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Easy and hard instances of arc ranking in directed graphs

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Abstract

In this paper we deal with the arc ranking problem of directed graphs. We give some classes of graphs for which the arc ranking problem is polynomially solvable. We prove that deciding whether $\chi'_r(G) \leqslant 6$, where G is an acyclic orientation of a 3-partite graph is an NP-complete problem. In this way we answer an open question stated by Kratochvil and Tuza in 1999.

Keywords: Graph ranking; Digraph; Computational complexity; Caterpillar

1. Introduction

An *edge k-ranking* of a simple graph is a coloring of its edges with k colors such that each path connecting two edges with the same color contains an edge with a bigger color. Parallel assembly of multipart products from their components is an example of a potential application of the edge ranking problem [2,3]. In the case of the edge ranking of trees the first result was given in [4] where an $O(n \log n)$ time approximation algorithm with a worst case performance ratio of 2 was described. Now, a linear time algorithm is known for optimal edge ranking of trees [8]. On the other hand, this problem remains NP-hard in the case of general graphs [7] of multitrees [2].

A function c mapping the set of vertices of a digraph $G = (V(G), E(G))$ into the set of integers $\{1, \ldots, k\}$ is a *vertex k-ranking* of G if each directed path between two vertices with the same color contains a vertex with a greater color, where a *directed path* connecting vertices u and v is a set of arcs $(uv_1), (v_1v_2), \ldots, (v_{i-1}v_i), (v_iv)$. The symbol $\chi_r(G)$ denotes the smallest number k such that there exists a vertex k-ranking of G. The vertex ranking problem of directed graphs was introduced in [6], where it was shown that it can be solved in polynomial time in the case of oriented trees. On the other hand, deciding whether $\chi_r(G) \leqslant 3$, where G is an acyclic orientation of a planar bipartite graph is an NP-complete problem [6].

In this paper we consider the arc ranking problem of directed graphs. A *directed path* between arcs (uv) and $(u'v')$ is any set of arcs $(v_1v_2), \ldots, (v_{i-1}v_i)$ such that $v_1 \in \{u, v\}$ and $v_i \in \{u', v'\}$, or $v_1 \in \{u', v'\}$ and $v_i \in \{u, v\}$. Then, function $c : E(G) \to \{1, \ldots, k\}$ is an *arc k-ranking* of a digraph G if each directed path connecting arcs with the same color i contains an arc with a color $j > i$. The smallest integer k such that G has an arc k-ranking is denoted by $\chi'_r(G)$.

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Section 2 gives an example of a family of graphs for which the arc ranking problem can be solved efficiently. In particular, a linear time algorithm for optimal coloring of caterpillars is described. This implies that some well-known classes of directed graphs like oriented paths or comets can be colored efficiently. An interesting question is whether the arc ranking problem can be solved in polynomial time for directed trees and we leave it as an open problem. In Section 3 we consider the complexity of the arc ranking problem. For an undirected graph deciding whether there exists an optimal edge ranking using a fixed number of colors can be done in constant time [1]. However, we prove in this paper that the decision problem

- input: G—an acyclic orientation of a 3-partite simple graph,
- question: $\chi'_r(G) \leqslant 6$?

is NP-complete. In this way we answer an open question stated in [6]. Moreover, this result gives a motivation for designing efficient algorithms for some special classes of acyclic digraphs—a nontrivial example is given in the next section.

2. A polynomial time algorithm for caterpillars

A color i is *visible* for $e \in E(G)$ (resp. $v \in V(G)$) if there exists a directed path between e (resp. v) and some arc with color i such that all arcs of this path have smaller colors than i. We say that arc e (vertex v) is *incident* to color i if e (resp. v) is adjacent (resp. incident) to some arc with color i. A *caterpillar* T is a tree containing subgraph P which is a path such that each vertex of T belongs to P or is adjacent to some vertex of P. The vertices of T which belong to P are denoted by $v_0, v_1, \ldots, v_{|V(P)|-1}$ and arcs by $e_1, \ldots, e_{|E(P)|}$, where $e_i = (v_i v_{i-1})$ or $e_i = (v_{i-1} v_i)$ and the arcs e_i, e_{i+1} are adjacent, $i = 1, \ldots, |E(P)| - 1$. The set of arcs in $E(T) \backslash E(P)$ incident to vertex $v_i \in V(P)$ is denoted by $E_i = \{e_i^1, \ldots, e_i^{k_i}\}$. The symbol $\deg_G(v)$ denotes the number of arcs (incoming and outgoing) adjacent to node v in digraph G.

We split P into the set of subpaths $P^1, \ldots, P^l$ such that each P^i is a directed subpath and P^i is not a proper subgraph of any other directed subpath in P. We say that arc e_i is the *first* arc of P^j if $e_i \in E(P^j)$ and $e_{i-1} \notin E(P^j)$. Similarly, e_i is the *last* arc of P^j if $e_i \in E(P^j)$ and $e_{i+1} \notin E(P^j)$. A path P^i is said to be *short* if it contains at most two arcs. Otherwise the subpath is *long*. Fig. 1 depicts an example of a caterpillar.

If G is a digraph and $S \subseteq V(G)$ then the subgraph of G *induced* by S is defined as $G[S] = (S, \{(uv) \in E(G) : u, v \in S\})$. Let $N(v)$ denote the set of neighbors of node v in T. We define

$$T_i = T[\{v_0\} \cup N(v_0) \cup \cdots \cup N(v_{i-1}) \cup (N(v_i) \backslash \{v_{i+1}\})], \quad 0 \leqslant i \leqslant |V(P)|,$$

$$T_{i,j} = T[V(T_j) \backslash V(T_{i-1})], \quad 0 \leqslant i \leqslant j \leqslant |V(P)|,$$

where $V(T_{-1}) = \emptyset$. Assume that we have an arc ranking c of T_i, where e_i is the last arc of some subpath. Define two sets $A_i(c)$, $B_i(c)$ so that $A_i(c)$ contains all colors of arcs which are incident to v_i and $B_i(c)$ contains all colors visible for e_{i+1} which do not belong to $A_i(c)$. In other words, the set $B_i(c)$ contains colors which are forbidden for the arc e_{i+1} and each color s in $A_i(c)$ is forbidden for each arc e of $T_{i,|V(P)|}$ such that e is connected to e_i by a directed path in T and all arcs of this directed path get smaller colors than s. We say that an arc ranking c' of T_j *extends* an arc ranking c of T_i, $j > i$ if c' is valid and $c'|_{E(T_i)} = c|_{E(T_i)}$, i.e. $c'(e) = c(e)$ for each $e \in E(T_i)$. Observe that an arc ranking of

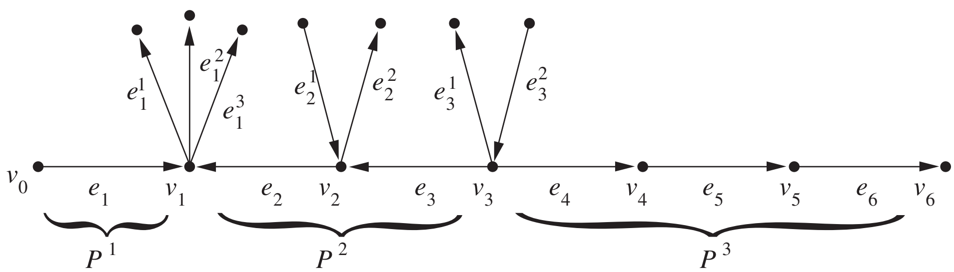

Fig. 1. A caterpillar containing two short subpaths P^1, P^2 and one long subpath P^3.

a directed graph G does not depend on the orientation of an arc (uv), such that $\deg_G(u) = 1$ or $\deg_G(v) = 1$ so we do not have to take into consideration the orientations of e_i^j, $i = 1, \ldots, |E(P)|$, $j = 1, \ldots, k_i$.

In the following we describe an efficient algorithm for arc ranking of caterpillars. First, we give three lemmas showing how to assign colors to the arcs of a long path. For the purposes of the next three lemmas define for an arc ranking c of T_i the set $F_i(c)$, $i = 0, \ldots, |V(P)| - 1$, which contains all the colors assigned to the arcs in $E(T_i)$, which are visible for v_i. In particular, if e_i is the last arc of a path then $F_i(c) = A_i(c) \cup B_i(c)$. For two sets X, Y, containing integers, we say that X is *lexicographically smaller* than Y (Y is *lexicographically bigger* than X) if there exists $x \notin X$ such that $x \in Y$ and for each $x' > x$ we have $x' \in X$ if and only if $x' \in Y$. In that case we write $X <_l Y$. Moreover $X \leqslant_l Y$ if $X <_l Y$ or $X = Y$.

Lemma 1. *Let the arcs $e_i, \ldots, e_j$ form a long subpath of P. Assume that $\widetilde{c}$ is an optimal arc ranking of T and c is an arc ranking of T_l, $i \leqslant l \leqslant j - 2$, using at most $\chi_r'(T)$ colors. If $F_l(c) \leqslant_l F_l(\widetilde{c})$ then c can be extended to an optimal arc ranking of T.*

Proof. We extend c to T in such a way that

$$c(e_{l+1}) = \max((\{\widetilde{c}(e_{l+1})\} \cup F_l(\widetilde{c})) \backslash F_l(c)) \tag{1}$$

and $c(e) = \widetilde{c}(e)$ for each $e \in E(T) \backslash (E(T_l) \cup \{e_{l+1}\})$.

Observe that for two sets X, Y we have that $X \subseteq Y$, $X \neq Y$ imply $X <_l Y$. So $\{\widetilde{c}(e_{l+1})\} \cup F_l(\widetilde{c}) \subseteq F_l(c)$ would imply that $F_l(\widetilde{c}) \subseteq F_l(c)$ and $F_l(\widetilde{c}) \neq F_l(c)$, because $\widetilde{c}(e_{l+1}) \notin F_l(\widetilde{c})$. By assumption this is not possible. So $(\{\widetilde{c}(e_{l+1})\} \cup F_l(\widetilde{c})) \backslash F_l(c) \neq \emptyset$, which means that the definition of $c(e_{l+1})$ given in (1) is correct.

We have that

$$F_l(\widetilde{c}) \cap \{c(e_{l+1}) + 1, \ldots, \chi_r'(T)\} = F_l(c) \cap \{c(e_{l+1}) + 1, \ldots, \chi_r'(T)\}, \tag{2}$$

because $(F_l(\widetilde{c}) \backslash F_l(c)) \cap \{c(e_{l+1}) + 1, \ldots, \chi_r'(T)\} \neq \emptyset$ contradicts (1), while $(F_l(c) \backslash F_l(\widetilde{c})) \cap \{c(e_{l+1}) + 1, \ldots, \chi_r'(T)\} \neq \emptyset$ gives a contradiction with (1) and the inequality $F_l(c) \leqslant_l F_l(\widetilde{c})$. Moreover, $c(e_{l+1}) \geqslant \widetilde{c}(e_{l+1})$ which follows directly from (1) in the case when $\widetilde{c}(e_{l+1}) \notin F_l(c)$ and if $\widetilde{c}(e_{l+1}) \in F_l(c)$ then assuming $c(e_{l+1}) < \widetilde{c}(e_{l+1})$ we have by (2) that $\widetilde{c}(e_{l+1}) \in F_l(\widetilde{c})$ which violates the definition of arc ranking in the case of $\widetilde{c}$.

We have that c uses at most $\chi_r'(T)$ colors, so it remains to show that c is a valid arc ranking of T. Assume for a contradiction that two arcs e' and e'' have the same color d and there exists a directed path P' between them, such that all arcs of this path have colors smaller than d. Clearly, it is not possible that $e', e'' \in E(T_l)$ or $e', e'' \in E(T) \backslash (E(T_l) \cup \{e_{l+1}\})$. The first case to consider is when one arc, say e'', equals e_{l+1}. It is not possible that $e' \in E(T_l)$ because by (1) $c(e_{l+1}) \notin F_l(c)$. Thus, $e' \in E(T) \backslash (E(T_l) \cup \{e_{l+1}\})$. Clearly $c(e_{l+1}) \neq \widetilde{c}(e_{l+1})$ because otherwise we have a contradiction with the fact that $\widetilde{c}$ is a valid arc ranking of T. So, by (1) we have that $c(e_{l+1}) \in F_l(\widetilde{c})$—a contradiction. Now assume that $e' \in E(T_l)$ and $e'' \in E(T) \backslash (E(T_l) \cup \{e_{l+1}\})$.Since $e_{l+1} \in E(P')$ we have that $d > c(e_{l+1})$. By (2) $d \in F_l(c)$ if and only if $d \in F_l(\widetilde{c})$ which leads to a contradiction with the assumption that $\widetilde{c}$ is an arc ranking of T. □

In the next lemma $\overline{S} = \{l : l \notin S\}$.

Lemma 2. *Let the arcs $e_i, \ldots, e_j$ form a long subpath of P and assume that an arc ranking c of T_{i-1} can be extended to an optimal solution for T. Define an arc ranking c' of T_{j-2} in such a way that $c'|_{E(T_{i-1})} = c|_{E(T_{i-1})}$:*

$$c'(e_i) = \begin{cases} \min(X \cap \overline{F_{i-1}(c')}) & \text{if } X \cap \overline{F_{i-1}(c')} \neq \emptyset, \\ \min(\overline{F_{i-1}(c')}) & \text{if } X \cap \overline{F_{i-1}(c')} = \emptyset, \end{cases} \tag{3}$$

where $X = \{c'(e_{i-1}), \ldots, k_i + 2\}$,

$$c'(e_t) = \min\{l : l \notin F_{t-1}(c') \cup \{1, \ldots, k_t\}\} \quad \text{for } t = i+1, \ldots, j-2 \tag{4}$$

and the arcs in E_t, $t = i, \ldots, j-2$, get the smallest and pairwise different colors which do not belong to $(F_{t-1}(c') \backslash \{1, \ldots, c'(e_t) - 1\}) \cup \{c'(e_t)\}$. Then, c' can be extended to an optimal solution for T.

Proof. We prove by induction on $t = i, \ldots, j-2$ that $F_t(c')$ is lexicographically minimal among all arc rankings c'' such that $c''|_{E(T_{i-1})} = c|_{E(T_{i-1})}$. By Lemma 1, c' can be extended to an optimal solution for T.

Let $t=i$. Observe that we may w.l.o.g. assume that if there exists a color $d \notin F_{i-1}(c)$ then no arc $e \in E_{i-1}$ gets a color bigger than d. This follows from the fact that if $c(e_i) > d$ then $c(e)$ can be redefined to d, and if $c(e_i) \leqslant d$ then we can exchange the colors of e and e_i because e_i is the first arc of a long path. This in particular implies that no color in $F_{i-1}(c')$ except $c'(e_{i-1})$ can be visible for the arcs in E_i. Following (3) we consider two cases.

Case 1: $\{c'(e_{i-1}), \ldots, k_i+2\} \cap \overline{F_{i-1}(c')} \neq \emptyset$. Clearly $c'(e_i) > c'(e_{i-1})$. If $c'(e_i) \leqslant k_i+1$ then $F_i(c') = \{1, \ldots, k_i+1\}$ and this set is lexicographically minimal, because $F_i(c')$ must contain at least k_i+1 elements since all arcs in $E_i \cup \{e_i\}$ get pairwise different colors in each proper arc ranking. If $c'(e_i) = k_i+2$ then

$$F_i(c') = \{1, \ldots, k_i, k_i+2\}. \tag{5}$$

If we assign to e_i a color bigger than k_i+2 then we clearly obtain a lexicographically bigger set than in (5). If e_i gets in c' a color smaller than k_i+2 then by (3) $c'(e_i) < c'(e_{i-1})$ and $F_i(c')$ must contain k_i+2 elements (since each arc $e \in E_i$ get a different color than $c'(e_i)$ and $c'(e_{i-1})$), which means that $F_i(c')$ is lexicographically bigger than in (5).

Case 2: $\{c'(e_{i-1}), \ldots, k_i+2\} \cap \overline{F_{i-1}(c')} = \emptyset$. If $c'(e_i) > c'(e_{i-1})$ then $F_i(c') = \{1, \ldots, k_i, c'(e_i)\}$ and $F_i(c')$ is lexicographically minimal, because by (3) all colors smaller than $c'(e_i)$ belong to $F_{i-1}(c')$. If $c'(e_i) < c'(e_{i-1})$ then no arc in E_i gets a color bigger than k_i+2, because all colors different than $c'(e_i)$ and $c'(e_{i-1})$ are available for the arcs in E_i. The only way to remove $c'(e_{i-1})$ from $F_i(c')$ is to assign to e_i a color d bigger than $c'(e_{i-1})$, and by (3) we have that $d > k_i+2$. Thus, F_i is lexicographically minimal.

Let $i < t \leqslant j-2$. By the induction hypothesis we have that the set $F_{t-1}(c')$ is lexicographically minimal. According to (4) we assign to e_t a color bigger than k_t. Thus,

$$F_t(c') = \{1, \ldots, k_t\} \cup \{c'(e_t)\} \cup \{l \in F_{t-1}(c') : l > c'(e_t)\}.$$

Assume that c'' is an arc ranking of T_t, such that $c''|_{E(T_{i-1})} = c|_{E(T_{i-1})}$. If $c'(e_t) = c''(e_t)$ then by assumption $F_{t-1}(c') \setminus \{1, \ldots, c'(e_t)\} \leqslant_l F_{t-1}(c'') \setminus \{1, \ldots, c'(e_t)\}$ which proves the thesis.

If $c''(e_t) > c'(e_t)$ then there are two possibilities: (i) $c''(e_t) \in F_{t-1}(c')$ implies that $F_{t-1}(c') \setminus \{1, \ldots, c''(e_t)\} \neq F_{t-1}(c'') \setminus \{1, \ldots, c''(e_t)\}$, because $c''(e_t) \notin F_{t-1}(c'')$. By the minimality of $F_{t-1}(c')$, $F_{t-1}(c') \setminus \{1, \ldots, c''(e_t)\} <_l F_{t-1}(c'') \setminus \{1, \ldots, c''(e_t)\}$. Since the arcs in $E_t \cup \{e_t\}$ get colors smaller than $c''(e_t)$ in arc ranking c', we have that $F_{t-1}(c') \setminus \{1, \ldots, c''(e_t)\} = F_t(c') \setminus \{1, \ldots, c''(e_t)\}$, which proves that $F_t(c') <_l F_t(c'')$—a contradiction; (ii) $c''(e_t) \notin F_{t-1}(c')$. By the minimality of $F_{t-1}(c')$ we have that $F_{t-1}(c') \setminus \{1, \ldots, c''(e_t)\} \leqslant_l F_{t-1}(c'') \setminus \{1, \ldots, c''(e_t)\}$, which means that $F_t(c') \setminus \{1, \ldots, c''(e_t)-1\} <_l F_t(c'') \setminus \{1, \ldots, c''(e_t)-1\}$, because $c''(e_t) \notin F_t(c')$ since $k_t < c'(e_t) < c''(e_t)$. So, $F_t(c') <_l F_t(c'')$—a contradiction.

If $k_t < c''(e_t) < c'(e_t)$ then by (4) $\{k_t, \ldots, c'(e_t)-1\} \subseteq F_{t-1}(c')$ which together with the minimality of $F_{t-1}(c')$ implies that $F_t(c') \setminus \{1, \ldots, c'(e_t)-1\} <_l F_t(c'') \setminus \{1, \ldots, c'(e_t)-1\}$. So, $F_t(c') <_l F_t(c'')$.

Finally, we consider the case $c''(e_t) \leqslant k_t$. Observe that $k_t+1-c''(e_t)$ arcs in E_t require bigger than $c''(e_t)$ and pairwise different colors which do not belong to $F_{t-1}(c'')$. Let $e \in E_t$ be such an arc that $c''(e) = \max(c''(E_t))$. Since $i < t \leqslant j-2$ we may exchange the colors of e and e_t in c''. The new set $F_t(c'')$ is not lexicographically bigger that the previous one and clearly $c''(e_t)$ is now bigger than k_t, so we have reduced this situation to one of the cases described above. □

Lemma 3. *Let $e_i, \ldots, e_j$ form a long subpath of P and let c be an optimal arc ranking of T_{i-1} such that $A_{i-1}(c) = \{1, \ldots, k_{i-1}+1\}$ and $k_{i-1}+2 \notin B_{i-1}(c)$. Then c can be extended to an optimal arc ranking of T.*

Proof. We assign color $k_{i-1}+2$ to the arc e_i and the arcs in E_i get the smallest and pairwise different colors which do not belong to $(F_{i-1}(c) \cup \{k_{i-1}+2\}) \setminus \{1, \ldots, k_{i-1}+1\} = \{k_{i-1}+2\}$, which results in an arc ranking c with lexicographically the smallest set $F_i(c)$. By Lemma 1, c can be extended to an optimal solution for T. □

Below we describe a procedure for optimal arc ranking of a sequence of short paths. Lemma 4 gives a bound for the number of colors required to label such a subgraph.

Lemma 4. *Let $P^r, \ldots, P^t$ be a sequence of short paths containing arcs $e_i, \ldots, e_j$ and let $d = \max\{k_{i-1}, \ldots, k_j\}$. Then*

$$d+1 \leqslant \chi'_r(T_{i-1,j}) \leqslant d+3. \tag{6}$$

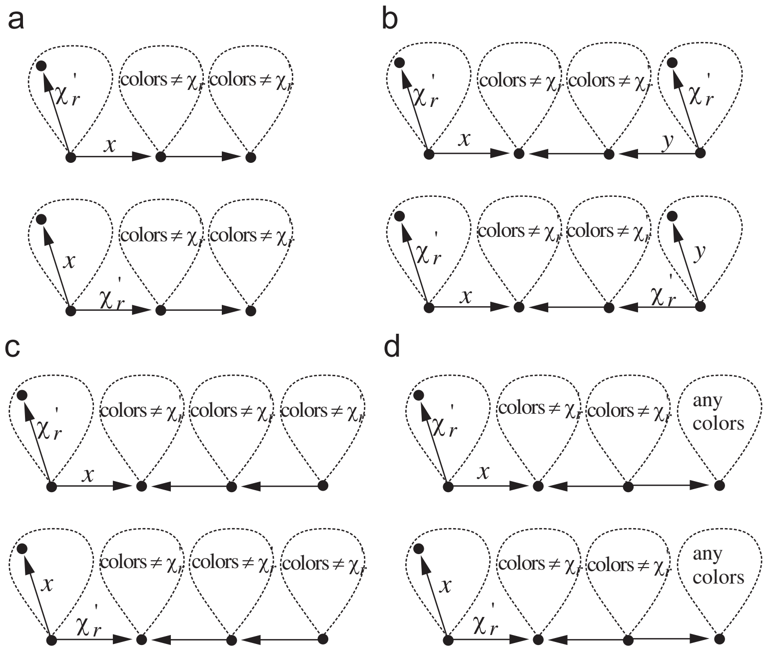

Fig. 2. Transformations of c when $\chi'_r \notin E_{p'+2}$.

Proof. The first inequality follows from the fact that for each digraph G, $\chi'_r(G) \geqslant \max\{\deg_G(v) : v \in V(G)\}$. Now we define an arc $(d+3)$-ranking of $T_{i-1,j}$. Let e_q^p get color p for each $p = 1, \ldots, k_q$ and $q = i-1, \ldots, j$. We label the arcs $e_i, \ldots, e_j$ such that for $r = 0, \ldots, j-i$ the arc e_{i+r} gets color

$$\begin{array}{ll} d+1 & \text{if } r \bmod 4 \in \{0,2\}, \\ d+2 & \text{if } r \bmod 4 = 1, \\ d+3 & \text{if } r \bmod 4 = 3. \end{array}$$

The length of each path P^q, $q = r, \ldots, t$ is bounded by 2 therefore this is a correct arc $(d+3)$-ranking. □

Lemma 5. *If $e_i, \ldots, e_j$ form a sequence of short paths then there exists an optimal arc ranking c of $T_{i-1,j}$ such that $c(e_p) \geqslant \chi'_r(T_{i-1,j}) - 14$ for each $p = i, \ldots, j$.*

Proof. Denote for brevity $\chi'_r = \chi'_r(T_{i-1,j})$. Assume that c is such an arc χ'_r-ranking of $T_{i-1,j}$ that as many arcs in $\{e_i, \ldots, e_j\}$ as possible get color χ'_r. Consider a sequence of arcs $e_p, \ldots, e_q$ such that $p \geqslant i$, $q \leqslant j$, $q - p > 12$ and none of these arcs is labeled with χ'_r by c. Find the smallest index $p' \in \{p-1, \ldots, q\}$ such that $e \in E_{p'}$ and $c(e) = \chi'_r$. If no such an arc e exists then for each arc $e \in \{e_{p+3}, \ldots, e_{q-3}\} \neq \emptyset$ there is no directed path connecting e to an arc labeled with χ'_r. So, we can modify c in such a way that one arc in $\{e_{p+3}, \ldots, e_{q-3}\}$ gets color χ'_r—a contradiction. If $p' > p+3$ then we can exchange the colors of e and $e_{p'}$ which gives a proper arc ranking, because e_{p+3} is not connected by a directed path to v_{p-1} (since each path is short) and the sets ofvisible colors for the arcs of $T_{p',q}$ do not change—a contradiction. Thus, $p' \leqslant p+3$ and $\chi'_r \in c(E_{p'})$. Clearly $\chi'_r \notin c(E_{p'+1})$. If $\chi'_r \notin c(E_{p'+2})$ then there are four cases to consider, shown in Fig. 2. In each case we can modify c so that some arc colored with x or y, where $x, y < \chi'_r$ gets color χ'_r, which leads to a contradiction. Note that we assumed that $e_{p'+1} = (v_{p'} v_{p'+1})$. The cases where $e_{p'+1} = (v_{p'+1} v_{p'})$ are analogous. As we have already mentioned, we do not have to take into consideration the orientations of the arcs not in $E(P)$. Since $q - p > 12$, we can similarly prove that $\chi'_r \in c(E_{p'+4})$ and $\chi'_r \in c(E_{p'+6})$.

Now we show that c can be modified so that some arc in $\{e_p, \ldots, e_q\}$ gets color χ'_r. Let the colors x, w, z, y be assigned to $e_{p'+1}, e_{p'+2}, e_{p'+3}$ and $e_{p'+4}$, respectively. We have two cases to consider: $y \notin c(E_{p'+1})$ and $y \in c(E_{p'+1})$ shown in Fig. 3(a) and (b), respectively. In both cases we have a contradiction, assuming that $z < y$ (this implies that $w \neq y$ and $y \notin c(E_{p'+2})$, which is required to obtain a proper arc ranking). Note that if $\chi'_r \in c(E_s)$, $s = p', p'+2, p'+4, p'+6$ then by

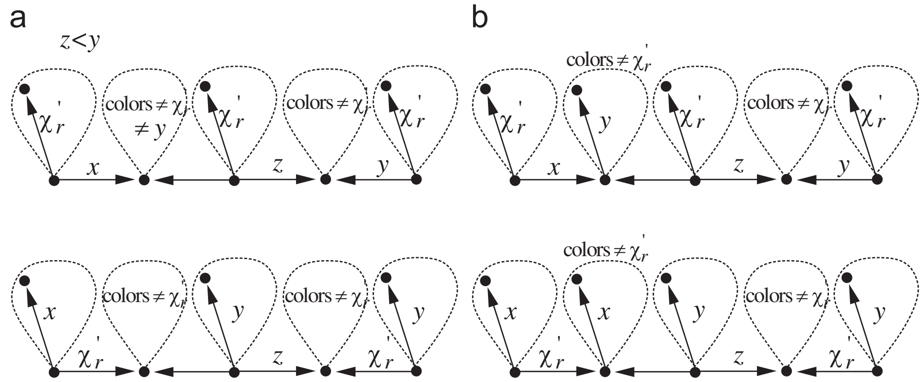

Fig. 3. Transformation of c which increases the number of arcs in $E(P)$ labeled with χ_r'.

the definition of ranking $c(e_{p'+3}) < c(e_{p'+4})$ or $c(e_{p'+3}) > c(e_{p'+4})$ and the corresponding subgraph in Fig. 3 is $T_{p',p'+4}$ or $T_{p'+2,p'+6}$, respectively. Note that if the arcs $e_{p'+2}, e_{p'+3}$ have the same orientation then $y \notin c(E_{p'+1} \cup \{e_{p'+1}\})$ or $y \in c(E_{p'+1} \cup \{e_{p'+1}\}) \wedge w > y$. In both cases let $e_{p'+4}$ get color χ_r' and the arc in $E_{p'+2}$ colored by χ_r' can be labeled with y. Observe that the arcs colored with x and w (resp. z and y) cannot have the same orientation. So, the situations described above and shown in Figs. 3(a) and (b) cover all possible cases obtained by changing the orientations of $e_{p'+1}, \ldots, e_{p'+4}$.

So, we have proved that w.l.o.g. we may assume that if there is a sequence of arcs $e_p, \ldots, e_q$ such that $\chi_r' \notin c(\{e_p, \ldots, e_q\})$ then $q-p \leqslant 12$. Let $S=\{e_{p'}, \ldots, e_{q'}\} \subseteq \{e_p, \ldots, e_q\}$, where $p=i$ or $c(e_{p-1})=\chi_r'$ and $q=j$ or $c(e_{q+1})=\chi_r'$, (possibly $p'=q'$) be a set of arcs such that

$$\min\{c(e_{p'-1}), c(e_{q'+1}), \chi_r'\} > \max(c(S)) + 1 \tag{7}$$

and $c(S)$ is a consecutive set of colors. If $p'=i$ (resp. $q'=j$) in (7) then let $c(e_{p'-1}) = +\infty$ ($c(e_{p'+1}) = +\infty$, resp.). We modify c in such a way that if $c(e) < \min\{c(e_{p'-1}), c(e_{q'+1}), \chi_r'\} - 1$ then $c(e) := c(e)+1$, and if $c(e) = \min\{c(e_{p'-1}), c(e_{q'+1}), \chi_r'\} - 1$ then $c(e) := 1$ for each $e \in E(T_{p'-1,q'})$. Note that c does not use more than χ_r' colors after the above modification. The function c is a valid arc ranking because no two arcs $e \in E(T_{p'-1,q'})$ and $e' \notin E(T_{p'-1,q'})$ violate the definition of ranking, while the coloring of $T_{p'-1,q'}$ remains valid, because an arc colored with label 1 does not belong to a directed path connecting any two arcs of T. We repeat the above step as long as there exists a subset S of $\{e_p, \ldots, e_q\}$ satisfying (7). This gives an arc ranking $c$such that $\min\{c(e_p), \ldots, c(e_q)\} \geqslant \chi_r' - 14$, because by assumption $|\{e_p, \ldots, e_q\}| \leqslant 14$ and $c(\{e_p, \ldots, e_q\})$ is a consecutive set of colors. □

Now we describe a procedure for optimal arc ranking of a sequence of short paths containing arcs $e_i, \ldots, e_j$. For each index $p=i-1, \ldots, j$ define set C_p containing the arc rankings of $T_{i-1,p}$. Initially, C_{i-2} contains an empty coloring. Given a set C_p, the algorithm computes C_{p+1}. We extend each function $c_p \in C_p$ to an arc ranking of $T_{i-1,p+1}$ in such a way that $c_p(e^s_{p+1}) = s$ for each $s = 1, \ldots, \min\{k_{p+1}, \chi_r'(T_{i-1,j}) - 15\}$. Then, e_{p+1} gets colors from $\{\chi_r'(T_{i-1,j}) - 14, \ldots, \chi_r'(T_{i-1,j})\}$ and for each choice of color the remaining arcs (if there are any) in E_{p+1} are labeled with all possible subsets of $\{\chi_r'(T_{i-1,j}) - 14, \ldots, \chi_r'(T_{i-1,j})\} \backslash \{c(e_{p+1})\}$. We insert into C_{p+1} all the valid arc rankings of $T_{i-1,p+1}$ obtained in this way. If C_{p+1} contains two functions c_1, c_2 such that the sets of visible colors for e_{p+2}, e_{p+3} are identical then we remove c_2 from C_{p+1}. In this way the size of each set C_p, $p=i-1, \ldots, j$, is bounded by a constant. Lemma 5 implies that C_j contains an arc $\chi_r'(T_{i-1,j})$-ranking. We do not know the value of $\chi_r'(T_{i-1,j})$ but by Lemma 4 we can compute it by running the above procedure at most three times, substituting $\chi_r'(T_{i-1,j}) = d+1, d+2, d+3$.

Lemma 6. *There exists a linear time algorithm for optimal arc ranking of a sequence of short paths $P^r, \ldots, P^t$.*

We used the sets C_p in the context of arc ranking of a sequence of short paths. In the following we will also use such sets in the case of long paths. If e_{j+1} is the first arc of a long path then it is possible that the set C_j does not contain an arc ranking of T_j which can be extended to an optimal solution for T. We add to the set C_j an additional arc ranking satisfying the assumptions of Lemma 3, which may be not optimal for T_j but may be extended to an optimal solution for T. We consider three cases.

Case 1: $k_j < \chi'_r(T_{i-1,j}) - 16$. For each ranking $c \in C_j$ color $k_j + 1$ is not incident to v_j and we can relabel e_j in such a way that $c(e_j) = k_j + 1$. If there is an arc in E_{j-1} labeled with $k_j + 1$ then we label this arc with the color previously assigned to e_j. Color $k_j + 2$ is now not visible for v_j under c, which implies that after the above modification, the partial arc ranking c has the property from Lemma 3.

Case 2: $k_j > \chi'_r(T_{i-1,j}) - 16$. If c^{o} is an optimal arc ranking of T then we may assume that $\{1, \dots, \chi'_r(T_{i-1,j}) - 15\} \subseteq A_j(c^{\mathrm{o}})$. The same property holds for each arc ranking stored in C_j. If c^{o} is optimal for the sequence of short paths $P^r, \dots, P^t$ then the set C_j contains an arc ranking c_j such that $A_j(c_j) \subseteq A_j(c^{\mathrm{o}})$ and $B_j(c_j) \subseteq B_j(c^{\mathrm{o}})$. Otherwise for $c \in C_j$ define function c' as follows:

$$c'|_{E(T_{j-2})} = c|_{E(T_{j-2})},$$
$$c'(\{e_j^1, \dots, e_j^{k_j}\}) = \{1, \dots, k_j\},$$
$$c'(e_j) = k_j + 1,$$

and $c'(e_{j-1}) = \chi'_r(T_{i-1,j}) + 1$. It remains to assign the colors to the arcs of E_{j-1} in such a way that $k_j + 2 \notin B_j(c')$. Thus, $c'(E_{j-1}) \subseteq X$, where $X = \{1, \dots, \chi'_r(T_{i-1,j})\} \setminus \{k_j + 1, k_j + 2\}$. If $\chi'_r(T_{i-1,j}) \geqslant d + 2$ then such an assignment is possible because $|X| \geqslant d \geqslant |E_{j-1}|$. If $\chi'_r(T_{i-1,j}) = d + 1$ then $k_j = d$ or $k_{i-1} = d$. If $k_j = d$ then $k_j + 2 = d + 2 \notin B_j(c)$ for each $c \in C_j$. If $k_j < d$ then $|E_{j-1}| \leqslant d - 1 \leqslant |X|$.

Case 3: $k_j = \chi'_r(T_{i-1,j}) - 16$. First, for each ranking $c \in C_j$ we perform the same operation as in *Case* 1, i.e. we change the color of e_j to $k_j + 1$. Then, if no function in C_j satisfies the assumptions of Lemma 3 we add an appropriate arc ranking in a similar way as in *Case* 2.

Assume now that we have a partial arc ranking c of T_{i-3}, where e_{i-1} is the last arc of a long path, and a set of arc rankings of short paths $P^r, \dots, P^t$ (containing arcs $e_i, \dots, e_j$). Observe that this situation is similar to the one described above. If we rename the vertices $v_0, \dots, v_j$ to $v_j, \dots, v_0$, respectively then we have to compute the set C_{j-i}. The algorithm for arc ranking of a sequence of short paths $e_i, \dots, e_j$ can be easily modified so that all arc rankings of $T_{i-1,j}$ with different sets of visible colors for the arcs of $T_{i-2} + e_{i-1}$ and $T_{j+1,|V(P)|} + e_{j+1}$ have been computed. Then for each arc ranking of a sequence of short paths we color arcs adjacent to v_{j-i+1}, v_{j-i+2} as in Lemma 2.

Let P^i, P^{i+1} be two long path, and P^i contains arcs $e_l, \dots, e_p$. It is sufficient if the set C_p contains at most two arc rankings. The first element of C_p is an optimal coloring c (with lexicographically smallest set of visible colors for e_{p+1}) obtained by means of a greedy coloring as shown in Lemma 2. If c does not satisfy the assumptions of Lemma 3 then we insert to C_p an arc ranking c' obtained from c in such a way that e_{p-1} gets the smallest available color but bigger than $k_p + 2$. Then, c' satisfies the assumptions of Lemma 3 but it may not be optimal in the case of T_p. In this way we can label a sequence of long paths.

Theorem 1. *There exists a linear time algorithm for finding an optimal arc ranking of a caterpillar.*

Proof. From Lemmas 2 and 3 we know how to color arcs of a long path. Rankings of long paths do not differ for simple and oriented subgraphs, so this step can be done in linear time [8]. We have also described the procedure for coloring a sequence of short paths, and we showed that this procedure creates a set of arc rankings such that at least one of them can be extended to an optimal solution for T. We have proved that for each $i = 1, \dots, |V(P)|$ $|C_i| = \Theta(1)$. Thus, the algorithm has linear running time. □

Since paths and comets are special cases of caterpillars, we have

Corollary 1. *The arc ranking problem for oriented paths and comets can be solved in linear time.*

3. Arc 6-ranking of acyclic orientations of 3-partite graphs is hard

We say that a digraph G is an *orientation* of a simple graph G' if $V(G) = V(G')$ and $\{u, v\} \in E(G')$ if and only if $(uv) \in E(G)$ or $(vu) \in E(G)$. The orientation is *acyclic* if digraph G does not contain a directed cycle

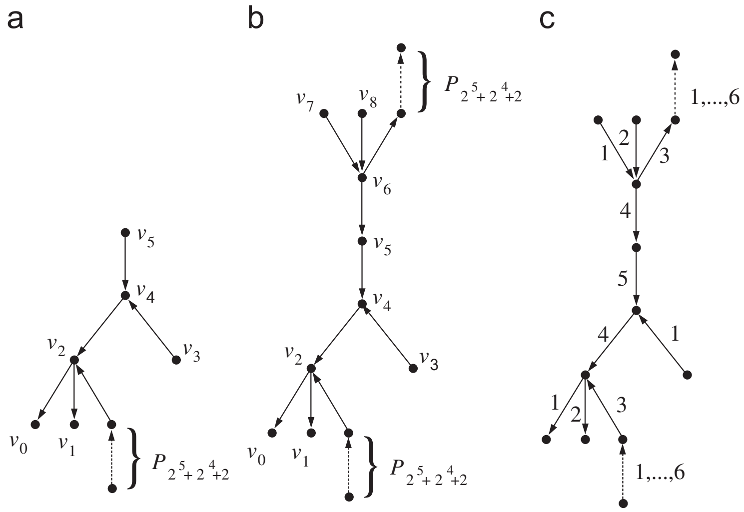

Fig. 4. Digraphs: (a) H_1; (b) H_2; and (c) an arc 6-ranking of H_2.

$(v_1v_2), \ldots, (v_{k-1}v_k), (v_kv_1)$. In order to prove that arc 6-ranking of acyclic orientations of 3-partite graphs is an NP-complete problem we will show a polynomial-time reduction from the 3-satisfiability problem (3-SAT). First, we define subgraphs H_1 and H_2, which will be used to create the digraph corresponding to the Boolean formula. Let P_n be the directed path:

$$P_n = (\{p_1, \ldots, p_n\}, \{(p_i p_{i+1}) : i = 1, \ldots, n-1\}).$$

Since the arc ranking problem of directed paths P_n is identical to the edge ranking problem of undirected paths we obtain that $\chi_r'(P_n) = \lceil \log n \rceil$ [5]. In particular, $\chi_r'(P_{2^5+1}) = 6$ and each arc of P_{2^5+1} can be colored with 6. Similarly, $\chi_r'(P_{2^5+2^4+2}) = 6$.

Digraphs H_1 and H_2 are defined as follows:

$$V(H_1) = \{v_0, \ldots, v_5\} \cup V(P_{2^5+2^4+2}), \tag{8}$$

$$E(H_1) = E(P_{2^5+2^4+2}) \cup \{(v_2v_0), (v_2v_1), (p_nv_2), (v_4v_2), (v_3v_4), (v_5v_4)\},$$

$$V(H_2) = V(H_1) \cup V(P_{2^5+2^4+2}) \cup \{v_6, v_7, v_8\}, \tag{9}$$

$$E(H_2) = E(H_1) \cup E(P_{2^5+2^4+2}) \cup \{(v_6v_5), (v_7v_6), (v_8v_6), (v_6p_1)\}.$$

The paths $P_{2^5+2^4+2}$ in (8) and (9) are different subgraphs (see Fig. 4). Figs. 4(a) and (b) present digraphs H_1 and H_2, respectively. Fig. 4(c) shows an optimal arc ranking of H_2.

Lemma 7. $\chi_r'(H_2) = 6$. *If c is an optimal arc ranking of H_2 then $c((v_5v_4)) \in \{5, 6\}$.*

Proof. Note that $\chi_r'(P_{2^5+2^4+2}) = 6$ and $P_{2^5+2^4+2}$ is a subgraph of H_2, which means that $\chi_r'(H_2) \geqslant 6$. Fig. 4(c) shows a proper arc 6-ranking of H_2. Thus $\chi_r'(H_2) \leqslant 6$, which completes the proof of the first part of the lemma.

Colors 5 and 6 assigned to some arcs of $P_{2^5+2^4+2}$ are visible for all arcs incident to v_6, which means that the arcs incident to node v_6 get colors from the set $\{1, 2, 3, 4\}$ in an optimal arc ranking c. Vertex v_2 and arcs adjacent to it have the same property. Thus, $c((v_5v_4)) > 4$ because otherwise there would exist a directed path connecting arc incident to v_6 with color 4 and arc incident to v_2 also with color 4, such that all arcs of this path have colors smaller than 4. □

For each variable of the Boolean formula we create a digraph G_k, where $k \geqslant 0$, which contains $k+1$ copies of H_2 denoted by $H_2^0, \ldots, H_2^k$ and k copies of subgraph H_1 denoted by $H_1^0, \ldots, H_1^{k-1}$. The subgraphs H_j^i are connected in

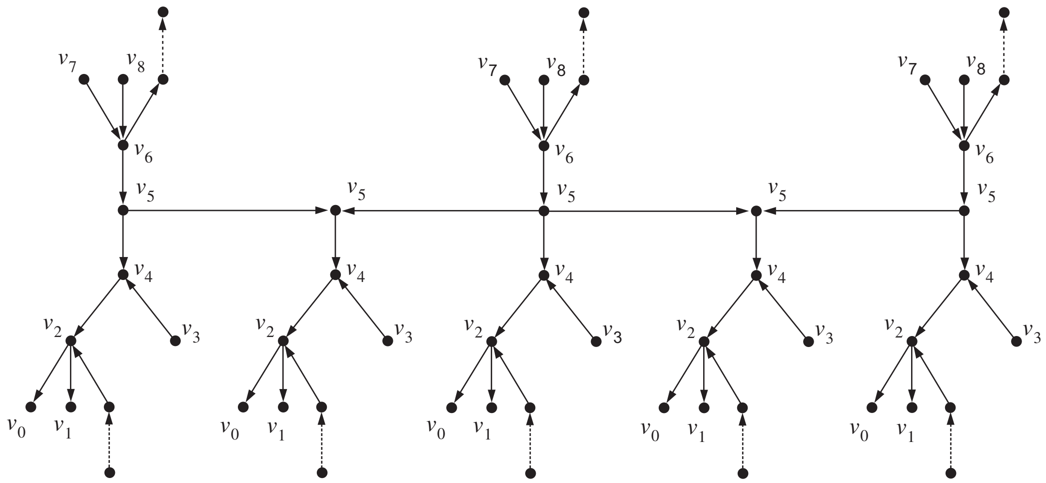

Fig. 5. Digraph G_2.

such a way that

$$(v_5^{i,2}v_5^{i,1}),\ (v_5^{i+1,2}v_5^{i,1}) \in E(G_k), \quad i = 0, \ldots, k-1,$$

where the symbols $v_5^{i,1}$, $v_5^{i,2}$ are used to denote vertices v_5 from subgraphs H_1^i and H_2^i, respectively. Fig. 5 presents digraph G_2.

Lemma 8. *For each $k \geqslant 0$ we have $\chi_r'(G_k) = 6$. If c is an optimal arc ranking of G_k then*

$$\forall_{i=0,\ldots,k,\, j=0,\ldots,k-1} \quad c((v_5^{j,1}v_4^{j,1})) = 5 \wedge c((v_5^{i,2}v_4^{i,2})) = 6$$

or

$$\forall_{i=0,\ldots,k,\, j=0,\ldots,k-1} \quad c((v_5^{j,1}v_4^{j,1})) = 6 \wedge c((v_5^{i,2}v_4^{i,2})) = 5.$$

Proof. The arcs of the path containing vertices $v_5^{0,2}, v_5^{0,1}, \ldots, v_5^{k-1,1}, v_5^{k,2}$ can be labeled with colors 1, 2, 3. If for each i, $c((v_5^{i,2}v_4^{i,2})) = 5$ and $c((v_5^{i,1}v_4^{i,1})) = 6$, then c is a valid 6-ranking of arcs of G_k, where all other arcs are colored according to the pattern shown in Fig. 4(c). This means that $\chi_r'(G_k) \leqslant 6$. From Lemma 7 and the fact that H_2 is a subgraph of G_k it follows that $\chi_r'(G_k) \geqslant 6$.

In order to prove the second part of the lemma we assume that c is an optimal arc ranking of G_k. By Lemma 7 we have that $c((v_5^{i,2}v_4^{i,2})) \geqslant 5$ for $i = 0, \ldots, k$. If $c((v_5^{i,1}v_4^{i,1})) < 5$ for some $i \in \{0, \ldots, k-1\}$ then the following two inequalities hold:

$$a = c((v_5^{i,2}v_5^{i,1})) > 4, \quad b = c((v_5^{i+1,2}v_5^{i,1})) > 4,$$

because $v_6^{i,2}$, $v_2^{i,1}$ are connected by a directed path in G_k and $v_6^{i+1,2}$, $v_2^{i,1}$ are connected by a directed path in G_k (as we have argumented in the proof of Lemma 7, each of these vertices is adjacent to an arc labeled with 4). We consider the case when $a = 5$ and $b = 6$ (the case when $a = 6$ and $b = 5$ is similar). Since c is valid, $c((v_5^{i,2}v_4^{i,2})) \neq 5$ and $c((v_5^{i,2}v_4^{i,2})) \neq 6$ because colors a and b are visible for the arc $(v_5^{i,2}v_4^{i,2})$. This implies that $c((v_5^{i,2}v_4^{i,2})) > 6$—a contradiction with optimality of c. Thus, we have $c((v_5^{i,1}v_4^{i,1})) \in \{5, 6\}$. It is possible that

$$c((v_5^{i,1}v_4^{i,1})) = c((v_5^{j,1}v_4^{j,1})), \quad c((v_5^{i,2}v_4^{i,2})) = c((v_5^{j,2}v_4^{j,2}))$$

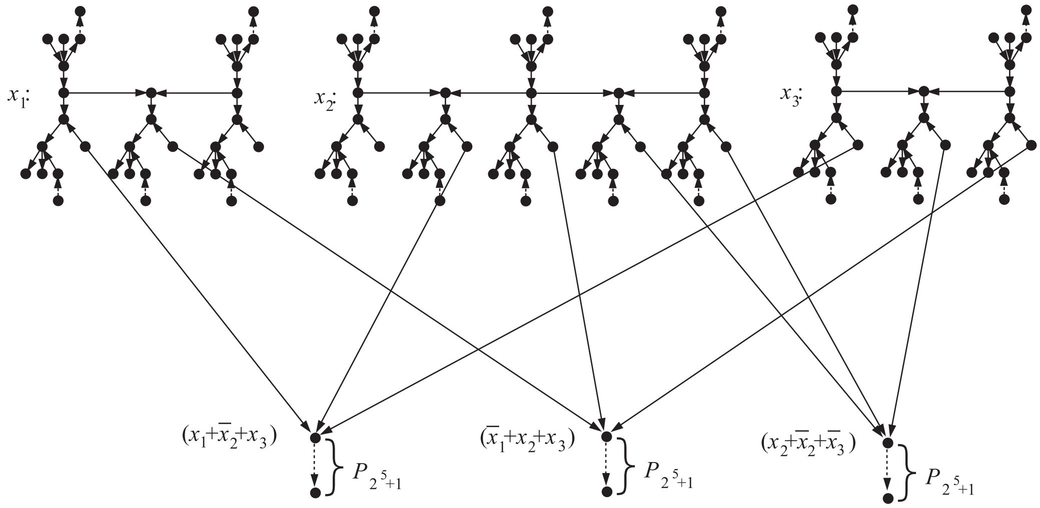

Fig. 6. Digraph G for a formula $(x_1 + \overline{x_2} + x_3)(\overline{x_1} + x_2 + x_3)(x_2 + \overline{x_2} + \overline{x_3})$.

for each $i = 0, \ldots, k$, $j = 0, \ldots, k - 1$ and we have proved that the following inequality:

$$c((v_5^{i,2} v_4^{i,2})) \neq c((v_5^{i,1} v_4^{i,1})), \quad i = 0, \ldots, k - 1$$

holds for any valid ranking which completes the proof. □

Let the Boolean formula $F = (l_{1,1} + l_{2,1} + l_{3,1}) \cdots (l_{1,q} + l_{2,q} + l_{3,q})$ contain variables $x_1, \ldots, x_r$. For each variable x_i of the formula F define numbers $f(x_i)$ and $\overline{f}(x_i)$ such that $f(x_i) = j$ ($\overline{f}(x_i) = j$) if variable x_i ($\overline{x_i}$, resp.) appears j times in F. Digraph G_k corresponding to the variable x_i of F will be denoted by G_k^i, where

$$k = \max\{f(x_i) - 1, \overline{f}(x_i)\}.$$

Digraph G corresponding to F contains subgraphs $G_{k_1}^1, \ldots, G_{k_r}^r$ and for each $i = 1, \ldots, q$ we add a directed path P_{2^5+1}, and for $j = 1, 2, 3$ we add an arc joining $v_3^{s,2} \in V(G_{k_d}^d)$ ($v_3^{s,1} \in V(G_{k_d}^d)$), $s \in \{0, \ldots, k_d\}$ with the vertex p_1 of this path if $l_{j,i} = x_d$, i.e. we add the arc $(v_3^{s,2} p_1)$ ($l_{j,i} = \overline{x_d}$, i.e. we add the arc $(v_3^{s,1} p_1)$, resp.). We add arcs between subgraphs $G_{k_i}^i$ and paths P_{2^5+1} in such a way that the following condition is true for each $i = 1, \ldots, r$:

$$\forall_{s=1,\ldots,k_i} \quad \deg(v_3^{s,1}), \deg(v_3^{s,2}) \leqslant 2.$$

Fig. 6 depicts an example of a digraph G corresponding to a formula $F = (x_1 + \overline{x_2} + x_3)(\overline{x_1} + x_2 + x_3)(x_2 + \overline{x_2} + \overline{x_3})$.

Lemma 9. $\chi_r'(G) \leqslant 6$ *if and only if formula F is satisfiable.*

Proof. We will find the lexicographically minimal set S of forbidden colors for an arc $(v_3^{s,z} u_i)$, assuming that no arc connecting u_i (symbol u_i is used to denote the vertex p_1 of the ith path P_{2^5+1}) with digraphs $G_{k_j}^j$ has been labeled. Arc $(v_3^{s,z} u_i)$ is incident to a path with $2^5 + 1$ vertices, so $6 \in S$. For each valid arc 6-ranking of this path, color 6 is visible for $(v_3^{s,z} v_4^{s,z})$, which means that $c((v_3^{s,z} v_4^{s,z})) < 6$. Thus, we have $c((v_5^{s,z} v_4^{s,z})) \in S$. Clearly, $c((v_3^{s,z} v_4^{s,z})) \in S$. According to Lemma 8 we need to consider two cases: $c((v_5^{s,z} v_4^{s,z})) = 5$ and $c((v_5^{s,z} v_4^{s,z})) = 6$. If $c((v_5^{s,z} v_4^{s,z})) = 5$ then $c((v_3^{s,z} v_4^{s,z})) \notin \{4, 5\}$, which implies that color 4, which is incident to $v_2^{s,z}$, belongs to S. In this case $S = \{c((v_3^{s,z} v_4^{s,z})), 4, 5, 6\}$. If $c((v_5^{s,z} v_4^{s,z})) = 6$ then $S = \{c((v_3^{s,z} v_4^{s,z})), 4, 6\}$ (or $S = \{5, 6\}$, when $c((v_3^{s,z} v_4^{s,z})) = 5$).

Now we are ready to prove the theorem. Assume that $\chi_r'(G) \leqslant 6$. Let the vertices which belong to subgraph $G_{k_j}^j$ and are adjacent to u_i be denoted by $v_3^{s_1,z_1}, v_3^{s_2,z_2}, v_3^{s_3,z_3}$. Some arc $(v_5^{s_j,z_j} v_4^{s_j,z_j})$, $j \in \{1, 2, 3\}$ has been labeled with color 6, because otherwise each arc $(v_3^{s_j,z_j}u_i)$, $j=1,2,3$ has the following set of visible colors: $S_i=\{c((v_3^{s_j,z_j}v_4^{s_j,z_j})),4,5,6\}$. This means that some arc incident to u_i requires a color greater than 6, a contradiction. If $c((v_5^{s_j,z_j}v_4^{s_j,z_j}))=6$ for some $j\in\{1,2,3\}$, where $(v_5^{s_j,z_j}v_4^{s_j,z_j})\in E(G_{k_t}^t)$ for $t\in\{1,\dots,r\}$, then we define

$$x_t=\begin{cases}1 & \text{if } z_j=2,\\ 0 & \text{if } z_j=1.\end{cases}$$

All the other variables x_j can get any Boolean value. In this way we have obtained the values of variables in F such that $F=1$.

Let us assume that $F=1$. Then for each $i=1,\dots,q$ we choose one variable x_j such that $x_j=1$ and $x_j\in\{l_{1,i},l_{2,i},l_{3,i}\}$ or $x_j=0$ and $\overline{x_j}\in\{l_{1,i},l_{2,i},l_{3,i}\}$. Then if x_j is a variable corresponding to the subgraph $G_{k_j}^j$, such that vertex $v_3^{s_j,x_j+1}$ of this subgraph is adjacent to u_i, then we define an arc ranking c: $c((v_5^{s_j,x_j+1}v_4^{s_j,x_j+1}))=6$ and the remaining arcs of $G_{k_j}^j$ are colored as in the proof of Lemma 8. Under such an arc ranking c, $(v_3^{s_j,x_j+1}u_i)$ has the following set of forbidden colors: $S=\{c((v_3^{s_j,x_j+1}v_4^{s_j,x_j+1})),4,6\}$. Each of two other arcs connecting subgraphs $G_{k_j}^j$ with vertex u_i has set S such that $|S|\leqslant 4$, which means that each arc incident to u_i can be labeled with a color smaller than 7, not belonging to the set of forbidden colors of this arc. Thus, c is a valid arc 6-ranking of G. □

Theorem 2. *The problem of arc* 6*-ranking of digraphs is* NP*-complete*.

Proof. The problem is clearly in NP. For a given instance F of the 3-SAT problem we create the digraph G. Lemma 9 implies that formula F is satisfiable if and only if G has an arc 6-ranking. Clearly, this is a polynomial-time reduction, and the thesis follows. □

Lemma 10. *Digraph G is an acyclic orientation of a* 3*-partite graph*.

Proof. Each subgraph $G_{k_i}^i$, $i=1,\dots,r$ is a tree, which means that if G has a cycle then this cycle contains arcs between vertices u_p and subgraphs $G_{k_i}^i$. All arcs connecting digraphs $G_{k_i}^i$ with u_p, $p=1,\dots,q$, have the following orientation: $(v_3^{s_j,x_j}u_p)$. Thus, G does not contain an oriented cycle. Subgraphs $G_{k_i}^i$ are bipartite and vertices $\{u_1,\dots,u_q\}$ form an independent set which means that G is 3-partite. □

From Theorem 2 and Lemma 10 we obtain the following result.

Theorem 3. *The problem of arc* 6*-ranking for acyclic orientations of* 3*-partite graphs is* NP*-complete*.

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