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EQUITABLE COLORING OF CORONA PRODUCTS OF GRAPHS

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Abstract

In many applications in sequencing and scheduling it is desirable to have an underlaying graph as equitably colored as possible. In this paper, we consider an equitable coloring of some corona products $G \circ H$ of two graphs *G* and *H*. In particular, we show that deciding the colorability of $G \circ H$ is NP-complete even if *G* is 4-regular and *H* is K_2. Next, we prove exact values or upper bounds on the equitable chromatic number $\chi = (G \circ H)$, where *G* is an equitably 3- or 4-colorable graph and *H* is an *r*-partite graph, a path, a cycle or a complete graph. Our proofs are constructive in that they lead to polynomial algorithms for equitable coloring of such graph products provided that an equitable coloring of *G* is given. As a by-product we obtain a new class of graphs that confirm Equitable Coloring Conjecture.

1. Introduction

All graphs considered in this paper are finite and simple, i.e., undirected, loopless and without multiple edges.

If the set of vertices of a graph *G* can be partitioned into *k* (possibly empty) classes $V_1, V_2, ..., V_k$ such that each V_i is an independent set and the condition $||V_i| - |V_j|| \leq 1$ holds for every pair (i, j), then *G* is said to be *equitably k-colorable*. The smallest integer *k* for which *G* is equitably *k*-colorable is known as the *equitable chromatic number* of *G* and denoted $\chi = (G)$ [13]. Since equitable coloring is a proper coloring with additional condition, so $\chi(G) \leq \chi = (G)$ for any graph *G*.

This model of graph coloring has many applications. Every time when we have to divide a system with binary conflict relations into equal or almost equal conflict-free subsystems, we can model this situation by means of equitable graph coloring. One motivation for equitable coloring suggested by Meyer [13] concerns scheduling problems. In this application, the vertices of a graph represent a collection of tasks to be performed, and an edge connects two tasks that should not be performed at the same time. A coloring of this graph represents a partition of tasks into subsets that may be performed simultaneously. Due to load balancing considerations, it is desirable to perform equal or nearly-equal numbers of tasks in each time slot, and this balancing is exactly what an equitable coloring achieves. Furmańczyk [7] mentioned a specific application of this type of scheduling problem, namely, assigning university courses to time slots in a way that avoids scheduling incompatible pairs of courses at the same time and spreads the courses evenly among the available time slots, since then the usage of additional resources (e.g. rooms) is maximal.

The notion of equitable colorability was introduced by Meyer [13]. However, an earlier work of Hajnal and Szemerédi [8] showed that a graph G with maximal degree Δ is equitably k-colorable if $k \geq \Delta + 1$. Recently, Kierstead et al. [10] have given an $O(\Delta |V(G)|^2)$-time algorithm for equitable $(\Delta + 1)$-coloring of graph G. In 1973, Meyer [13] formulated the following conjecture:

Conjecture 1 (Equitable Coloring Conjecture (ECC))**.** For any connected graph G, other than complete graph or odd cycle, $\chi = (G) \leq \Delta(G)$.

This conjecture has been verified for all graphs on six or fewer vertices. Lih and Wu [11] proved that the Equitable Coloring Conjecture is true for all bipartite graphs. Wang and Zhang [14] considered a broader class of graphs, namely, r-partite graphs. They proved that Meyer's conjecture is true for complete graphs from this class. Also, the conjecture was confirmed for outerplanar graphs [15] and planar graphs with maximum degree at least 13 [16].

There are very few papers on the complexity of equitable coloring. First of all, a straightforward reduction from graph coloring to equitable coloring by adding sufficiently many isolated vertices to a graph, proves that it is NP-complete to test whether a graph has an equitable coloring with a given number of colors (greater than two). Secondly, Bodlaender and Fomin [1] showed that equitable coloring can be solved to optimality in polynomial time for trees (previously known due to Chen and Lih [3]) and outerplanar graphs. A polynomial time algorithm is also known for equitable coloring of split graphs [2].

The *corona* of two graphs G and H is the graph $G \circ H$ formed from one copy of G and $|V(G)|$ copies of H, where the ith vertex of G is adjacent to every vertex in the ith copy of H. Such type of graph products was introduced by Frucht and Harary in 1970 [5]. For example, the corona $P_n \circ K_1$ is a comb graph. Another corona graph, namely, $L(K_4) \circ K_2$, where $L(G)$ is a line graph of graph G, is depicted in Figure 1(b).

The paper is organized as follows. We start the next section with a theorem concerning the complexity of equitable coloring of coronas. It turns out that the general problem is NP-hard even for the corona of line graphs of cubic graphs (i.e., 3-regular) and K_2. In Section 3, some results concerning the equitable colorability of corona products of r-partite graphs are given. Next, in Section 4, we consider corona products of graphs G with $\chi = (G) \leq 4$ and cycles. In Section 5, we study corona products of those graphs G and paths. In this way we establish a new class of graphs that can be colored optimally in polynomial time and confirm the ECC conjecture. Finally, we give summary in Section 6.

To the best of our knowledge this work is the first paper on equitable coloring of corona graphs.

2. NP-completeness Proof

The problem of equitable vertex coloring of corona graphs is NP-hard.

Theorem 2.1. *The problem of deciding if* $\chi = (G \circ K_2) \leq 3$ *is NP-complete even if G is the line graph of a cubic graph*.

Proof. Our reduction comes from the problem of edge coloring. We know from [9] that the latter problem is NP-complete even if the given graph is cubic.

Let G_3 be a cubic graph such that $G = L(G_3)$.

It is not hard to see that whenever $\chi'(G_3) = 4,$ then any 4-edge-coloring can be transformed to an equitable 4-coloring by using a recoloring technique [4]. If, however, $\chi'(G_3) = 3,$ then each color class constitutes a perfect matching and hence such a coloring is equitable. This follows that the equitable chromatic index $\chi' = (G_3) = 3$ or 4.

Let us assume that $\chi' = (G_3) = 3$. It follows that $\chi = (G) = 3.$ We claim that the graph obtained from G, namely, the corona $G \circ K_2,$ is also equitably 3-colorable. Indeed, given 3-coloring of G, we color the vertices in copies of K_2 with the two remaining colors. In such a coloring, the color classes have the same cardinality, thus the coloring is equitable.

Of course, if the edges of G_3 cannot be colored with 3 colors, then graph $G \circ K_2$ is not equitably 3-colorable, since its chromatic number is strictly greater than 3. So, $\chi = (G \circ K_2) = 3$ if and only if $\chi'(G_3) = 3.$ An example of the reduction is given in Figure 1.

Let us notice that $G \circ K_2$ can be obtained from G in polynomial time. As the verification of the equitability of a 3-coloring of $G \circ K_2$ is in NP, the thesis of the theorem follows. □

Although the problem of equitable coloring of coronas is NP-hard, it is possible to give an upper bound for $\chi = (G \circ H)$ in the case, when H is an arbitrary graph of order m, while G is properly colored with at most $m + 1$ colors. The bound is an easy consequence of the following proposition and the fact that if G and G' are simple graphs on the same set of vertices and $E(G) \subseteq E(G'),$ then $\chi = (G) \leq \chi = (G').$

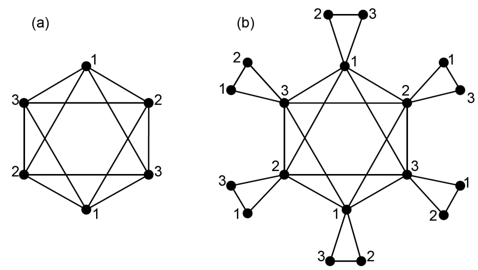

Figure 1. An example of reduction: (a) $L(K_4)$; (b) $L(K_4) \circ K_2$.

Proposition 2.2. *If G is a graph with* $\chi(G) \leq m+1$, *then* $\chi_= (G \circ K_m) = m+1$.

Proof. Let $V(G) = \{u_1, u_2, ..., u_n\}$ and $V(K_m) = \{v_1, v_2, ..., v_m\}$. By the definition of corona graph, each vertex of G is adjacent to every vertex of a copy of K_m. Since $\chi(G) \leq m+1$, we color properly vertices $u_1, u_2, ..., u_n$ of G with $m+1$ colors. Graph $G \circ K_m$ contains n cliques of order $m+1$, K^i_{m+1}, $i = 1, ..., n$, so we cannot use fewer colors than $m+1$.

Now, in every clique K^i_{m+1}, $i = 1, ..., n$, one of the vertices is colored. We extend this coloring to other vertices of the clique. In every case, we use $m+1$ different colors. In the whole graph $G \circ K_m$ every color j, $j = 1, ..., m+1$ is used exactly n times. Thus, our coloring is equitable. $\square$

In further part of the paper, we will assume that $|V(G)| = n$ and $|V(H)| = m$.

3. Equitable Coloring of Corona Graphs with *r*-partite Graphs

In this section, we consider corona products of graph G and r-partite graphs, where graph G fulfills some additional condition.

Theorem 3.1. *Let G be an equitably k-colorable graph on* $n \geq k$ *vertices and let H be a* $(k-1)$*-partite graph. If* $k \mid n$, *then*

$$\chi_= (G \circ H) \leq k.$$

Proof. Suppose $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$, where $V_1, ..., V_k$ are independent sets each of size n/k. This means that they form an equitable k-coloring of G. For each vertex $z \in V(G)$, let $H_z = (X_1^z, ..., X_{k-1}^z, E^z)$ be the copy of $(k-1)$-partite graph $H = (X_1, ..., X_{k-1}, E)$ in $G \circ H$ corresponding to z. Let

$$V_1' = V_1 \cup \bigcup_{z \in V_2} X_1^z \cup \cdots \cup \bigcup_{z \in V_k} X_{k-1}^z,$$

$$V_2' = V_2 \cup \bigcup_{z \in V_3} X_1^z \cup \cdots \cup \bigcup_{z \in V_k} X_{k-2}^z \cup \bigcup_{z \in V_1} X_{k-1}^z,$$

$$\vdots$$

$$V_{k-1}' = V_{k-1} \cup \bigcup_{z \in V_k} X_1^z \cup \bigcup_{z \in V_1} X_2^z \cup \cdots \cup \bigcup_{z \in V_{k-2}} X_{k-1}^z,$$

$$V_k' = V_k \cup \bigcup_{z \in V_1} X_1^z \cup \cdots \cup \bigcup_{z \in V_{k-1}} X_{k-1}^z.$$

Observe that $V(G \circ H) = V_1' \cup \cdots \cup V_k'$ is an equitable k-coloring of $G \circ H$. In this coloring each of k colors is used exactly $n(1 + |X_1| + \cdots + |X_{k-1}|)/k$ times. □

Now, we consider the equitable coloring of corona of K_1 and r-partite graph.

Theorem 3.2. *Let $H(X_1, ..., X_r)$ be a connected r-partite graph. Then*

$$\chi = (K_1 \circ H) = 1 + \lceil |X_1|/2 \rceil + \cdots + \lceil |X_r|/2 \rceil.$$

Proof. To prove the theorem it is enough to see that the color assigned to the universal vertex (the vertex of K_1) can be used at most once and any other color can be used at most twice. □

4. Equitable Coloring of Corona Graphs with Cycles

The class of graphs that we consider in this section is corona graphs with cycles. We begin with the following:

Theorem 4.1. *Let G be an equitably* 3*-colorable graph on* $n \geq 2$ *vertices and let* $k \geq 2$ *be a positive integer. If* $3|n$ *or* $k = 2$, *then*

$$\chi = (G \circ C_{2k}) = 3.$$

Proof. Suppose $V(G) = V_1 \cup V_2 \cup V_3$, where V_1, V_2, V_3 are independent sets, such that $||V_i| - |V_j|| \leq 1$ for each i, $j \in \{1, 2, 3\}$. This means that the sets form the equitable 3-coloring of G. Moreover, we assume $|V_1| \geq |V_2| \geq |V_3|$.

Case 1. n is divisible by 3.

Since each even cycle is a bipartite graph, the thesis follows immediately from Theorem 3.1.

Case 2. $k = 2$.

We color consecutively the copies of C_4. In each case, we use two available colors alternately (one color is forbidden). Dependly on the number of vertices in G we have one of the following situations:

(i) $n \bmod 3 = 1$

We have used $5\lfloor n/3 \rfloor + 1$ times color 1, $5\lfloor n/3 \rfloor + 2$ times color 2 and $5\lfloor n/3 \rfloor + 2$ times color 3.

(ii) $n \bmod 3 = 2$

We have used $5\lfloor n/3 \rfloor + 3$ times color 1, $5\lfloor n/3 \rfloor + 3$ times color 2 and $5\lfloor n/3 \rfloor + 4$ times color 3.

(iii) $n \bmod 3 = 0$

Each color is used $5\lfloor n/3 \rfloor + 5$ times.

In every case, the coloring is equitable.

To complete the proof note that $\chi = (G \circ C_{2k})$ cannot be less than 3, because the corona graph contains a triangle. □

It turns out that the above theorem includes all cases in which three colors are enough. In fact, if G is equitably 3-colorable with $n \neq 3l$, $n \geq 2$, vertices and $k \geq 3$, then $\chi = (G \circ C_{2k}) \neq 3$. The smallest example is $K_2 \circ C_6$ for which $\chi = (K_2 \circ C_6) = 4$. Now, we assume that G is equitably 4-colorable.

Theorem 4.2. *Let G be an equitably 4-colorable graph on $n \geq 2$ vertices, and let $k \geq 3$ be a positive integer. If $3 \nmid n$, then*

$$\chi = (G \circ C_{2k}) = 4.$$

Proof. Suppose graph G has been colored equitably with 4 colors and the cardinalities of color classes are arranged in non increasing way. We order the vertices of $G: v_1, v_2, \ldots, v_n$ in such a way that vertex v_i is colored with color $i \bmod 4$, and we use color 4 instead of 0. Let us consider two cases.

Case 1. n is even.

We color consecutive, with respect to the order $v_1, v_2, \ldots, v_n$ given previously, copies of C_{2k} in the following way. Every time we use two colors. If vertex in G is colored with 1 or 4, then we color the appropriate copy of C_{2k} with colors 2 and 3, alternately. In other cases, we color a copy of C_{2k} with colors 1 and 4. Since n is even, we have only two cases.

(i) $n \bmod 4 = 0$

Each color is used the same number of times, namely, $\lfloor n/4 \rfloor (2k+1)$ times.

(ii) $n \bmod 4 = 2$

Each of colors 1 and 2 is used $\lfloor n/4 \rfloor (2k+1) + k + 1$ times, and each of colors 3 and 4 is used $\lfloor n/4 \rfloor (2k+1) + k$ times.

This means that our coloring is equitable.

Case 2. n is odd.

(i) $n \bmod 4 = 1$

Let us start with graph G on 5 vertices. We color vertices in consecutive copies of C_{2k} in corona $G \circ C_{2k}$ as follows:

- we color the 1st and 5th copies using k times color 2, $\lceil k/2 \rceil$ times color 3, $\lfloor k/2 \rfloor$ times color 4,
- we color the 2nd copy using $\lfloor k/2 \rfloor$ times color 1, k times color 3 and $\lceil k/2 \rceil$ times color 4,
- we color the 3rd copy using k times color 1 and k times color 4,
- we color the 4th copy using k times color 1, $\lceil k/2 \rceil$ times color 2 and $\lfloor k/2 \rfloor$ times color 3.

In such a coloring of $G \circ C_{2k}$ we have used:

- $2 + 2k + \lfloor k/2 \rfloor$ times color 1,
- $1 + 2k + \lceil k/2 \rceil$ times color 2,
- $1 + k + 2\lceil k/2 \rceil + \lfloor k/2 \rfloor = 1 + 2k + \lceil k/2 \rceil$ times color 3,
- $1 + k + 2\lfloor k/2 \rfloor + \lceil k/2 \rceil = 1 + 2k + \lfloor k/2 \rfloor$ times color 4.

The difference between the number of appearances of each color does not exceed 1, so our coloring is equitable.

In cases when $n \geq 13$ (recall that the cases when n is divisible by 3 are excluded) we color the first $4((n-1)/4 - 1)$ copy of C_{2k} using in the ith copy k times color $(i \bmod 4 + 1) \bmod 4$ and k times color $(i \bmod 4 + 2) \bmod 4$. In this part each color is used the same number of times. Finally, we color the last five copies in the way given above. Graph $G \circ C_{2k}$ is colored equitably.

(ii) $n \bmod 4 = 3$

Let us start with graph G on 7 vertices. We color vertices in consecutive copies of C_{2k} in graph $G \circ C_{2k}$ as follows:

- the ith copy, $1 \le i \le 4,$ is colored using k times color $(i \bmod 4 + 1) \bmod 4$ and k times color $(i \bmod 4 + 2) \bmod 4,$
- the 5th copy is colored using k times color 2, $\lfloor k/2 \rfloor$ times color 3, $\lceil k/2 \rceil$ times color 4,
- the 6th copy is colored using k times color 1 and k times color 3,
- the 7th copy is colored using $\lceil k/2 \rceil$ times color 1, $\lfloor k/2 \rfloor$ times color 2 and k times color 4.

In such a coloring of $G \circ C_{2k}$ color 1 is used $2 + 3k + \lceil k/2 \rceil,$ colors 2 and 3 are used $2 + 3k + \lfloor k/2 \rfloor$ times, color 4 is used $1 + 3k + \lceil k/2 \rceil$ times. The coloring is equitable.

In cases when $n \ge 11$ we color first $4((n - 3)/4 - 1)$ copy of C_{2k} using in the ith copy k times color $(i \bmod 4 + 1) \bmod 4$ and k times color $(i \bmod 4 + 2) \bmod 4.$ In this part each color is used the same number of times. Finally, we color the remaining seven copies in the way given above. Graph $G \circ C_{2k}$ is colored equitably.

Now, we have to prove that we cannot use less than 4 colors. Since graph $G \circ C_{2k}$ includes cycle $C_3,$ $\chi = (G \circ C_{2k}) \ge 3.$ Let us consider whether our graph can be colored with three colors. Let us assume that it can be. Then the size of the smallest color class is equal to $\lfloor (2k + 1)n/3 \rfloor$ while the size of the largest independence set is equal to $\lceil n/3 \rceil + (n - \lceil n/3 \rceil)k.$ Since n is not divisible by 3, we have

$$\left\lfloor \frac{(2k+1)n}{3} \right\rfloor > \left\lceil \frac{n}{3} \right\rceil + \left(n - \left\lceil \frac{n}{3} \right\rceil \right) k.$$

Consequently, equitable 3-coloring does not exist, and the thesis of the theorem follows. □

Now, we consider cycles with odd number of vertices. It is easy to see that the corona of graph G and odd cycle needs at least four colors. It turns out that this number suffices for graphs under consideration.

Theorem 4.3. *Let G be equitably* 4-*colorable graph on* $n \geq 2$ *vertices, and let k be a positive integer*. *Then*

$$\chi = (G \circ C_{2k+1}) = 4.$$

Proof. We consider two main cases.

Case 1. n is divisible by 4.

Since each odd cycle is a 3-partite graph, the thesis follows immediately from Theorem 3.1. The corona can be equitably colored using four colors and each color is used exactly $n(2k+2)/4$.

Case 2. n is not divisible by 4.

Suppose G has been equitably colored with 4 colors so that the cardinalities of color classes are arranged in non increasing way. We order the vertices of $G : v_1, v_2, ..., v_n$ in such a way that vertex v_i is colored with color i mod 4, and, as previously, we use color 4 instead of 0. Let us consider two subcases.

(i) $n \bmod 4 = 1$

First, let us consider the first $4((n-1)/4 - 1)$ copy of C_{2k+1}. They can be equitably colored with four colors using each of colors the same number of times, namely, $((n-1)/4 - 1)(2k+2)$ (cf. Case 1).

The last five copies of C_{2k+1} are consecutively colored in the following way:

- the 1st and 5th copies are colored using k times color 2, $\lceil k/2 \rceil$ times color 3, $\lfloor k/2 \rfloor + 1$ times color 4,
- the 2nd copy is colored using $\lfloor k/2 \rfloor + 1$ times color 1, k times color 3 and $\lceil k/2 \rceil$ times color 4,
- the 3rd copy is colored using k times color 1, k times color 4 and 1 time color 2,

- the 4th copy is colored using k times color 1, $\lceil k/2 \rceil$ times color 2 and $\lfloor k/2 \rfloor + 1$ times color 3.

(ii) $n \bmod 4 = 2$ or $n \bmod 4 = 3$

First, let us consider the first $4\lfloor n/4 \rfloor$ copy of C_{2k+1}. They can be equitably colored with four colors using each of colors the same number of times, namely, $\lfloor n/4 \rfloor (2k+2)$ (cf. Case 1).

The last two or three copies of C_{2k+1} are consecutively colored in the following way:

- $n \bmod 4 = 2$
  - the 1st copy is colored using k times color 2, k times color 3, 1 time color 4,
  - the 2nd copy is colored using k times color 1, k times color 4, 1 time color 3.

In such a coloring of $G \circ C_{2k+1}$ every color is used $\lfloor n/4 \rfloor (2k+2) + k + 1$ times.

- $n \bmod 4 = 3$
  - the 1st copy is colored using k times color 2, $\lfloor k/2 \rfloor + 1$ times color 3, $\lceil k/2 \rceil$ times color 4,
  - the 2nd copy is colored using k times color 1, k times color 3 and 1 time color 4,
  - the 3rd copy is colored using $\lceil k/2 \rceil$ times color 1, $\lfloor k/2 \rfloor + 1$ times color 2 and k times color 4.

Calculations, similar to those given in the proof of Theorem 4.2, convince that such colorings are equitable.

For any k, C_{2k+1} is colored in the proper way with three colors. Since each vertex of G is adjacent to every vertex of the corresponding copy of

C_{2k+1}, this vertex needs an additional color. Hence, the minimum number of colors used for equitable coloring of $G \circ C_{2k+1}$ is 4. □

To complete our considerations, we give a simple result for corona of one vertex graph G and cycles. Since $K_1 \circ C_m = W_{m+1}$ for $m \geq 3$, the color assigned to the universal vertex (the vertex of K_1) cannot be used more times than one and any other color may be used at most twice. Thus, for $m \geq 3$ we have:

$$\chi = (K_1 \circ C_m) = \begin{cases} 4, & \text{if } m = 3, \\ \left\lceil \frac{m}{2} \right\rceil + 1, & \text{if } m > 3. \end{cases}$$

5. Equitable Coloring of Corona Graphs with Paths

The next class of graphs that we consider is corona graphs with paths. Theorems 3.1 and 3.2 can be used to draw the following conclusions:

Corollary 5.1. *Let* $m \geq 3$ *be a positive integer. Then*

$$\chi = (K_1 \circ P_m) = \left\lceil \frac{m}{2} \right\rceil + 1.$$

Corollary 5.2. *Let* G *be an equitably* 3*-colorable graph on* $n \geq 2$ *vertices and let* $m \geq 2$ *be a positive integer. If* $3 | n$ *or* $m = 4$, *then*

$$\chi = (G \circ P_m) = 3.$$

It turns out that these are not the only coronas of paths that need 3 colors for equitable coloring.

Theorem 5.3. *Let G be an equitably* 3*-colorable graph on* $n \geq 2$ *vertices and let* $m \geq 2$ *be a positive integer. If* $m = 2, 3, 5$, *then*

$$\chi = (G \circ P_m) = 3.$$

Proof. Suppose G has been equitably colored with 3 colors. We color consecutive copies of P_m, $m = 2, 3, 5$. In each case, we use two available colors, alternately (one color is forbidden). Calculations, similar to those given in the proof of Theorem 4.1, convince that such coloring is equitable. □

In the remaining cases of $G \circ P_m$, we have to use four colors. In the way similar to that given in the proof of Theorem 4.2 we can prove the following proposition.

Proposition 5.4. *Let G be an equitably* 4*-colorable graph on* $n \geq 2$ *vertices and let* $m \geq 6$ *be a positive integer*. *If* $3 \nmid n$, *then*

$$\chi = (G \circ P_m) \leq 4.$$

In fact, this number is equal to 4, which is the subject of the following theorem.

Theorem 5.5. *Let G be an equitably* 4*-colorable graph on* $n \geq 2$ *vertices and let* $m \geq 6$ *be a positive integer*. *If* $3 \nmid n$, *then*

$$\chi = (G \circ P_m) = 4.$$

Proof. It suffices to show that $\chi = (G \circ P_m) \neq 3$. Suppose to the contrary that our graph can be colored with 3 colors. Then the size of the smallest color class is equal to $\lfloor (m+1)n/3 \rfloor$ while the size of the largest independence set is equal to $\lceil n/3 \rceil + (n - \lceil n/3 \rceil)\left\lfloor \frac{m}{2} \right\rfloor$. Since n is not divisible by 3 and $m \geq 6$, we have

$$\left\lfloor \frac{(m+1)n}{3} \right\rfloor > \left\lceil \frac{n}{3} \right\rceil + \left(n - \left\lceil \frac{n}{3} \right\rceil\right)\left\lfloor \frac{m}{2} \right\rfloor.$$

An equitable 3-coloring does not exist. This contradiction concludes the proof. □

6. Final Remarks

In this paper, we have given an NP-completness proof for equitable coloring of corona graphs and established some special cases of such products, for which equitable coloring can be found in polynomial time. Of course, the complexity of equitable coloring of $G \circ H$ depends on the complexity of equitable 3- or 4-coloring of graph G, which is generally NP-hard. More precisely, since the time required to color a single vertex of H is constant, such a coloring of graphs under consideration can be done in time $O(g(n) \cdot m)$, where $g(n)$ is the complexity of equitable 3- or 4-coloring of graph G. However, the following graphs:

- broken spoke wheels [6],
- reels [6],
- cubic graphs except K_4 [4],
- some graph products (including corona products themselves) [7, 12]

admit equitable 3-coloring in polynomial time, and so do the corresponding coronas. In addition to this, since $\Delta(G \circ H) = \Delta(G) + |V(H)| \leq |V(G)| \cdot |V(H)|$, we have confirmed Equitable Coloring Conjecture for these graphs. The main of our results are summarized in Table 1.

Table 1. Possible values of the equitable chromatic number of coronas $G \circ H$

<table>
<tr><th colspan="2" rowspan="2">H / G</th><th rowspan="2">Bipartite graphs</th><th colspan="2">Even cycles C_{2k}</th><th rowspan="2">Odd cycles</th><th colspan="2">path P_k</th></tr>
<tr><th>$k = 2$</th><th>$k \geq 3$</th><th>$2 \leq k \leq 5$</th><th>$k \geq 6$</th></tr>
<tr><td rowspan="2">Equitably 3-colorable graph G on $n \geq 2$ vertices</td><td>$3 \mid n$</td><td>3</td><td rowspan="2">3</td><td>3</td><td rowspan="2">4</td><td rowspan="2">3</td><td>3</td></tr>
<tr><td>$3 \nmid n$</td><td></td><td>4</td><td>4</td></tr>
<tr><td>Equitably 4-colorable graph G on $n \geq 2$ vertices</td><td>$3 \nmid n$</td><td></td><td>≤ 4</td><td>4</td><td>4</td><td>≤ 4</td><td>4</td></tr>
</table>

Among others, we have shown that $\chi = (K_1 \circ C_m) = \chi = (K_1 \circ P_m) = 1 + m - \left\lfloor \frac{m}{2} \right\rfloor$ for $m \geq 5$. This can be generalized as follows:

Theorem 6.1. *If H is a graph on m vertices such that its complement* $\overline{H}$ *has a maximum matching of size s, then*

$$\chi = (K_1 \circ H) = 1 + m - s.$$

Proof. This is because K_1 must form a color class and so the other color classes are of size at most two. To make it optimal, we need to find as many pairs of nonadjacent vertices in *H*, or equivalently adjacent vertices in $\overline{H}$, as possible. □

For example: $\chi = (K_1 \circ W_m) = \chi = (K_1 \circ F_m) = 1 + m - \left\lfloor \frac{m-1}{2} \right\rfloor$ for wheels and fans, respectively.

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