$$(u_i - v_i)^{(r+1,m)} \le h_0 \sum_{j=1}^{n} \partial_{q_j} f_{h.i}(P) (1 - s_{ij}) \delta_j (u_i - v_i)^{(r+1,m)},$$

where $x^{(m)} \in B_h$. It follows from (2.4) and from Theorem 2.1 that $(u_i - v_i)^{(r+1,m)} \le 0$ for $x^{(m)} \in \partial_0 B_h$. Then we obtain (2.5) by induction and the theorem follows. □

3. APPROXIMATE SOLUTIONS OF DIFFERENCE FUNCTIONAL EQUATIONS

We first prove that there exists exactly one solution $u_h : E_h^* \to \mathbb{R}^k$ of the problem (1.6), (1.7). For each $x^{(m)} \in B_h$ we put

$$\Delta^{(m)} = \{x^{(m+e_j)} : 1 \le j \le \kappa\} \cup \{x^{(m-e_j)} : \kappa + 1 \le j \le n\}.$$

Lemma 3.1. *If Assumption* H$[f_h]$ *is satisfied and* $\varphi_h : E_{h.0} \cup \partial_0 E_h \to \mathbb{R}^k$ *then there exists exactly one solution* $u_h : E_h^* \to \mathbb{R}^k$ *of* (1.6), (1.7).

Proof. Is follows from (1.7) that u_h is defined on $E_{h.0}$. Suppose that $0 \le r < K$ is fixed and that $u_{h.i}$ is defined on $(t^{(\tau)}, x^{(m)}) \in E_{h.r}^*$ for $1 \le i \le k$. We assume also that i is fixed, $1 \le i \le k$. Consider the problem

$$z_i^{(r+1,m)} = u_{h.i}^{(r,m)} + h_0 f_{h.i}(t^{(r)}, x^{(m)}, u_h^{(r,m)}, (u_h)_{[r,m]}, s_i \delta u_{h.i}^{(r,m)} + (1 - s_i)\delta z_i^{(r+1,m)}) \tag{3.1}$$

$$z_i^{(r+1,m)} = \varphi_{h.i}^{(r+1,m)} \quad \text{for} \quad x^{(m)} \in \partial_0 B_h. \tag{3.2}$$

Suppose now that the numbers $u_{h.i}(t^{(r+1)}, y)$ where $y \in \Delta^{(m)}$ are known. Write

$$\psi_i(\tau) = u_{h.i}^{(r,m)} + h_0 f_{h.i}(t^{(r)}, x^{(m)}, u_h^{(r,m)}, (u_h)_{[r,m]}, Q_i^{(r+1,m)}(\tau)),$$

where

$$\begin{aligned} Q_i^{(r+1,m)}(\tau) = \Big(&\frac{1}{h_1}\Big(s_{i1}(u_{h.i}^{(r,m+e_1)} - u_{h.i}^{(r,m)}) + (1 - s_{i1})(u_{h.i}^{(r+1,m+e_1)} - \tau)\Big), \dots, \\ &\frac{1}{h_\kappa}\Big(s_{i\kappa}(u_{h.i}^{(r,m+e_\kappa)} - u_{h.i}^{(r,m)}) + (1 - s_{i\kappa})(u_{h.i}^{(r+1,m+e_\kappa)} - \tau)\Big), \\ &\frac{1}{h_{\kappa+1}}\Big(s_{i\kappa+1}(u_{h.i}^{(r,m)} - u_{h.i}^{(r,m-e_{\kappa+1})}) + (1 - s_{i\kappa+1})(\tau - u_{h.i}^{(r+1,m-e_{\kappa+1})})\Big), \dots, \\ &\frac{1}{h_n}\Big(s_{in}(u_{h.i}^{(r,m)} - u_{h.i}^{(r,m-e_n)}) + (1 - s_{in})(\tau - u_{h.i}^{(r+1,m-e_n)})\Big)\Big), \end{aligned}$$

Then $\psi = (\psi_1, \dots, \psi_k) : \mathbb{R} \to \mathbb{R}^k$ is of class C^1 and

$$\psi_i'(\tau) = -h_0 \sum_{j=1}^{n} \frac{1}{h_j}(1 - s_{ij})|\partial_{q_j} f_{h.i}(t^{(r)}, x^{(m)}, u_h^{(r,m)}, (u_h)_{[r,m]}, Q_i^{(r+1,m)}(\tau))| \le 0$$

for $\tau \in \mathbb{R}$. Then equation $\tau = \psi_i(\tau)$ has exactly one solution and consequently the number $u_{h.i}^{(r+1,m)}$ can be calculated. Since $u_{h.i}^{(r+1,m)}$ is given for $x^{(m)} \in \partial_0 B_h$ it follows that there exists exactly one solution $u_{h.i}^{(r+1,m)}$ of (3.1), (3.2) for $x^{(m)} \in B_h$. Then $u_{h.i}$ is defined on $E^*_{h.r+1}$. Then by induction the solution u_h exists and it is unique on E^*_h.

Suppose that the functions $v_h : E^*_h \to \mathbb{R}^k$ and $\alpha_0, \gamma : H \to \mathbb{R}_+$ are such that for $i = 1, \ldots, k$ we have

$$\|\delta_0 v_h^{(r,m)} - F_h[v_h]^{(r,m)}\|_\infty \leq \gamma(h) \;\text{ on }\; E'_h, \tag{3.3}$$

$$\|\varphi_h^{(r,m)} - v_h^{(r,m)}\|_\infty \leq \alpha_0(h) \;\text{ on }\; E_{h.0} \cup \partial_0 E_h \tag{3.4}$$

and

$$\lim_{h\to 0} \gamma(h) = 0, \;\; \lim_{h\to 0} \alpha_0(h) = 0. \tag{3.5}$$

The function v_h satisfying the above relations can be considered as an approximate solution to the problem (1.6), (1.7). □

Assumption H$[f_h, \sigma_h]$. Suppose that Assumption H$[f_h]$ is satisfied and there is $\sigma_h : I'_h \times \mathbb{R}_+ \to \mathbb{R}_+$ such that:

1) for each $t \in I'_h$ the function $\sigma_h(t, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing,
2) $\sigma_h(t, 0) = 0$ for $t \in I'_h$ and the difference problem,

$$\eta^{(r+1)} = \eta^{(r)} + h_0 \sigma_h(t^{(r)}, \eta^{(r)}), \;\; 0 \leq r \leq K-1, \tag{3.6}$$

$$\eta^{(0)} = 0 \tag{3.7}$$

is stable in the following sense: if $\tilde{\gamma}, \tilde{\alpha}_0 : H \to \mathbb{R}_+$ are functions such that

$$\lim_{h\to 0} \tilde{\gamma}(h) = 0, \;\; \lim_{h\to 0} \tilde{\alpha}_0(h) = 0$$

and $\eta_h : I_h \to \mathbb{R}_+$ is a solution of the problem

$$\eta^{(r+1)} = \eta^{(r)} + h_0 \sigma_h(t^{(r)}, \eta^{(r)}) + h_0 \tilde{\gamma}(h), \;\; 0 \leq r \leq K-1, \tag{3.8}$$

$$\eta^{(0)} = \tilde{\alpha}_0(h), \tag{3.9}$$

then there is $\alpha : H \to \mathbb{R}_+$ such that

$$\eta_h^{(r)} \leq \alpha(h) \tag{3.10}$$

for $0 \leq r \leq K$ and $\lim_{h\to 0} \alpha(h) = 0$,
3) if $w \geq \overline{w}$, $p \geq \overline{p}$ for $w, \overline{w} \in F(D_h, \mathbb{R}^k)$, $p, \overline{p} \in \mathbb{R}^k$ then for $i = 1, \ldots, k$ we have

$$f_{h.i}(t^{(r)}, x^{(m)}, p, w, q) - f_{h.i}(t^{(r)}, x^{(m)}, \overline{p}, \overline{w}, q) \leq$$
$$\leq \sigma_h(t^{(r)}, \max\{\|p - \overline{p}\|_\infty, \|w - \overline{w}\|_{D_h}\}).$$

We give the theorem on the estimate of the difference between the exact and approximate solutions to the problem (1.6), (1.7).

Theorem 3.2. *Suppose that Assumption* $\mathrm{H}[f_h, \sigma_h]$ *is satisfied and:*

1) $\varphi_h : E_{h.0} \cup \partial_0 E_h \to \mathbb{R}^k$ *is a given function and* $u_h : E_h^* \to \mathbb{R}^k$ *is a solution of the problem* (1.6), (1.7),
2) *the functions* $v_h : E_h^* \to \mathbb{R}^k$ *and* $\gamma, \alpha_0 : H \to \mathbb{R}_+$ *are such that relations* (3.3)–(3.5) *hold.*

Then there exists a function $\alpha : H \to \mathbb{R}_+$ *such that*

$$\|u_h^{(r,m)} - v_h^{(r,m)}\|_\infty \le \alpha(h) \ \text{ on } E_h \ \text{ and } \lim_{h\to 0} \alpha(h) = 0. \tag{3.11}$$

Proof. Let the function $C_h = (C_{h.1}, \ldots, C_{h.k}) : E'_h \to \mathbb{R}^k$ be defined by

$$\delta_0 v_h^{(r,m)} = F_h[v_h]^{(r,m)} + C_h^{(r,m)}. \tag{3.12}$$

It follows from (3.3) that $\|C_h^{(r,m)}\|_\infty \le \gamma(h)$ on E'_h.

Suppose that i, $1 \le i \le k$, is fixed. Let the function $\omega_h : I_h \to \mathbb{R}$ be the solution of (3.8), (3.9) with $\tilde{\gamma}(h) = \gamma(h)$ and $\tilde{\alpha}_0(h) = \alpha_0(h)$ for $h \in H$. Write

$$\widetilde{v}_{h.i}^{(r,m)} = v_{h.i}^{(r,m)} + \alpha_0(h) \ \text{ on } E_{h.0},$$

$$\widetilde{v}_{h.i}^{(r,m)} = v_{h.i}^{(r,m)} + \omega_h^{(r)} \ \text{ on } E_h \cup \partial_0 E_h.$$

We prove that $\widetilde{v}_{h.i}$ satisfies the difference inequality

$$\delta_0 \widetilde{v}_{h.i}^{(r,m)} \ge F_{h.i}[\widetilde{v}_h]^{(r,m)} \ \text{ on } E'_h. \tag{3.13}$$

We conclude from Assumption $\mathrm{H}[f_h, \sigma_h]$, the mean value theorem and (3.12) that

$$\begin{aligned}
\delta_0 \widetilde{v}_{h.i}^{(r,m)} &= F_{h.i}[\widetilde{v}_h]^{(r,m)} + \frac{1}{h_0}\Big(\omega_h^{(r+1)} - \omega_h^{(r)}\Big) + C_{h.i}^{(r,m)} + \\
&\quad + \Big[f_{h.i}(t^{(r)}, x^{(m)}, v_h^{(r,m)}, (v_h)_{[r,m]}, s_i \delta v_{h.i}^{(r,m)} + (1 - s_i)\delta v_{h.i}^{(r+1,m)}) - \\
&\quad - f_{h.i}(t^{(r)}, x^{(m)}, \widetilde{v}_h^{(r,m)}, (\widetilde{v}_h)_{[r,m]}, s_i \delta v_{h.i}^{(r,m)} + (1 - s_i)\delta v_{h.i}^{(r+1,m)}) \Big] \ge \\
&\ge F_{h.i}[\widetilde{v}_h]^{(r,m)} + \frac{1}{h_0}\Big(\omega_h^{(r+1)} - \omega_h^{(r)}\Big) - \sigma_h(t^{(r)}, \omega_h^{(r)}) - \gamma(h),
\end{aligned}$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. The above inequality and (3.8) imply (3.13). By the initial boundary estimate

$$u_{h.i}^{(r,m)} \le \widetilde{v}_{h.i}^{(r,m)} \ \text{ on } E_{h.0} \cup \partial_0 E_h \tag{3.14}$$

and (3.13) and Theorem 2.3 we obtain

$$u_{h.i}^{(r,m)} \le v_{h.i}^{(r,m)} + \omega_h^{(r)} \ \text{ on } E_h. \tag{3.15}$$

In a similar manner we can see that

$$v_{h.i}^{(r,m)} - \omega_h^{(r,m)} \le u_{h.i}^{(r,m)} \ \text{ on } E_h. \tag{3.16}$$

Now we obtain the assertion of Theorem 3.2 from (3.15), (3.16) and from the stability of problem (3.6), (3.7). □

4. CONVERGENCE OF THE DIFFERENCE METHOD

We formulate sufficient conditions for the convergence of the method (1.6), (1.7). We consider a class of difference problems (1.6), (1.7) where f_h is a superposition of f and a suitable interpolating operator.

Assumption H$[T_h]$. Suppose that the operator $T_h : F(D_h, \mathbb{R}^k) \to C(D, \mathbb{R}^k)$ satisfies the conditions:

1) if $w, \widetilde{w} \in F(D_h, \mathbb{R}^k)$ then
$$\|T_h[w] - T_h[\widetilde{w}]\|_D \le \|w - \widetilde{w}\|_{D_h},$$
2) if $w, \widetilde{w} \in F(D_h, \mathbb{R}^k)$ and $w \le \widetilde{w}$, then $T_h[w] \le T_h[\widetilde{w}]$,
3) if $w : D \to \mathbb{R}^k$ is of class C^1 then there is $\gamma : H \to \mathbb{R}_+$ such that
$$\| T_h[w_h] - w \|_D \le \gamma(h) \ \text{ and } \ \lim_{h\to 0} \gamma(h) = 0,$$
where w_h is the restriction of w to the set D_h.

Later in this section we give a suitable example of the interpolating operator which satisfies the above Assumptions.

Write
$$F_h[z]^{(r,m)} = (F_{h.1}[z]^{(r,m)}, \ldots, F_{h.k}[z]^{(r,m)}),$$
$$F_{h.i}[z]^{(r,m)} = f_i(t^{(r)}, x^{(m)}, z^{(r,m)}, T_h z_{[r,m]}, s_i \delta z_i^{(r,m)} + (1 - s_i)\delta z_i^{(r+1,m)}), \quad 1 \le i \le k.$$
We will approximate solutions of (1.1), (1.2) with solutions of the difference functional equation
$$\delta_0 z^{(r,m)} = F_h[z]^{(r,m)} \tag{4.1}$$
with initial condition (1.7).

Assumption H$[f, \sigma]$. Suppose that the function $f : \Omega \to \mathbb{R}^k$, $f = (f_1, \ldots, f_k)$, of the variables (t, x, p, w, q) is such that:

1) f is nondecreasing with respect to the functional variable and for each $P = (t, x, p, w, q) \in \Omega$ there exist partial derivatives
$$\partial_p f(P) = (\partial_{p_1} f_1(P), \ldots, \partial_{p_k} f_k(P)), \quad \partial_q f(P) = [\partial_{q_j} f_i(P)]_{i=1,\ldots,k, j=1,\ldots,n}$$
and
$$\partial_{p_i} f_i(t, x, \cdot, w, q) \in C(\mathbb{R}^k, \mathbb{R}), \ 1 \le i \le k,$$
$$\partial_q f(t, x, p, w, \cdot) \in C(\mathbb{R}^n, \mathbb{R}^{k\times n}),$$
2) for each $P \in \Omega$ and for $i = 1, \ldots, k$ we have
$$\partial_q f_i(P) \diamond \theta \ge 0$$
and
$$1 + h_0 \partial_{p_i} f_i(P) - h_0 \sum_{j=1}^{n} \frac{1}{h_j} s_{ij} |\partial_{q_j} f_i(P)| \ge 0, \tag{4.2}$$

3) the function f satisfies the following monotonicity condition with respect to variable p: for each i, $1 \le i \le k$, if $p \le \bar{p}$, $p = (p_1, \ldots, p_k)$, $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_k)$, and $p_i = \bar{p}_i$, then $f_i(t, x, p, w, q) \le f_i(t, x, \bar{p}, w, q)$,
4) there is $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that:
   (i) σ is continuous and it is nondecreasing with respect to both variables,
   (ii) $\sigma(t, 0) = 0$ for $t \in [0, a]$ and for any ε, $\varepsilon_0 \ge 0$ the maximal solution of the Cauchy problem
$$\omega'(t) = \sigma(t, \omega(t)) + \varepsilon, \quad \omega(0) = \varepsilon_0 \tag{4.3}$$
   exists on $[0, a]$; moreover $\tilde{\omega}(t) = 0$, $t \in [0, a]$, is the maximal solution of this problem for $\varepsilon = \varepsilon_0 = 0$,
   (iii) for $i = 1, \ldots, k$ we have
$$f_i(t, x, p, w, q) - f_i(t, x, \overline{p}, \overline{w}, q) \le \sigma(t, \max\{\|p - \overline{p}\|_\infty, \|w - \overline{w}\|_D\})$$
   on Ω for $w \ge \overline{w}$ and $p \ge \overline{p}$.

Theorem 4.1. *Suppose that Assumption* H$[f, \sigma]$ *are satisfied and:*

1) $h \in H$ *and the function* $u_h : E_h^* \to \mathbb{R}^k$ *is a solution of* (1.7), (4.1) *and there is* $\alpha_0 : H \to \mathbb{R}_+$ *such that*
$$\|\varphi^{(r,m)} - \varphi_h^{(r,m)}\|_\infty \le \alpha_0(h) \ \text{ on } \ E_{h.0} \cup \partial_0 E_h \ \text{ and } \ \lim_{h \to 0} \alpha_0(h) = 0, \tag{4.4}$$
2) $v : E^* \to \mathbb{R}^k$ *is a solution of* (1.1), (1.2) *and* v *is of class* C^1 *on* E^*.

Then there is $\alpha : H \to \mathbb{R}_+$ *such that*
$$\|u_h^{(r,m)} - v_h^{(r,m)}\|_\infty \le \alpha(h) \ \text{ and } \ \lim_{h \to 0} \alpha(h) = 0. \tag{4.5}$$

Proof. Write
$$f_h(t, x, p, z, q) = f(t, x, p, T_h z, q) \ \text{ on } \ \Omega_h.$$
Then each coordinate of $f_h = (f_{h.1}, \ldots, f_{h.k})$ is nondecreasing with respect to the functional variable. It follows easily that for $w \in F(D_h, \mathbb{R}^k)$ we have $\|T_h w\|_D = \|w\|_{D_h}$. Then condition 3) of Assumption H$[f_h, \sigma_h]$ is satisfied with $\sigma_h(t, \omega) = \sigma(t, \omega)$, $(t, \omega) \in I'_h \times \mathbb{R}_+$. Let the function $C_h : E'_h \to \mathbb{R}^k$ be defined by (3.12). Then from Assumption H$[T_h]$ and from assumption 2) of the theorem it follows that there is $\gamma : H \to \mathbb{R}_+$ such that
$$\|C_h^{(r,m)}\|_\infty \le \gamma(h) \quad \text{on} \quad E'_h \quad \text{and} \quad \lim_{h \to 0} \gamma(h) = 0.$$
It follows that v_h is an approximate solution to (1.7), (4.1). Now we prove that the difference problem
$$\eta^{(r+1)} = \eta^{(r)} + h_0 \sigma(t^{(r)}, \eta^{(r)}) \ \text{ for } \ 0 \le r \le K - 1, \ \eta^{(0)} = 0 \tag{4.6}$$
is stable in the sense of Assumption H$[f_h, \sigma_h]$. Suppose that $\tilde{\gamma}, \tilde{\alpha}_0 : H \to \mathbb{R}_+$ are such functions that
$$\lim_{h \to 0} \tilde{\gamma}(h) = 0, \ \lim_{h \to 0} \tilde{\alpha}_0(h) = 0.$$

Consider the solution $\eta_h : I_h \to \mathbb{R}_+$ of the problem

$$\eta^{(r+1)} = \eta^{(r)} + h_0\sigma(t^{(r)}, \eta^{(r)}) + h_0\tilde{\gamma}(h), \quad 0 \le r \le K-1,$$
$$\eta^{(0)} = \tilde{\alpha}_0(h).$$

Lets denote by $\omega_h : [0,a] \to \mathbb{R}_+$ the maximal solution of the problem (4.3) with $\varepsilon := \tilde{\gamma}(h)$, $\varepsilon_0 := \tilde{\alpha}_0(h)$. It is easily seen that $\eta_h^{(r)} \le \omega_h^{(r)}$ for $0 \le r \le K$ and

$$\lim_{h\to 0} \omega_h(t) = 0 \text{ uniformly on } [0,a].$$

Then we have $\eta_h^{(r)} \le \omega_h(a)$ for $0 \le r \le K$ and the problem (4.6) is stable with $\alpha(h) := \omega_h(a)$. Then all the conditions of Theorem 3.2 are satisfied and the assertion (4.5) follows with $\alpha(h) := \omega_h(a)$ for $\tilde{\gamma} = \gamma$, $\tilde{\alpha}_0 = \alpha_0$. □

Now we give an example of the interpolating operator which satisfies the Assumption H$[T_h]$. Write

$$S_* = \{s = (s_1, \ldots, s_n) : s_i \in \{0,1\} \text{ for } 1 \le i \le n\}.$$

Let $w \in F(D_h, \mathbb{R}^k)$ and $(t,x) \in D$. There exists $(t^{(r)}, x^{(m)}) \in D_h$ such that $t^{(r)} \le t \le t^{(r+1)}$, $x^{(m)} \le x \le x^{(m+1)}$ and $(t^{(r+1)}, x^{(m+1)}) \in D_h$. Write

$$T_h[w](t,x) = \left(1 - \frac{t - t^{(r)}}{h_0}\right) \sum_{s\in S_*} w^{(r,m+s)} \left(\frac{x - x^{(m)}}{h'}\right)^s \left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-s} +$$
$$+ \frac{t - t^{(r)}}{h_0} \sum_{s\in S_*} w^{(r+1,m+s)} \left(\frac{x - x^{(m)}}{h'}\right)^s \left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-s},$$

where

$$\left(\frac{x - x^{(m)}}{h'}\right)^s = \prod_{i=1}^n \left(\frac{x_i - x_i^{(m_i)}}{h_i}\right)^{s_i},$$
$$\left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-s} = \prod_{i=1}^n \left(1 - \frac{x_i - x_i^{(m_i)}}{h_i}\right)^{1-s_i},$$

in the above formulas we take $0^0 = 1$. It is easy to see that $T_h[w] \in C(D, \mathbb{R}^k)$. The above interpolating operator has been defined in [5, Chapter 5].

Lemma 4.2. *Suppose that $w : D \to \mathbb{R}^k$ is of class C^2. Let $\tilde{C}$ be a constant such that $\|\partial_{tt} w\|_D$, $\|\partial_{tx_i} w\|_D$, $\|\partial_{x_i x_j} w\|_D \le \tilde{C}$, $i,j = 1,\ldots,n$. Then*

$$\|T_h[w_h] - w\|_D \le \tilde{C}(h_0^2 + \|h'\|^2)$$

where $\|h'\| = h_1 + \ldots + h_n$ and w_h is the restriction of w to the set D_h.

The above lemma is a consequence of Theorem 5.27 presented in [5]. We omit the proof.

Lemma 4.3. *Suppose that*

1) *the solution* $v : E^* \to \mathbb{R}^k$ *of differential problem* (1.1), (1.2) *is of class* C^2 *and the assumptions of Theorem 4.1 are satisfied with* $\sigma(t,p) = Lp$, $L > 0$,
2) *there exist* $\tilde{M}$, $\tilde{C} \in \mathbb{R}_+$ *such that*

$$\|\partial_q f(t,x,p,z,q)\| \le \tilde{M}$$

and

$$\|\partial_{x_j} v(t,x)\|, \quad \|\partial_{tt} v(t,x)\|_\infty, \quad \|\partial_{tx_j} v(t,x)\|_\infty, \quad \|\partial_{x_j x_k} v(t,x)\|_\infty \le \tilde{C}$$

on Ω *for* j, $k = 1, \ldots, n$.

Then there is a solution $u_h : E_h^* \to \mathbb{R}^k$ *of* (1.7), (4.1) *and*

$$\|u_h^{(r,m)} - v_h^{(r,m)}\|_\infty \le \widetilde{\eta}_h^{(r)} \quad \text{on } E_h, \tag{4.7}$$

where

$$\widetilde{\eta}_h^{(r)} = \alpha_0(h) e^{aL} + \bar{\gamma}(h) \frac{e^{aL} - 1}{L}$$

and

$$\bar{\gamma}(h) = \tilde{C}\Big[\frac{1}{2} h_0 + L(h_0^2 + \|h'\|^2) + (h_0 + \|h'\|)\tilde{M}\Big].$$

Proof. From the assumptions of lemma we conclude that the operators δ_0, δ satisfy the following conditions

$$\|\delta_0 v_h^{(r,m)} - \partial_t v^{(r,m)}\|_\infty \le \frac{1}{2}\tilde{C} h_0,$$

$$\|\delta_j v_h^{(r,m)} - \partial_{x_j} v^{(r,m)}\|_\infty \le \frac{1}{2}\tilde{C}\|h'\|, \quad j = 1, \ldots, n.$$

It follows from the above estimates and from Lemma 4.2 that

$$\begin{aligned}
|C_{h.i}^{(r,m)}| = |\delta_0 v_{h.i}^{(r,m)} &- \partial_t v_i^{(r,m)} + \\
&+ f_i(t^{(r)}, x^{(m)}, v^{(r,m)}, v_{(t^{(r)}, x^{(m)})}, \partial_x v_i^{(r,m)}) - \\
&- f_i(t^{(r)}, x^{(m)}, v_h^{(r,m)}, T_h[v_h]_{[r,m]}, s_i \delta v_{h.i}^{(r,m)} + (1 - s_i)\delta v_{h.i}^{(r+1,m)})| \le \bar{\gamma}(h).
\end{aligned}$$

The function $\widetilde{\eta}_h$ is a solution of the problem

$$\eta^{(r+1)} = \eta^{(r)}(1 + h_0 L) + h_0 \bar{\gamma}(h), \quad 0 \le r \le K - 1, \;\; \eta^{(0)} = \alpha_0(h).$$

Then from Theorem 3.2 we get the assertion (4.7). □

5. NUMERICAL EXAMPLES

Write

$$\begin{aligned} E &= [0, 0.5] \times [-1, 1), \\ E_0 &= \{0\} \times [-1, 1], \\ \tilde{E} &= [0, 0.25] \times [-1, 1) \times (-1, 1], \\ \tilde{E}_0 &= \{0\} \times [-1, 1] \times [-1, 1]. \end{aligned}$$

We consider the initial boundary value problems for differential integral equations with solutions defined on E and for differential equations with deviated variables with solutions defined on $\tilde{E}$.

We give the results of the approximation of solutions to the above problems by solutions of the difference methods with respect to the value of parameter.

Example 5.1. Consider the differential integral equation

$$\begin{aligned} \partial_t z(t,x) = \partial_x z(t,x) &+ \sin\left(\partial_x z(t,x) - t z(t,x)\right) + \\ &+ t \int_x^1 z(t,\zeta) d\zeta + (x-1) \int_0^t z(\tau, x) d\tau + (x - 1 - t) e^{t(x-1)} \end{aligned}$$

with the initial boundary condition

$$z(0,x) = 1, \quad x \in [-1,1], \qquad z(t,1) = 1, \quad t \in [0, 0.5].$$

The exact solution of this problem is known. It is $z(t,x) = e^{t(x-1)}$.

The difference method of the above initial boundary value problem has the form

$$\begin{aligned} z^{(r+1,m)} = z^{(r,m)} + h_0 \Bigg[& \left(s\delta z^{(r,m)} + (1-s)\delta z^{(r+1,m)} \right) + \\ & + \sin\left(\left(s\delta z^{(r,m)} + (1-s)\delta z^{(r+1,m)} \right) - t^{(r)} z^{(r,m)} \right) + \\ & + t^{(r)} \int_{x^{(m)}}^{1} T_h\left[z_{[r,m]} \right]\left(t^{(r)}, \xi \right) d\xi + \\ & + \left(x^{(m)} - 1 \right) \int_0^{t^{(r)}} T_h\left[z_{[r,m]} \right]\left(\tau, x^{(m)} \right) d\tau + \\ & + \left(x^{(m)} - t^{(r)} - 1 \right) e^{t^{(r)}\left(x^{(m)}-1\right)} \Bigg]. \end{aligned} \tag{5.1}$$

If we put in (5.1) $s = 1$ we get the explicit difference method, with $s = 0$ we have the strong implicit difference method.

The Tables 1 and 2 show the maximal values of errors for several step sizes with respect to the value of parameter s.

Table 1. Maximal values of errors

(h_0, h_1)	$s = 1$	$s = 0.7$	$s = 0.5$	$s = 0$
$(2^{-10}, 2^{-7})$	6.3686e-4	5.2564e-4	4.5320e-4	6.4022e-4
$(2^{-11}, 2^{-8})$	3.1865e-4	2.6316e-4	2.2702e-4	3.2278e-4
$(2^{-12}, 2^{-9})$	1.5938e-4	1.3166e-4	1.1362e-4	1.6214e-4
$(2^{-13}, 2^{-10})$	7.9706e-5	6.5855e-5	5.6837e-5	7.9706e-5

Table 2. Maximal values of errors, violated (CFL) condition

(h_0, h_1)	$s = 1$	$s = 0.7$	$s = 0.5$	$s = 0$
$(2^{-5}, 2^{-8})$	4.7380e+7	1.2966e-1	1.5559e-2	1.1231e-2
$(2^{-6}, 2^{-9})$	2.3352e+25	1.6332e+2	8.1333e-3	6.3779e-3
$(2^{-7}, 2^{-10})$	9.9753e+61	2.3574e+9	4.1554e-3	3.2818e-3
$(2^{-8}, 2^{-11})$	1.6760e+136	2.6961e+24	2.0996e-3	1.6763e-3

Table 2 contains results in the case when the (CFL) condition (4.2) is violated. We can observe that the explicit difference scheme ($s = 1$) is divergent, the implicit difference method with $s \in (0.5, 1)$ is not stable but with values of the weights from the interval $[0, 0.5]$ we get a difference method which is convergent.

Example 5.2. Consider the differential equation with deviated variables

$$\begin{aligned}\partial_t z(t,x,y) = \partial_x z(t,x,y) - \partial_y z(t,x,y) + \\ &+ \arctan\Big(\partial_x z(t,x,y) - \partial_y z(t,x,y) - 2t(x+y)z(t,x,z)\Big) + \\ &+ \Big(z(t, 0.5(x+y), 0.5(x-y))\Big)^3 + (x^2 - y^2 - 2tx - 2ty)z(t,x,y) - e^{3txy}\end{aligned}$$

with the initial boundary condition

$$\begin{aligned} z(0,x,y) &= 1, \quad (x,y) \in [-1,1] \times [-1,1], \\ z(t,1,y) &= e^{t(1-y^2)}, \quad t \in [0, 0.25], \ \ y \in [-1,1], \\ z(t,x,-1) &= e^{t(x^2-1)}, \quad t \in [0, 0.25], \ \ x \in [-1,1]. \end{aligned}$$

The exact solution of this problem is known. It is $z(t,x,y) = e^{t(x^2-y^2)}$. In the difference method for the above problem we put $s_1 = s_2 = s$.

The Tables 3 and 4 show maximal values of errors for several step sizes with respect to the value of weight s.

Table 3. Maximal values of errors

(h_0, h_1, h_2)	$s = 1$	$s = 0.7$	$s = 0.5$	$s = 0$
$(2^{-8}, 2^{-5}, 2^{-5})$	9.3550e-4	2.1550e-3	3.6082e-3	7.8235e-3
$(2^{-9}, 2^{-6}, 2^{-6})$	4.9769e-4	1.1258e-3	1.8542e-3	4.0655e-3
$(2^{-10}, 2^{-7}, 2^{-7})$	2.5674e-4	5.7656e-4	9.4274e-4	2.0794e-3
$(2^{-11}, 2^{-8}, 2^{-8})$	1.2840e-4	2.8932e-4	4.7148e-4	1.0420e-3

Table 4. Maximal values of errors, violated (CFL) condition

(h_0, h_1, h_2)	$s = 1$	$s = 0.7$	$s = 0.5$	$s = 0$
$(2^{-5}, 2^{-7}, 2^{-7})$	5.7513e+1	1.8724e-2	2.9517e-2	5.3660e-2
$(2^{-6}, 2^{-8}, 2^{-8})$	2.5674e+21	4.9418e-2	1.6878e-2	3.1569e-2
$(2^{-7}, 2^{-9}, 2^{-9})$	∞	3.6084e-0	8.7001e-3	1.7227e-2
$(2^{-8}, 2^{-10}, 2^{-10})$	∞	2.1976e+26	4.4728e-3	8.7446e-3

From Tables 3 and 4 we get the same conclusions as in Example 5.1.

Remark 5.3. The both examples show that the difference method we present in the paper is stable for $s \in [0, 0.5]$. We can conclude that the assumed (CFL) condition in our analysis is only sufficient but not necessary for the stability of the considered method for $s \in (0, 0.5]$. In the case when $s \in (0.5, 1]$ we have stability only if the (CFL) condition is satisfied. For $s = 0$ we have unconditional stability.

REFERENCES

[1] P. Brandi, Z. Kamont, A. Salvadori, *Approximate solutions of mixed problems for first order partial differential-functional equations*, Atti. Sem. Mat. Fis. Univ. Modena **39** (1991) 1, 277–302.

[2] P. Brandi, Z. Kamont, A. Salvadori, *Differential and differential-difference inequalities related to mixed problems for first order partial differential-functional equations*, Atti. Sem. Mat. Fis. Univ. Modena **39** (1991) 1, 255–276.

[3] E. Godlewski, P.-A. Reviart, *Numerical Approximation of Hyperbolic Systems of Conservation Laws*, Springer, New York, NY, USA, 1996.

[4] D. Gottlieb, E. Tadmor, *The CFL condition for spectral approximations to hyperbolic initial-boundary value problems*, Math. of Computat. **56** (1991), 565–588.

[5] Z. Kamont, *Hyperbolic Functional Differential Inequalities and Applications*, Kluver Acad. Publ., Dordrecht, 1999.

[6] A. Kępczyńska, *Implicit difference functional inequalities and applications*, Math. Balk. **20** (2006) 2, 167–184.

[7] Z. Kowalski, *A difference method for the non-linear partial differential equations of the first order*, Ann. Polon. Math. **18** (1966), 235–242.

[8] M. Malec, *Sur une famille bi-paramétrique de schémas des différences finies pour l'équation parabolique sans dérivées mixtes*, Ann. Polon. Math. **31** (1975), 47–54.

[9] M. Malec, *Sur une famille bi-paramétrique des schémas des différences pour les systémes paraboliques*, Bull. Acad. Polon. Sci. Sèr. Sci. Math. Astronom. Phys. **23** (1975) 8, 871–875.

[10] M. Malec, *Sur une famille biparamétrique de schémas des différences finies pour un systéme d'équations paraboliques aux dérivées mixtes et avec des conditions aux limites du type de Neumann*, Ann. Polon. Math. **32** (1976), 33–42.

[11] M. Malec, M. Rosati, *A convergent scheme for nonlinear systems of differential functional equations of parabolic type*, Rendiconti di Matematica. Serie VII **3** (1983) 2, 211–227.

[12] M. Malec, C. Mączka, W. Voigt, *Weak difference-functional inequalities and their application to the difference analogue of non-linear parabolic differential-functional equations*, Beiträge zur Numerischen Mathematik **11** (1983), 69–79.

[13] M. Malec, M. Rosati, *Weak monotonicity for nonlinear systems of functional-finite difference inequalities of parabolic type*, Rendiconti di Matematica. Serie VII **3** (1983) 1, 157–170.

[14] M. Malec, A. Schiaffino, *Méthode aux diffÞerences finies pour une équation non-linéaire differentielle fonctionnelle du type parabolique avec une condition initiale de Cauchy*, Bollettino della Unione Matemática Italiana. Serie VII. B **1** (1987) 1, 99–109.

[15] K.W. Morton, D.F. Mayers, *Numerical Solution of Partial Differential Equations*, Cambridge University Press, 1994.

[16] A. Pliś, *On difference inequalities corresponding to partial differential inequalities of the first order*, Ann. Polon. Math. **20** (1968), 179–181.

[17] K. Prządka, *Difference methods for non-linear partial differential-functional equations of first order*, Math. Nachr. **138** (1988), 105–123.

[18] A.A. Samarskii, *The theory of difference schemes*, vol. 249 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, NY, USA, 2001.

[19] A.A. Samarskii, P.P. Matus, P.N. Vabishchevich, *Difference Schemes with Operator Factors*, vol. 546 of Mathematics and Its Applications, Kluver Acad. Publ., Dordrecht, The Netherlands, 2002.

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