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Equitable colorings of l-corona products of cubic graphs

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A graph G is equitably k-colorable if its vertices can be partitioned into k independent sets in such a way that the number of vertices in any two sets differ by at most one. The smallest integer k for which such a coloring exists is known as the *equitable chromatic number* of G and it is denoted by $\chi_=(G)$.

In this paper the problem of determining the value of equitable chromatic number for multicoronas of cubic graphs $G \circ^l H$ is studied. The problem of ordinary coloring of multicoronas of cubic graphs is solvable in polynomial time. The complexity of equitable coloring problem is an open question for these graphs. We provide some polynomially solvable cases of cubical multicoronas and give simple linear time algorithms for equitable coloring of such graphs which use at most $\chi_=(G \circ^l H) + 1$ colors in the remaining cases.

Key words: corona graph, l-corona products, cubic graph, equitable chromatic number, polynomial algorithm, 1-absolute approximation algorithm

1. Introduction

All graphs considered in this paper are connected, finite and simple, i.e. undirected, loopless and without multiple edges. Many of them are cubic, i.e. 3-regular graphs.

The paper concerns one of popular graph coloring models, namely equitable coloring. If the set of vertices of a graph G can be partitioned into k (possibly empty) classes $V_1, V_2, \ldots, V_k$ such that each V_i is an independent set and the

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condition $||V_i| - |V_j|| \leqslant 1$ holds for every pair (i, j), then G is said to be *equitably k-colorable*. The smallest integer k for which G is equitably k-colorable is known as the *equitable chromatic number* of G and denoted by $\chi_=(G)$ [19]. When the condition $||V_i| - |V_j|| = 0$ holds for every pair (i, j), graph G is said to be *strong equitably k-colorable*. Given a k-coloring of G, a vertex with color i is called an *i-vertex*.

It is interesting to note that if a graph G is equitably k-colorable, it does not imply that it is equitably $(k + 1)$-colorable. A counterexample is the complete bipartite graph (also cubic graph) $K_{3,3}$, which can be equitably colored with two colors, but not with three. The smallest integer k, for which G is equitably k'-colorable for all $k' \geqslant k$, is called the *equitable chromatic threshold* of G and denoted by $\chi_=^*(G)$.

We use also the concept of semi-equitable coloring. Graph G has a *semi-equitable k-coloring*, if there exists a partition of its vertices into independent sets $V_1, \ldots, V_k \subset V$ such that one of these subsets, say V_i, is of size $\notin \{\lfloor n/k \rfloor, \lceil n/k \rceil\}$, and the remaining subgraph $G - V_i$ is equitably $(k - 1)$-colorable. Note that not all graphs have such a coloring, for example K_4 does not have. In the following we will say that graph G has $(V_1, \ldots, V_k)$-coloring to express explicitly a partition of V into k independent sets. If, however, only cardinalities of color classes are important, we will use the notation of $[|V_1|, \ldots, |V_k|]$-coloring.

The model of equitable graph coloring has many practical applications (creating timetables, task scheduling, transport problems, networks, etc.). Every time when we have to divide a system with binary conflict relations into equal or almost equal conflict-free subsystems we can model this situation by means of equitable graph coloring (see for example [8, 10]).

The problem of equitable coloring has attracted attention of many graph theory specialists for almost 50 years. The conducted studies are mainly focused on the proving of known conjectures for particular graph classes (cf. [3, 16]), analysis of the problem's complexity (cf. [11]), designing exact algorithms for polynomial cases (cf. [17]), and approximate algorithms or heuristics for hard cases (cf. [6]). We know that the equitable coloring problem is NP-complete in general case, as a particular case of vertex coloring. Very recently a few papers investigating the parameterized complexity of equitable coloring have been published (cf. [4, 13, 14]).

In this paper we consider the problem of equitable vertex-coloring for one of known graph products, namely for corona products of cubic graphs. Graph products are interesting and useful in many situations. The complexity of many problems, also equitable coloring, that deal with very large and complicated graphs is reduced greatly if one can fully characterize the properties of less complicated prime factors. Moreover, corona graphs lie often close to the boundary between easy and hard coloring problems [9]. More formally, the *corona* of two graphs, n_G-vertex graph G and n_H-vertex graph H, is a graph $G \circ H$ formed from one copy of G, called the *center graph*, and n_G copies of H, named the *outer graph*, where the i-th vertex of G is adjacent to every vertex in the i-th copy of H. Such type of graph product was introduced by Frucht and Harary [5]. In this paper we extend this concept to l-corona products as follows. For any integer $l \geqslant 2$, we define the graph $G \circ^l H$ as $G \circ^l H = (G \circ^{l-1} H) \circ H$, where $G \circ^1 H = G \circ H$. Graph $G \circ^l H$ is also named as *l-corona product* of G and H.

The problem of equitable coloring of corona products of cubic graphs was considered in [9]. The authors showed that although the problem of ordinary coloring of coronas of cubic graphs is solvable in polynomial time, the problem of equitable coloring becomes NP-hard for such graphs. Moreover, they provided polynomially solvable instances of cubical coronas in some cases and 1-absolute approximation algorithms in the remaining cases. In this paper we extend the results from [9] to cubical multicoronas. Note that the product $G \circ H$ of two cubic graphs is no longer cubic.

Now, let us recall some facts concerning cubic graphs. In 1994, Chen et al. [2] proved that for every connected cubic graph, the chromatic number of which is 3, the equitable chromatic number of it is also equal to 3. Moreover, since connected cubic graph G, for which $\chi(G) = 2$ is a bipartite graph $G(A, B)$ such that $|A| = |B|$, we have:

$$\chi(G) = \chi_=(G)$$

and due to Brooks Theorem [1], since the only cubic 4-chromatic graph is K_4 which is easily seen to be equitably colorable using 4 colors:

$$2 \leqslant \chi_=(G) \leqslant 4,$$

for any cubic graph G.

Let

- Q_2 denote the class of 2-chromatic cubic graphs,
- Q_3 denote the class of 3-chromatic cubic graphs,
- Q_4 denote the class of 4-chromatic cubic graphs.
  Clearly, $Q_4 = \{K_4\}$.

Next, let $Q_2(t) \subset Q_2$ ($Q_3(t) \subset Q_3$) denote the class of 2-chromatic (3-chromatic) cubic graphs with partition sets of cardinality t, and let $Q_3(u, v, w) \subset Q_3$ denote the class of 3-chromatic graphs with color classes of cardinalities u, v and w, respectively, where $u \geqslant v \geqslant w \geqslant u - 1$.

Hajnal and Szemeredi [15] proved

Theorem 1. *If G is a graph satisfying $\Delta(G) \leqslant k$ then G has an equitable $(k + 1)$-coloring.*

This theorem implies that every subcubic graph G, i.e. a graph with $\Delta(G) \leqslant 3$, has an equitable k-coloring for every $k \geqslant 4$. In other words,

$$\chi^*_=(G) \leqslant 4. \tag{1}$$

This result was extended in [11] into a semi-equitable coloring of cubic graphs. We showed that, given a n-vertex subcubic graph G and constants $\epsilon > 0$, $k \geqslant 4$, it is NP-complete to obtain a semi-equitable k-coloring of G whose non-equitable color class is of size s if $s \geqslant n/3 + \epsilon n$, and it is polynomially solvable if $s \leqslant n/3$. In particular, we proved

Theorem 2 ([11]). *Given a n-vertex subcubic graph G not including K_4 neither $K_{3,3}$, a constant $k \geqslant 4$, and an integer function $s = s(n)$, a semi-equitable k-coloring of G of type $[s, \lceil \frac{n-s}{k-1} \rceil, \ldots, \lfloor \frac{n-s}{k-1} \rfloor]$ can be found in $O(n^2)$ time, if only $s \leqslant \lceil n/3 \rceil$.*

The remainder of the paper is organized as follows. In Section 2 we provide some auxiliary tools while in Section 3 we give our main results. Namely, we give in some cases polynomial algorithms for optimal equitable coloring of cubical coronas $G \circ^l H$, $l \geqslant 1$, while in the remaining cases we give sharp bounds on the equitable chromatic number of l-corona products of such graphs. Section 4 summarizes our results in a tabular form and remains as an open question the complexity status of equitable coloring of graphs under consideration.

2. Some auxiliaries

In this section we prove lemmas, which are used very often in the further part of the paper.

Lemma 1. *Let $k \geqslant 5$, G be a strong equitably k-colorable graph, and let H be a cubic graph. Then $G \circ^l H$ is strong equitably k-colorable for every $l \geqslant 1$.*

Proof. First, notice that every cubic graph H can be seen as a $(k-1)$-partite graph $H(X_1, X_2, \ldots, X_{k-1})$, $k \geqslant 5$ (due to inequality (1)). Next, let $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$, where $V_1, \ldots, V_k$ are independent sets each of size n_G/k (due to the assumption of strong equitable k-colorability of G).

Now, we determine an equitable k-coloring of $G \circ^1 H$, starting from the strong equitable k-coloring of G: $c : V(G) \to \{1, 2, \ldots, k\}$. We extend it to the copies of H in $G \circ H$ in the following way:

- color vertices of each copy of H linked to an i-vertex of G using color $(i+j) \bmod k$ for vertices in X_j (we use color k instead of 0), for $i = 1, \ldots, k$ and $j = 1, \ldots, k-1$.

Let us notice that this k-coloring of $G \circ^1 H$ is strong equitable. Indeed, every color is used $n_G/k + n_G/k(|X_1| + |X_2| + \cdots + |X_{k-1}|) = n_G(n_H + 1)/k$ times. The thesis follows from induction on l. □

Lemma 2. *Let G be a strong equitably 4-colorable graph, and let $H \in Q_2 \cup Q_3$. Then $G \circ^l H$ is strong equitably k-colorable for every $l \geqslant 1$.*

Proof. Let $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, where $V_1, \ldots, V_4$ are independent sets of size $n_G/4$ each. Now, we determine an equitable 4-coloring of $G \circ^1 H$, starting from the strong equitable 4-coloring of G: $c : V(G) \to \{1, 2, 3, 4\}$. We extend it to the copies of H in $G \circ H$ in the following way:

Case 1: $H \in Q_2$

Let $H = H(X_1, X_2)$. We color the vertices of each copy of H linked to an i-vertex of G using color $(i + j)$ mod 4 for vertices in X_j (we use color 4 instead of 0), for $i = 1, \ldots, 4$ and $j = 1, 2$.

Case 2: $H \in Q_3$

Let $H = H(X_1, X_2, X_3)$. We color vertices of each copy of H linked to an i-vertex of G using color $(i + j)$ mod 4 for vertices in X_j (we use color 4 instead of 0), for $i = 1, \ldots, 4$ and $j = 1, 2, 3$.

Notice that the 4-coloring of $G \circ^1 H$ is strong equitable. Indeed, every color is used $n_G(n_H + 1)/4$ times. The thesis follows from induction on l. □

Lemma 3. *Let G be a strong equitably 3-colorable graph, and let $H \in Q_2$. Then $G \circ^l H$ is strong equitably 3-colorable for every $l \geqslant 1$.*

Proof. Let $V(G) = V_1 \cup V_2 \cup V_3$, where V_1, V_2, V_3 are independent sets of size $n_G/3$ each. Now, we determine an equitable 3-coloring of $G \circ^1 H$, starting from the strong equitable 3-coloring of G: $c : V(G) \to \{1, 2, 3\}$. We extend it to the copies of H in $G \circ H$ in the following way. Let $H = H(X_1, X_2)$. We color the vertices of each copy of H linked to an i-vertex of G using color $(i + j)$ mod 3 for vertices in X_j (we use color 3 instead of 0), for $i = 1, 2, 3$ and $j = 1, 2$. Notice that the 3-coloring of $G \circ^1 H$ is strong equitable. Indeed, every color is used $n_G(n_H + 1)/3$ times. The thesis follows from induction on l. □

Actually, we have proved the following corollaries.

Corollary 1. *Let H be a cubic graph different from K_4 and let $l \geqslant 1$. If G has a strong equitable k-coloring then*

$$\chi_=(G \circ^l H) \leqslant k,$$

for any $k \geqslant 4$.

Corollary 2. *Let H be a cubic graph, $H \in Q_2$ and let $l \geqslant 1$. If G has a strong equitable k-coloring then*

$$\chi_=(G \circ^l H) \leqslant k,$$

for any $k \geqslant 3$.

3. Main results

3.1. Case $H \in Q_2$

In this subsection we obtain exact values of $\chi_=(G \circ^l H)$, where $H \in Q_2$. We give also polynomial-time algorithms for the corresponding colorings.

First, let us recall a known result.

Proposition 1 ([9]). *If G is any cubic graph and $H \in Q_2$, then*

$$\chi_=(G \circ H) = \begin{cases} 3 & \text{if } G \neq K_{3,3} \text{ and } 6|n_G, \\ 4 & \text{otherwise.} \end{cases}$$

Theorem 3. *If G and H are cubic graphs, then for any $l \geqslant 1$ $\chi_=(G \circ^l H) = 3$ if and only if G has a strong equitable 3-coloring and $H \in Q_2$.*

Proof. ($\Leftarrow$) Since n_G is even, as G is cubic, and $3|n_G$, as G has a strong equitable 3-coloring so we have that $6|n_G$. Of course, $G \neq K_{3,3}$, as $K_{3,3}$ is not equitably 3-colorable. Thus, the truth for $l = 1$ follows from Proposition 1. Please note, that the equitable 3-coloring is strong. Indeed, the number of vertices of $G \circ H$ is $n_G(n_H + 1)$ and since $6|n_G$, any equitable 3-coloring implies three color classes each of size $n_G(n_H+1)/3$. For $l > 1$, we get the thesis inductively due to Lemma 3.
($\Rightarrow$) Assume that $\chi_=(G \circ^l H) = 3$. This implies that:

- H must be 2-chromatic,
- G must be 3-colorable (not necessarily equitably), i.e. $\chi(G) \leqslant 3$, which implies $G \in Q_2 \cup Q_3$.

Otherwise, we would have $\chi(G \circ^l H) \geqslant 4$, which is a contradiction.

We begin with $l = 1$. Since $H \in Q_2$ is connected, its bipartition is determined. Let $H \in Q_2(t)$, $t \geqslant 3$. Observe that every 3-coloring of G determines a 3-partition of $G \circ H$. Let us consider any 3-coloring of G with color classes of cardinality n_1, n_2 and n_3, respectively, where $n_G = n_1 + n_2 + n_3$ and $n_1 \geqslant n_2 \geqslant n_3$. Then the cardinalities of color classes in the 3-coloring of $G \circ H$ form a sequence $(n_1^1, n_2^1, n_3^1) = (n_1 + (n_2 + n_3)t, n_2 + (n_1 + n_3)t, n_3 + (n_1 + n_2)t)$. Such a 3-coloring of $G \circ H$ is equitable if and only if $n_1 = n_2 = n_3$. This means that G must have a strong equitable 3-coloring.

For greater values of l the cardinalities of color classes in the determined 3-coloring of $G \circ^l H$, (n_1^l, n_2^l, n_3^l), can be computed from the recursion:

$$\begin{cases} n_1^0 = n_1, \\ n_2^0 = n_2, \\ n_3^0 = n_3. \end{cases}$$

For $l \geqslant 1$:

$$\begin{cases} n_1^l = n_1^{l-1} + (n_2^{l-1} + n_3^{l-1})t, \\ n_2^l = n_2^{l-1} + (n_1^{l-1} + n_3^{l-1})t, \\ n_3^l = n_3^{l-1} + (n_1^{l-1} + n_2^{l-1})t. \end{cases}$$

One can observe that in the determined 3-coloring of $G \circ^l H$ the following statements are true:

- the color classes with the biggest difference between their cardinalities are classes of colors 1 and 3,
- the order relation between cardinalities of color classes of colors 1 and 3 changes alternately, namely $n_1^l \geqslant n_3^l$ for odd l while $n_1^l \leqslant n_3^l$ for even l,
- the absolute value of the difference between cardinalities of color classes for colors 1 and 3 does not decrease as l goes to infinity.

Due to the above, the 3-coloring of $G \circ^l H$ is equitable if only $n_1 = n_2 = n_3$, which completes the proof. □

Theorem 4. *If G is an arbitrary cubic graph and $H \in Q_2$, $l \geqslant 1$, then*

$$\chi_=(G \circ^l H) \leqslant 4.$$

Proof. We start from an equitable 4-coloring of graph G - this is possible due to inequality (1). If it is a strong equitable 4-coloring of G, i.e. if $n_G = 4s$ for some $s \geqslant 1$, then the thesis follows immediately from Lemma 2. So, let $n_G = 4s + 2$ for some $s \geqslant 1$. Note that in any equitable 4-coloring of G exactly two color classes include one more vertex than the remaining two classes. Let us denote them as V_x, V_y, $x \neq y$, and $x, y \in \{1, \ldots, 4\}$, i.e. $|V_x| = |V_y| = s + 1$. Let $v_x \in V_x$ and $v_y \in V_y$. It is easy to see that $(G - \{v_x, v_y\}) \circ H$ has a strong equitable 4-coloring (due to Lemma 2 for $l = 1$), named by c_1. Now, we show that $G[\{v_x, v_y\}] \circ H$ is equitably 4-colorable, where $G[\{v_x, v_y\}]$ is the subgraph of G induced by vertex set $\{v_x, v_y\}$. Let H_x (H_y) be a copy of H linked to v_x (v_y). Furthermore, let X_x (X_y) and Y_x (Y_y) be the partition sets of H_x (H_y). Color the vertices of X_x with color y, vertices of Y_x with color r where $r \neq x$ and $r \neq y$, vertices of X_y with color x, and vertices of Y_y with color t, where $t \notin \{x, y, r\}$. One can easily check that this results in an equitable 4-coloring of $G[\{v_x, v_y\}] \circ H$, named by c_2. Merging the colorings c_1 and c_2 we get an equitable 4-coloring of the whole $G \circ H$.

By applying the above procedure recursively for bigger l we get an equitable 4-coloring of $G \circ^l H$. □

3.2. Case $H \in Q_3$

In this subsection we obtain some polynomially solvable cases concerning optimal equitable coloring of multicoronas $G \circ^l H$, where $H \in Q_3$. In the remaining cases we give 1-absolute approximation algorithms.

Theorem 5. *Let G be any cubic graph and let $H \in Q_3$. If G has a strong equitable 4-coloring, then*

$$\chi_=(G \circ^l H) = 4$$

for any $l \geqslant 1$.

Proof. It is clear that $\chi_=(G \circ^l H) \geqslant 4$, for G and H under assumption. On the other hand, $\chi_=(G \circ^l H) \leqslant 4$ by Lemma 2, and the thesis follows. □

Proposition 2. *If G is a subgraph of cubic graph on $n_G \geqslant 4$ vertices, where $4|n_G$ and $H \in Q_3$, then there is an equitable 5-coloring of $G \circ^l H$ for any $l \geqslant 1$.*

Proof. Let $n_G = 4x$ for some integer $x \geqslant 1$. First, let us notice that there is a strong equitable 4-coloring of $G \circ^l H$ due to inequality (1) and Corollary 1. We color equitably $G \circ^l H$ with 4 colors in the way described in the proof of Lemma 2. In such a coloring every color is used exactly $x(n_H + 1)^l$ times. Now, we have to choose vertices in each of the four color classes which should be recolored to 5 so that the resulting 5-coloring of $G \circ^l H$ is equitable. It turns out that we can choose a proper number of i-vertices, $i = 1, 2, 3$, and 4, that should be recolored to 5 from the partition sets X_1 of $H(X_1, X_2, X_3)$ linked to vertices of $G \circ^{l-1} H$ during creating l-corona product $G \circ^l H$ from $G \circ^{l-1} H$. Moreover, we need only copies of H from this l-th step that were linked to vertices of G. Since $n_G = 4x$, we have exactly x i-vertices in G, $i \leqslant 4$. We will see that we need at most $x|X_1|$ i-vertices that should be recolored to 5. We choose them from X_1's linked to $(i-1)$-vertices in G, $i = 1, 2, 3, 4$ (we use color 4 instead of color 0). To prove this, let us consider three cases.

Case 1: $H(X_1, X_2, X_3) \in Q_3(t+1, t, t)$ *for some odd* $t \geqslant 3$.

In 4-coloring of $G \circ^l H$ each of four colors is used $x(3t+2)$ times, while in every equitable 5-coloring of the corona, each of five colors must be used $\lceil (12xt + 8x)/5 \rceil = 2xt + x + \lceil (2xt + 3x)/5 \rceil$ or $2xt + x + \lfloor (2xt + 3x)/5 \rfloor$ times.

This means that the number of vertices that should be recolored to 5 in each of the four color classes is equal to at most

$$3xt+2x-2xt-x-\lfloor(2xt+3x)/5\rfloor = x(t+1)-\lfloor(2xt+3x)/5\rfloor < x(t+1) = x|X_1|.$$

Case 2: *$H(X_1, X_2, X_3) \in Q_3(t+1, t+1, t)$ for some even $t \geqslant 2$.*

In 4-coloring of $G \circ H$ each of four colors is used $x(3t+3)$ times, while in every equitable 5-coloring of the corona each of five colors must be used $\lceil(12xt+12x)/5\rceil = 2xt+2x+\lceil(2xt+2x)/5\rceil$ or $2xt+2x+\lfloor(2xt+2x)/5\rfloor$ times. This means that the number of vertices that should be recolored to 5 in each of the four color classes is equal to at most

$$3xt+3x-2xt-2x-\lfloor(2xt+2x)/5\rfloor = x(t+1)-\lfloor(2xt+2x)/5\rfloor < x(t+1) = x|X_1|.$$

Case 3: *$H(X_1, X_2, X_3) \in Q_3(t, t, t)$ for some even $t \geqslant 2$.*

In 4-coloring of $G \circ H$ each of four colors is used $x(3t+1)$ times, while in every equitable 5-coloring of the corona each of five colors must be used $\lceil(12xt+4x)/5\rceil = 2xt+\lceil(2xt+4x)/5\rceil$ or $2xt+\lfloor(2xt+4x)/5\rfloor$ times. This means that the number of vertices that should be recolored to 5 in each of the four color classes is equal to at most

$$3xt+x-2xt-\lfloor(2xt+4x)/5\rfloor = x(t+1)-\lfloor(2xt+4x)/5\rfloor \leqslant xt = x|X_1|.$$

This completes the proof. □

Theorem 6. *If G is a cubic graph on $n_G \geqslant 8$ vertices and $H \in Q_3$, then*

$$\chi_=(G \circ^l H) \leqslant 5$$

for any $l \geqslant 1$.

Proof. If $5|n_G$, then G has a strong equitable 5-coloring (due to inequality (1)) and the thesis follows from Lemma 1 for $k=5$. We need to consider the cases where $n_G \bmod 5 \neq 0$.

Case 1: *$n_G \bmod 5 = 1$ and $n_G \geqslant 16$.*

We start from a semi-equitable 5-coloring of cubic graph G of type $[(n+4)/5, (n+4)/5, (n+4)/5, (n+4)/5, (n-16)/5]$ - this is possible due to Theorem 2 for $k=5$. Next, we choose four 1-vertices, four 2-vertices, four 3-vertices, and four 4-vertices from the center graph G. They form a set $V^{16}(G)$. We consider the subgraph of G induced by this vertex set - subcubic graph $G[V^{16}]$, and corona graph $G[V^{16}] \circ^l H$ being subgraph of $G \circ^l H$. $G[V^{16}] \circ^l H$ has an equitable 5-coloring due to Proposition 2. Note, that this equitable 5-coloring of l-corona, described in the proof of

Proposition 2, starts from a strong equitable 4-coloring of the center graph. Thus, it is possible to extend the strong 4-coloring of $G[V^{16}]$ into equitable 5-coloring of $G[V^{16}] \circ^l H$. Next, we consider subgraph $G[V \backslash V^{16}]$, strong equitably 5-colored, as a center graph of l-corona $G[V \backslash V^{16}] \circ^l H$. Due to Lemma 1 for $k = 5$, $G[V \backslash V^{16}] \circ^l H$ has an equitable 5-coloring and this coloring is strong equitable. Furthermore, also this coloring is based on strong equitable 5-coloring of $G[V \backslash V^{16}]$. This means, that equitable 5-colorings of $G[V^{16}] \circ^l H$ and $G[V \backslash V^{16}] \circ^l H$ may be combined into one proper equitable 5-coloring of $G \circ^l H$.

Case 2: $n_G \bmod 5 = 2$.

The idea is similar to that presented in the previous case. This time we start from a semi-equitable 5-coloring of G of type $[(n+3)/5, (n+3)/5, (n+3)/5, (n+3)/5, (n-12)/5]$ – this is possible due to Theorem 2.

Analogously, we choose three 1-vertices, three 2-vertices, three 3-vertices, and three 4-vertices from the graph G. They form a set $V^{12}(G)$. First we extend the coloring of G into $G[V^{12}] \circ^l H$, and then into $G[V \backslash V^{12}] \circ^l H$. Finally, we obtain an equitable 5-coloring of $G \circ^l H$.

Case 3: $n_G \bmod 5 = 3$.

This time we start from a semi-equitable 5-coloring of G of type $[(n+2)/5, (n+2)/5, (n+2)/5, (n+2)/5, (n-8)/5]$ (possible due to Theorem 2). Next, we choose, analogously to the previous case, 8 vertices of G, forming set $V^8(G)$. We extend the coloring of G into $G[V^8] \circ^l H$, and then into $G[V \backslash V^8] \circ^l H$. Finally, we obtain an equitable 5-coloring of $G \circ^l H$.

Case 4: $n_G \bmod 5 = 4$.

In the last case we start from a semi-equitable coloring of G of type $[(n+1)/5, (n+1)/5, (n+1)/5, (n+1)/5, (n-4)/5]$. We choose one vertex of each color i, $1 \leqslant i \leqslant 4$, from graph G. The vertices form the set V^4. First, we extend the coloring of $G[V^4]$ into an equitable 5-coloring of $G[V^4] \circ^l H$, in the way described in the proof of Proposition 2. Next, we extend the strong equitable 5-coloring of $G[V \backslash V^4]$ into strong equitable 5-coloring of $G[V \backslash V^4] \circ^l H$ (due to method described in the proof of Lemma 1). Finally, we obtain an equitable 5-coloring of $G \circ^l H$. □

3.3. Case $H = K_4$

Proposition 3 ([7]). *If G is a graph with $\chi(G) \leqslant m+1$, then $\chi_=(G \circ K_m) = m+1$.*

Theorem 7. *If G is cubic and $l \geqslant 1$, then*

$$\chi_=(G \circ^l K_4) = 5.$$

Proof. Since the following inequalities hold for every cubic graph G:

$$2 \leqslant \chi(G) \leqslant 4,$$

due to Brooks theorem [1], so cubic graph G fulfills the assumption of Proposition 3 for $m = 4$ and we have $\chi_=(G \circ K_4) = 5$. As $G \circ^l K_4 = (G \circ^{l-1} K_4) \circ K_4$ and $\chi_=(G \circ^2 K_4) = \chi_=(G \circ^3 K_4) = \cdots = \chi_=(G \circ^{l-1} K_4) = 5$, we get immediately the thesis. □

4. Conclusion

In the paper we have given some results concerning the equitable coloring of l-corona products $G \circ^l H$, where G and H are cubic graphs. Our main results are summarized in Table 1. In the table the entry '3 or 4' means that we have identified all the cases for which $\chi_=(G \circ^l H) = 3$ and/or $\chi_=(G \circ^l H) = 4$. The entry '$\leqslant 5$' means merely that $\chi_=(G \circ^l H) \leqslant 5$.

Table 1: Possible values of $\chi_=(G \circ^l H)$ for cubical multicoronas

G \ H	Q_2	Q_3	Q_4
$Q_2(t)$	3 or 4 [Thm. 3, 4]	4 for t even [Thm. 5] $\leqslant 5$ for t odd [Thm. 6]	5 [Thm. 7]
Q_3	3 or 4 [Thm. 3, 4]	$\leqslant 5^*$ [Thm. 6]	5 Thm. 7]
Q_4	4 [Thm. 4]	4 [Thm. 5]	5 [Thm. 7]

*: we remind to the reader the case, where $n_G = 6$. One should check easily that the bound holds also for such center graphs G (there are only two cubic graphs on 6 vertices).

Note that our results confirm the Equitable Coloring Conjecture for graphs under consideration. This conjecture was posed by Meyer [19] in 1973.

What about the complexity of equitable coloring of cubical multicoronas? From [9] we know that this problem is NP-hard for coronas $G \circ^l H$, $l = 1$. We remain as an open question whether this result can be extended to arbitrary cubical coronas $G \circ^l H$, $l \geqslant 2$.

We know that ordinary coloring of cubical multicoronas can be determined in polynomial time. The exact values of ordinary chromatic number of l-corona products under consideration are given in Table 2. The appropriate coloring of $G \circ H$ is obtained by coloring G with $\chi(G)$ colors and extending this coloring into copies of r-partite cubic graph H linked to i-vertex of G by coloring r partition sets with $(i + 1) \bmod \chi(G \circ^1 H), \ldots, (i + r) \bmod \chi(G \circ^1 H)$, respectively (we use color $\chi(G \circ^1 H)$ instead of color 0). Such a coloring can be extended into copies of H for bigger l in the similar way.

Table 2: The exact values of $\chi(G \circ^l H)$ for cubical multicoronas

G \ H	Q_2	Q_3	Q_4
Q_2	3	4	5
Q_3	3	4	5
Q_4	4	4	5

Simple comparison of Tables 1 and 2 leads us to the conclusion that our results miss the exact values by at most one color.

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