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# Equivariant Morse equation

**Research Article**

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**Abstract:** The paper is concerned with the Morse equation for flows in a representation of a compact Lie group. As a consequence of this equation we give a relationship between the equivariant Conley index of an isolated invariant set of the flow given by $\dot{x} = -\nabla f(x)$ and the gradient equivariant degree of $\nabla f$. Some multiplicity results are also presented.

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**Keywords:** Morse equation • Conley index • Equivariant gradient degree • Critical orbit • Group action • Poincaré polynomial • Euler ring



## 1. Introduction

Let $X$ be a locally compact metric space and assume that there is a given flow on $X$. In [3] Conley and Zehnder proved that if a finite collection of compact and invariant sets $\{M_\pi : \pi \in \mathcal{D}\}$ forms a so-called Morse decomposition of an isolated invariant set $S$, then the equality

$$\sum_{\pi\in\mathcal{D}} \mathcal{P}(t, h(M_\pi)) = \mathcal{P}(t, h(S)) + (1+t)\mathcal{Q}(t) \tag{ME}$$

holds true. Here $\mathcal{Q}$ is a polynomial in $t$ having non-negative integer coefficients and $\mathcal{P}$ stands for the Poincaré polynomial of the homotopy Conley index of $M_\pi$ and $S$. In the following we will refer to equations of type (ME) as the Morse equation.

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This result is a generalization of the classical Morse relations, which give a relationship between the set of critical points of a Morse function $f\colon M \to \mathbb{R}$ defined on a Riemannian closed $n$-manifold and the topology of the underlying domain. Namely, let $c_k$ denote the number of critical points of $f$ of the Morse index $k$ and let $\beta^k(M)$ be the $k$-th Betti number of $M$. Then one can consider the negative gradient flow of $f$ and apply the above equality to obtain the formula $\sum_{k=0}^{n} c_k t^k = \sum_{k=0}^{n} \beta^k(M) t^k + (1+t)\mathcal{Q}(t)$. A survey exposition of this material is presented in the joint paper of Izydorek and the author [14].

The purpose of this paper is to present a generalization of the above mentioned result of Conley and Zehnder to the equivariant setting, i.e., when the equivariant flows acting on the representation of a compact Lie group are taken into considerations. Namely, we prove the following.

### Theorem 1.1.

*Let $V$ be an orthogonal representation of a compact Lie group. If $S$ is an isolated invariant set of an equivariant flow on $V$ and $(M_1, \dots, M_m)$ is a Morse decomposition of $S$, then there exists*

$$\mathcal{Q}_G(t) = \sum_{(H)} \left( \sum_q \rho^q_{(H)} t^q \right) \mathfrak{u}^G_{(H)} \tag{1}$$

*with all integer coefficients $\rho^q_{(H)} \geq 0$ such that*

$$\sum_{j=1}^{m} \mathcal{P}_G(t, h_G(M_j)) = \mathcal{P}_G(t, h_G(S)) + (1+t)\mathcal{Q}_G(t). \tag{EME}$$

This generalization is consistent with the evolution of the equivariant counterparts of such tools as the topological mapping degree and the Conley index. The main reference for this subject are the papers of Gęba [8], Gęba & Rybicki [9] and the papers of Rybicki and his collaborators [10, 17, 19, 20]. The equivariant version of the Conley index is the subject of the papers of Floer [6], Floer & Zehnder [7]. In [13] Izydorek extends the equivariant Conley index to the situations where the local compactness property of the phase space fails, and he successfully applies it to the strongly indefinite problems.

We will use the equation (EME) to derive some multiplicity results for critical-point orbits of invariant functions defined on the $G$-representation ($G = \mathrm{SO}(2), \mathbb{Z}_p$, $p$ prime). We also obtain a relationship between the equivariant Conley index and the gradient equivariant degree. This should be seen as an illustration of application of the Morse equation (EME), because the ideas are contained in the work of Gęba [8].

The equation (ME) is expressed in terms of the Poincaré polynomials with integer coefficients being the Betti numbers of certain index pairs, for details we refer to [3, 14]. We are going to define the Betti numbers of the equivariant Conley index and then to define the Poincaré polynomial appropriate for our purposes. One can expect that in the case of the trivial group, the obtained equation will coincide with the classical one. As a matter of fact, a proper definition of the Poincaré polynomial of the index is actually crucial on the way to obtain our result. The motivations of such definition become clear, having regard to the form of the elements in the Euler ring $U(G)$ of a compact Lie group $G$. Let us mention only that in our approach to the equivariant theory there is no equivariant cohomology at all.

After this introduction the paper is organized as follows. Section 2 outlines the material from equivariant topology including the concept of the Euler ring of the compact Lie group. Sections 3 and 4 are devoted to the equivariant versions of the mapping degree and the Conley index theory. Simple examples are presented. Section 5 contains the proof of the equivariant Morse equation. Then we give a calculation of the Poincaré polynomial of an isolated orbit and deduce a relationship between the gradient equivariant degree and the equivariant Conley index. The last part of this work is devoted to some simple multiplicity results for critical orbits of invariant functions. The presented work is a part of the author's PhD thesis [22].

## 2. Preliminaries

Let $G$ be a compact Lie group. A subgroup $H \subset G$ is called *conjugate* to a subgroup $K \subset G$ if there is $g \in G$ such that $H = g^{-1}Kg$. The conjugacy defines an equivalence relation, and we will write $(H)$ for a conjugacy class of $H$. The set of all conjugacy classes of closed subgroups of $G$ will be denoted by $\Phi(G)$. The set $\Phi(G)$ is partially ordered. We write $(H) \leq (K)$ if $gHg^{-1} \subset K$ for some $g \in G$.

An *action of a group* $G$ on a topological space $X$ (also called $G$-action) is a continuous map $G \times X \to X$ sending $(g, x)$ to $gx$ and satisfying the following properties:

- $ex = x$ for all $x \in X$, where $e$ stands for the identity of $G$;
- $h(gx) = (hg)x$ for all $g, h \in G$ and $x \in X$.

A $G$-*space* is a pair consisting of an underlying space $X$ and a given $G$-action. A *linear representation* of a group $G$ is a pair $(V, \rho)$, where $V$ is a vector space and $\rho$ is a $G$-action such that for all $g \in G$ the map $\rho_g = \rho(g, \cdot)$ is a linear automorphism of $V$. If $V$ is equipped with an inner product $\langle \cdot, \cdot \rangle$, then we say that the representation is *orthogonal* if $\langle \rho_g x, \rho_g y \rangle = \langle x, y \rangle$ for all $x, y \in V$ and $g \in G$. Throughout the paper, all representations are assumed to be finite dimensional, real and orthogonal.

Let $X$ be a $G$-space. For $x \in X$, the set $G_x = \{g \in G : gx = x\}$ is the *isotropy subgroup* of $G$ at $x$ and the set $Gx = \{gx : g \in G\}$ is the *orbit* of $G$ through $x$. For each $x \in X$, the group $G_x$ is closed in $G$.

Points $x, y \in X$ are of the same *orbit type* if the isotropy subgroups $G_x$ and $G_y$ are conjugate subgroups of $G$. Since $G_{gx} = g^{-1}G_x g$, the points on the same orbit are of the same orbit type. Hence, the set $\Phi(G)$ of all conjugacy classes also will be called the set of orbit types.

Let $H$ be a closed subgroup of $G$. We will use the following notation:

$$X^H = \{x \in X : H \subset G_x\} = \{x \in X : hx = x,\ h \in H\}, \qquad X^{(H)} = GX^H = \{x \in X : (H) = (K) \text{ for } K \subset G_x\},$$
$$X_H = \{x \in X : G_x = H\}, \qquad X_{(H)} = GX_H = \{x \in X : (G_x) = (H)\},$$
$$X^{>(H)} = X^{(H)} \setminus X_{(H)} = \bigcup_{(K)>(H)} X^{(K)}.$$

***Definition 2.1.***
An orbit $Gx$ and its orbit type $(G_x)$ are called *principal* if $Gx$ has a $G$-invariant open neighbourhood that contains no orbit of smaller orbit type with respect to the partial order $\leq$ in $\Phi(G)$.

***Theorem 2.2 ([1, Theorem 3.1]).***
*Let $(H)$ be a principal orbit type. Then the union $X_{(H)}$ of orbits of principal type is open and dense in $X$.*

A subset $\Omega$ of a $G$-space $X$ is called a $G$-*invariant* (a $G$-set) provided that $x \in \Omega$ and $g \in G$ imply $gx \in \Omega$. If $X$ and $Y$ are $G$-spaces, then a continuous map $f \colon X \to Y$ is called a $G$-*equivariant map* (a $G$-map) if the relation $f(gx) = gf(x)$ holds for all $x \in X$ and $g \in G$.

### 2.1. $G$-complexes

The object of our interest, the Conley index, is a homotopy type of a pointed space which supports the structure of CW-complex. Although the notion of CW-complex is well known in topology, we present here some basic definitions, since the $G$-equivariant Conley index joins the notion of CW-complex and $G$-space. The definitions are borrowed from the paper by Gęba and Rybicki [9].

We use the standard notation $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ and $D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ for the unit $(n-1)$-sphere and the unit $n$-ball in $\mathbb{R}^n$ respectively. In what follows we assume that $D^n$ carries the trivial $G$-action, i.e. $gx = x$ for all $x \in D^n$ and $g \in G$. We set $B^n = D^n \setminus S^{n-1}$.

***Definition 2.3.***

Let $(X, A)$ be a compact pair of $G$-spaces and $\{H_j\}$, $j = 1, 2, \dots, q$, a family of closed subgroups of $G$. We say that *$X$ is obtained from $A$ by simultaneously attaching a family of equivariant $k$-cells of orbit type* $\{(H_j) : j = 1, \dots, q\}$ if there exists a $G$-map $\varphi\colon \bigsqcup_{j=1}^{q} D^k \times (G/H_j) \to X$ which maps $\bigsqcup_{j=1}^{q} B^k \times (G/H_j)$ homeomorphically onto $X \setminus A$. We call $\varphi(D^k \times (G/H_j))$ a closed $k$-dimensional cell of orbit type $(H_j)$.

***Definition 2.4.***

Let $X$ be a compact $G$-space. A *finite equivariant CW-decomposition of* $X$ consists of an increasing family of $G$-subsets $X^0 \subset X^1 \subset \dots \subset X^n = X$ and a family $\bigcup_{k=0}^{n} \{H_{j,k} : j = 1, \dots, q(k)\}$ of closed subgroups of $G$ such that

- $X^0 = \bigsqcup_{j=1}^{q(0)} G/H_{j,0}$;
- the space $X^k$ is obtained from $X^{k-1}$ by simultaneously attaching a family of equivariant $k$-cells of orbit type $\{(H_{j,k}) : j = 1, \dots, q(k)\}$ for each $1 \le k \le n$.

A *pointed $G$-space* is a pair $(X, x_0)$, where $X$ is a $G$-space with a distinguished point $x_0$ called the base point and such that the action of $G$ leaves the base point fixed. The pointed $G$-spaces are the objects of the category whose morphisms are $G$-maps preserving the base point. If $X$ is a $G$-space, then the superscript plus $X^+$ means that $X$ is considered as a pointed space with a separate base point added.

***Definition 2.5.***

Let $(X, x_0)$ be a pointed compact $G$-space. A *pointed finite equivariant CW-decomposition of* $(X, x_0)$ consists of an increasing family of $G$-subsets $X^{-1} \subset X^0 \subset X^1 \subset \dots \subset X^n = X$ and a family $\bigcup_{k=0}^{n} \{H_{j,k} : j = 1, \dots, q(k)\}$ of closed subgroups of $G$ such that

- $X^{-1} = \{x_0\}$;
- $X^0 = \{x_0\} \sqcup \bigsqcup_{j=1}^{q(0)} G/H_{j,0}$;
- the space $X^k$ is obtained from $X^{k-1}$ by simultaneously attaching a family of equivariant $k$-cells of orbit type $\{(H_{j,k}) : j = 1, \dots, q(k)\}$ for each $1 \le k \le n$.

The family $\bigcup_{k=0}^{n} \{H_{j,k} : j = 1, \dots, q(k)\}$ is called the *orbit type of the decomposition of* $X$. For short we use the term $G$-complex (pointed $G$-complex) for a (pointed) $G$-space if there exists a (pointed) finite equivariant CW-decomposition of $X$ (resp. $(X, x_0)$).

## 2.2. Euler ring $U(G)$

If $(X, x_0)$ and $(Y, y_0)$ are pointed $G$-spaces ($gx_0 = x_0$ and $gy_0 = y_0$ for all $g \in G$), then we say that $(X, x_0)$ and $(Y, y_0)$ have the same *$G$-homotopy type* iff there exists a pair of $G$-maps $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (Y, y_0) \to (X, x_0)$ such that $gf \sim_G \mathrm{id}_{(X,x_0)}$ and $fg \sim_G \mathrm{id}_{(Y,y_0)}$. The symbol $\sim_G$ means that if $H_t$ is a homotopy joining two $G$-equivariant maps, then for all $t \in [0, 1]$ the map $H_t$ is a $G$-map as well. Of course, the relation $\sim_G$ is an equivalence and the equivalence class under relation $\sim_G$ is denoted by $[X]_G$. We say that $[X]_G$ is the $G$-homotopy type of $X$.

Let us introduce the symbol $\mathcal{F}(G)$ for the category whose objects are pointed $G$-complexes and $\mathcal{F}[G]$ for the set of all $G$-homotopy types of pointed $G$-complexes. For $(X, x_0), (Y, y_0) \in \mathcal{F}(G)$ we define its *wedge sum* to be $X \vee Y = \big((X \times \{y_0\}) \cup (\{x_0\} \times Y), (x_0, y_0)\big) \in \mathcal{F}(G)$, and its *smash product* $X \wedge Y = (X \times Y/X) \vee Y$. Of course we also have $X \wedge Y \in \mathcal{F}(G)$.

Let $\mathbf{F} = \mathbb{Z}[\mathcal{F}[G]]$ be the free abelian group generated by the $G$-homotopy classes of pointed $G$-complexes and let $\mathbf{N}$ be the subgroup of $\mathbf{F}$ generated by all elements $[A]_G - [X]_G + [X/A]_G$, where $A$ is a pointed $G$-subcomplex of $X$. Define $U(G) = \mathbf{F}/\mathbf{N}$. The class of $[X]_G \in \mathcal{F}[G]$ under this identification will be denoted by $\mathfrak{u}(X)$. Directly from the definition of $U(G)$ we see that the addition can be obtained via the wedge sum $\mathfrak{u}(X)+\mathfrak{u}(Y)=\mathfrak{u}(X\vee Y)$. Moreover, the assignment $(X,Y)\mapsto X\wedge Y$ induces the multiplication in $U(G)$, cf. [4], that is $\mathfrak{u}(X)\mathfrak{u}(Y)=\mathfrak{u}(X\wedge Y)$.

### Definition 2.6.

The set $U(G)$ with the composition laws defined as above is called the *Euler ring* of the group $G$.

It is a classical result that the coset space $G/H$ of a compact Lie group over the closed subgroup $H$ is a smooth compact $G$-manifold, cf. for instance [15]. Hence, due to a theorem of Illman [12], it is a $G$-complex. Therefore $G/H^+\in\mathcal{F}(G)$ and one can consider the element $\mathfrak{u}(G/H^+)\in U(G)$. In what follows we will write $\mathfrak{u}^G_{(H)}$ instead of $\mathfrak{u}(G/H^+)$. The abelian group structure of the ring $U(G)$ is fairly easy and its description is given in the following statement.

### Proposition 2.7 ([4]).

*As a group $U(G)$ is the free abelian group with basis $\mathfrak{u}^G_{(H)}$, where $(H)\in\Phi(G)$. If $X\in\mathcal{F}(G)$, then*

$$\mathfrak{u}(X)=\sum_{(H)\in\Phi(G)}\chi\left(X^{(H)}/G,X^{>(H)}/G\right)\mathfrak{u}^G_{(H)}.$$

*Here $\chi$ stands for the Euler characteristic of the pair of CW-complexes. As a ring $U(G)$ is commutative with the unit $\mathfrak{u}^G_{(G)}$.*

The Euler characteristic of a cell complex $K$ can be expressed as an alternating sum $\chi(K)=\sum_{k=0}^{\infty}(-1)^k s_k$, where $s_k$ is the number of $k$-cells in the complex $K$. This formula holds in an equivariant setting as well and we have a nice tool for computations.

### Proposition 2.8 ([9]).

*Let $X\in\mathcal{F}(G)$ and let $\bigcup\limits_{k=0}^{n}\{H_{j,k}:j=1,\dots,q(k)\}$ be an orbit type of the decomposition of $X$. Then*

$$\mathfrak{u}(X)=\sum_{(H)\in\Phi(G)}n_{(H)}(X)\mathfrak{u}^G_{(H)},\tag{2}$$

*where $n_{(H)}(X)=\sum\limits_{k=0}^{n}(-1)^k\nu((H),k)$ and $\nu((H),k)$ is the number of equivariant $k$-cells of orbit type $(H)$.*

## 3. Degree for equivariant gradient maps

In this section we briefly recall a definition of the degree for gradient $G$-maps presented in [8]. The paper [8] is the main reference for this section, where the reader can find proofs of theorems discussed below.

Let $V$ be an orthogonal $G$-representation, $G$ a compact Lie group. We say that a function $\varphi\colon V\to\mathbb{R}$ is *$G$-invariant* if $\varphi$ is constant on the orbits of $G$, i.e., $\varphi(gx)=\varphi(x)$ for $x\in V$ and $g\in G$. If $f\colon V\to V$ is a gradient of a continuously differentiable $G$-invariant function $\varphi$, i.e. $f=\nabla\varphi$, then we call it *a $G$-equivariant gradient map*. As an immediate consequence of the above definition and the chain rule we get the property that $f(gx)=gf(x)$ for all $x\in V$ and $g\in G$.

In the same manner we define a homotopy joining two equivariant gradient maps. Namely, a map $h:V\times[0,1]\to V$ is a *gradient $G$-homotopy* if there exists a $G$-invariant function $q\colon V\times[0,1]\to\mathbb{R}$ of class $C^1$ ($q(gx,t)=gq(x,t)$) such that $h(x,t)=\nabla q(x,t)$ for all $t\in[0,1]$ and $x\in V$. The gradient is taken with respect to the $x$ variable.

### Definition 3.1.

Let $\Omega \subset V$ be an open bounded and $G$-invariant subset and $f\colon V \to V$ a gradient $G$-map.

- We say that a pair $(f, \Omega)$ is $\nabla_G$-*admissible* provided that $f(x) \neq 0$ for $x \in \partial\Omega$. In other words, a $\nabla_G$-admissible pair is an equivariant map of pairs $f\colon (V, \partial\Omega) \to (V, V \setminus \{0\})$.
- Two $\nabla_G$-admissible pairs $(f_0, \Omega)$ and $(f_1, \Omega)$ are $\nabla_G$-*homotopic* if there exists a gradient $\nabla_G$-homotopy $h\colon V \times [0,1] \to V$ connecting them, i.e., $h_i = f_i$, $i = 0, 1$, and such that the pair $(h_t, \Omega)$ is $\nabla_G$-admissible for $t \in [0,1]$.

From now on, $f\colon V \to V$ will always mean an equivariant gradient map. Let $x$ be a fixed point in $V$ of an orbit type $(H)$, i.e., $H = G_x$. We have an orthogonal splitting

$$V = T_x(Gx) \oplus W_x \oplus N_x, \tag{3}$$

where $W_x$ is the orthogonal complement of $T_x(Gx)$ in the tangent space $T_x(V_{(H)})$ and $N_x = T_x(V_{(H)})^{\perp}$. Assume that $x \in f^{-1}(0)$ and $f$ is differentiable at $x$. With respect to the decomposition (3) the derivative $Df(x)$ is of the form (for details, see [8])

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & Kf(x) & 0 \\ 0 & 0 & Lf(x) \end{bmatrix},$$

i.e., $Kf(x) = Df(x)|_{W_x}$ and $Lf(x) = Df(x)|_{N_x}$.

### Definition 3.2.

An orbit $Gx$ is called a *regular zero orbit* of $f$, if $f(x) = 0$ and $\ker Df(x) = T_x(Gx)$. It means that the map $Kf(x) \oplus Lf(x)\colon W_x \oplus N_x \to W_x \oplus N_x$ is an isomorphism. The *Morse index* of the regular zero orbit $Gx$ is defined to be the number of negative eigenvalues of $Kf(x)$, $k = \dim W_x^-$. We set $\sigma(Gx) = (-1)^k$.

For an open $G$-set $U$ such that $\overline{U}$ is a compact subset of $V_{(H)}$ and $\varepsilon > 0$, define

$$N(U, \varepsilon) = \{v \in V : v = x + n,\ x \in U,\ n \in N_x,\ |n| < \varepsilon\}.$$

The set $N(U, \varepsilon)$ will be called a *tubular neighbourhood of type* $(H)$ provided that the decomposition $v = x + n$ is unique. Let $\varepsilon > 0$ be small enough so that $N(U, \varepsilon)$ is a tubular neighbourhood of type $(H)$. A gradient equivariant map $f$ is $(H)$-*normal* on $N(U, \varepsilon)$ if for all $v = x + n \in N(U, \varepsilon)$,

$$f(v) = f(x) + n.$$

### Definition 3.3 (generic pair).

We say that a $\nabla_G$-admissible pair $(f, \Omega)$ is *generic* if there exists an open $G$-subset $\Omega_0 \subset \Omega$ such that

(a) $f^{-1}(0) \cap \Omega \subset \Omega_0$;

(b) $f_{|\Omega_0}$ is of class $C^1$;

(c) $f^{-1}(0) \cap \Omega_0$ is composed of regular zero orbits;

(d) for each $H$ with $Z = f^{-1}(0) \cap \Omega_{(H)} \neq \emptyset$ there exists a tubular neighbourhood $N(U, \varepsilon)$ of type $(H)$ such that $Z \subset N(U, \varepsilon) \subset \Omega$ and $f$ is $(H)$-normal on $N(U, \varepsilon)$.

The next theorem allows us to define the gradient degree for a $\nabla_G$-admissible pair $(f, \Omega)$.

### Theorem 3.4 (generic approximation theorem, [8]).

*For any $\nabla_G$-admissible pair $(f, \Omega)$ there exists a generic pair $(f_1, \Omega)$ such that $(f, \Omega)$ and $(f_1, \Omega)$ are $\nabla_G$-homotopic.*

**Lemma 3.5 ([8]).**
*If $(f,\Omega)$ is a $\nabla_G$-admissible pair then there exists a gradient $G$-map $f_1\colon V\to V$ such that* **(i)** $f_1(x)=f(x)$ *for* $x\in V\setminus\Omega$ *and* **(ii)** $(f_1,\Omega)$ *is* $\nabla_G$*-admissible and generic.*

**Proof of Theorem 3.4.** Let $(f_1,\Omega)$ be a $\nabla_G$-admissible and generic pair from Lemma 3.5. Define $G$-homotopy $h\colon V\times[0,1]\to V$ as $h(x,t)=(1-t)f(x)+tf_1(x)$. Clearly the pair $(h(\cdot,t),\Omega)$ is $\nabla_G$-admissible for all $t\in[0,1]$. □

**Definition 3.6.**
Let $(f,\Omega)$ be a $\nabla_G$-admissible pair. The *$G$-equivariant gradient degree* of $(f,\Omega)$ is an element of the Euler ring $U(G)$ defined as

$$\deg_G^{\nabla}(f,\Omega) = \sum_{(H)\in\Phi(G)} n_{(H)}\mathfrak{u}_{(H)}^G,$$

where

$$n_{(H)} = \sum_{(G_{x_i})=(H)} \sigma(Gx_i)$$

and $Gx_i$ are the disjoint orbits of type $(H)$ in $f_1^{-1}(0)\cap\Omega$. Here $(f_1,\Omega)$ is any generic pair $G$-homotopic to $(f,\Omega)$.

**Theorem 3.7 ([8]).**
*If two generic pairs $(f_0,\Omega)$ and $(f_1,\Omega)$ are $G$-homotopic, then* $\deg_G^{\nabla}(f_0,\Omega)=\deg_G^{\nabla}(f_1,\Omega)$.

**Example 3.8.**
Let $\Omega=\{(x,y)\in\mathbb{R}^2 : 1/2<x^2+y^2<3/2\}$ and the action of $G=SO(2)$ on $\mathbb{R}^2$ be given by

$$\gamma_\theta = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\in SO(2), \qquad \theta\in[0,2\pi), \tag{4}$$
$$\gamma_\theta(x,y) = (x\cos k\theta - y\sin k\theta,\ x\sin k\theta + y\cos k\theta), \qquad k\in\mathbb{N}.$$

Hence $V$ is the plane $\mathbb{R}^2$ with rotations by the angle $k\theta$. Define $\varphi\colon V\to\mathbb{R}$ by the formula $\varphi(x,y)=-(x^2+y^2-1)^2$. It is easy to check that $\varphi$ is an SO(2)-invariant function, and the pair $(\nabla\varphi,\Omega)$ is $\nabla_G$-admissible. Each point except the origin has an orbit type $(\mathbb{Z}_k)$, that is $V_{(\mathbb{Z}_k)}=\mathbb{R}^2\setminus\{(0,0)\}$. The map $f=\nabla\varphi$ vanishes at the point $(x_0,y_0)=(1,0)$, and consequently the whole orbit $G(1,0)\approx S^1$ is the set of zeros of $f$. The derivative at $(1,0)$ is a map $(u,v)\mapsto(-8u,0)$ with the kernel $\ker Df(1,0)=\operatorname{span}([0,1])$ which is exactly the tangent space $T_{(1,0)}G(1,0)$. The Morse index of $G(1,0)$ is 1 and hence $\sigma(G(1,0))=-1$. Directly from the definition one obtains $\deg_G^{\nabla}(f,\Omega)=-\mathfrak{u}_{(\mathbb{Z}_k)}^G$.

## 4. Equivariant Conley index

With a locally Lipschitz vector field $v\colon\mathbb{R}^n\to\mathbb{R}^n$ one can associate a *local flow* by integration of a differential equation. More precisely, through each point $x\in\mathbb{R}^n$ there passes a maximal integral curve $\phi_x\colon(\alpha_x,\beta_x)\to\mathbb{R}^n$ satisfying $\dot\phi_x(t)=v(\phi_x(t))$ and $\phi_x(0)=x$. Setting $D=\{(t,x)\in\mathbb{R}\times V : t\in(\alpha_x,\beta_x)\}$ and $\phi(t,x)=\phi_x(t)$ we obtain a local flow on $V$, that is **(i)** $D\subset\mathbb{R}\times V$ is an open neighbourhood of $\{0\}\times V$ and $\phi\colon D\to\mathbb{R}^n$ is continuous; **(ii)** if $(t,x)\in D$ and $(s,\phi(t,x))\in D$ then $(s+t,x)\in D$ and $\phi(s,\phi(t,x))=\phi(s+t,x)$; **(iii)** $\phi(0,x)=x$.

We will be concerned with an equivariant vector field.

**Lemma 4.1 (cf. [5]).**
*Let $V$ be a representation of a compact Lie group $G$ and $v\colon V\to V$ a $G$-equivariant, locally Lipschiz vector field. Then the differential equation*

$$\dot x(t)=v(x(t))$$

*defines a local $G$-flow. That is:* **(i)** *the set $D\subset\mathbb{R}\times V$ is a $G$-set, i.e., if $(t,x)\in D$ then $(t,gx)\in D$ for all $g\in G$;* **(ii)** $\phi(t,gx)=g\phi(t,x)$ *for all $(t,x)\in D$ and $g\in G$.*

From now on we will consider local flows generated by vector fields at least of class $C^1$. Without loss of generality, for our purposes, we can assume that the equation $\dot{x} = v(x)$ generates a flow, i.e., $D = \mathbb{R} \times V$.

We will give some basic definitions and notions which are necessary for the definition of the Conley index in the presence of an action of a Lie group $G$. For the classical notion of the Conley (homotopy) index we refer the reader to [2]. Let $\phi$ be a $G$-flow on $V$. For a $G$-set $X \subset V$ the *maximal invariant subset* under the flow $\phi$ in $X$ is given by

$$\operatorname{inv}(X) = \{x \in X : \phi^t(x) \in X \text{ for all } t \in \mathbb{R}\}.$$

Since $X$ is $G$-invariant so is $\operatorname{inv}(X)$. If $X$ is in addition compact and $\operatorname{inv}(X) \subset \operatorname{int} X$, then $X$ is called an *isolating neighbourhood* and $\operatorname{inv}(X)$ is an *isolated invariant set*. For an isolated invariant set there exists a $G$-*index pair* $(N, L)$, i.e., the pair of compact $G$-invariant subsets of $V$ such that

(i) the closure of $N \setminus L$ is an isolating neighbourhood;

(ii) $L$ is positively invariant rel. $N$; *and*

(iii) if $x \in N$ and $\phi^{[0,t]}(x) \not\subset N$ for some $t > 0$, then $\phi^s(x) \in L$ for some $s \in [0, t]$.

For the existence of a $G$-index pair we refer to [6, 8].

The $G$-homotopy type of the quotient $N/L$ does not depend on the particular choice of the index pair. Recall that $N/L$ is obtained from $N$ by collapsing all points in $L$ to the point $[L]$ which is distinguished in $N/L$. The action of $G$ on $N/L$ is induced from the action on $N$ and $g[L] = [L]$ for all $g \in G$.

Assume $X \subset V$ is an isolated neighbourhood of a flow $\phi$.

### Definition 4.2.

The $G$-*equivariant Conley index* of $S = \operatorname{inv}(X)$, denoted by $h_G(S)$ (or sometimes $h_G(X, \phi)$, to indicate the isolating neighbourhood and the flow), is defined to be a $G$-homotopy type of a pointed $G$-space $N/L$, where $(N, L)$ is an arbitrary $G$-index pair for $S$. That is, $h_G(X, \phi) = [N/L]_G$.

In fact, this index is a homotopy type of some finite $G$-CW-complex, cf. [22]. The equivariant Conley index has the same properties as the ordinary one. In particular the continuation property holds. We say that $\phi\colon \mathbb{R} \times V \times [0,1] \to V$ is a continuous family of $G$-flows on $V$ if $\phi^\lambda\colon \mathbb{R} \times V \to V$ is a $G$-flow on $V$ for all $\lambda \in [0, 1]$, where $\phi^\lambda(t, x) = \phi(t, x, \lambda)$. Notice that we do not restrict the class of flows to the gradient one if it is not specified otherwise.

### Proposition 4.3.

*Suppose that $X$ is a compact $G$-subset of $V$ and $\phi$ is a continuous family of $G$-flows on $V$. If $X$ is an isolating neighbourhood for $\phi^\lambda$, $\lambda \in [0, 1]$, then $h_G(X, \phi^0) = h_G(X, \phi^1)$.*

### Example 4.4.

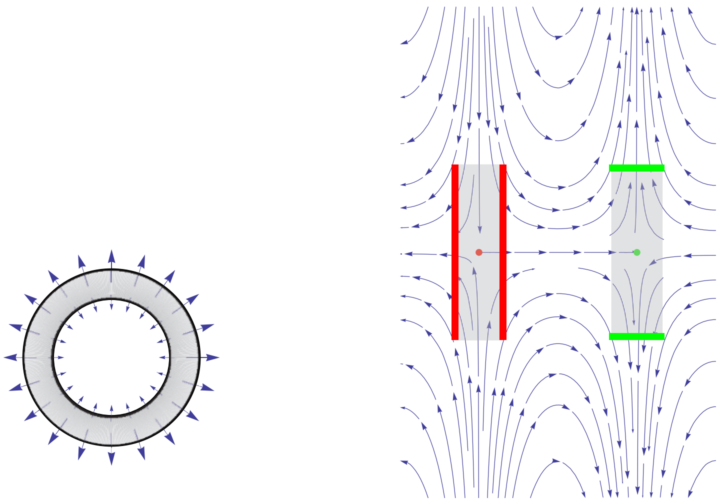
Let $G = \mathrm{SO}(2)$ and $V$ be the real plane with the action of $G$ given by rotation, i.e., for

$$\gamma_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in G, \qquad \theta \in [0, 2\pi),$$
$$\gamma_\theta(x, y) = (x\cos k\theta - y\sin k\theta,\ x\sin k\theta + y\cos k\theta), \qquad k \in \mathbb{N}.$$

Consider the $G$-flow on $V$ given by the vector field

$$v(x, y) = \big(x(x^2+y^2-1),\ y(x^2+y^2-1)\big).$$

The set $N = \{(x, y) \in V : 1/2 \le x^2+y^2 \le 3/2\}$ is an isolating $G$-invariant neighbourhood and $\operatorname{inv}(N) = \{(x, y) \in V : x^2+y^2 = 1\}$. The index pair can be chosen to be $(N, \partial N)$, see Figure 1(a). The Conley index is a $G$-homotopy type of a $G$-complex consisting of one 0-cell of orbit type $(G)$ (as a distinguished point with the trivial action) and one 1-cell of orbit type $(\mathbb{Z}_k)$. According to formula (2) we have $\mathfrak{u}(h_G(N)) = -\mathfrak{u}^G_{(\mathbb{Z}_k)}$. Notice that we do not take into account the distinguished point.



(a) (b)

**Figure 1.**

We give another simple example that shows the difference between the classical Conley index and the equivariant one.

***Example 4.5.***
Let $G = \mathbb{Z}_2 = \{e, \gamma\}$ and let $V$ equal $\mathbb{R}^2$ with the action given by $\gamma(x, y) = (x, -y)$. Let $\phi$ be a $G$-flow given by the system of two equations

$$\dot{x} = \sin x, \qquad \dot{y} = -y \cos x.$$

It is easily seen that $(k\pi, 0)$, $k \in \mathbb{Z}$, are equilibrium points. These are isolated, $\phi$- as well as $G$-invariant sets of the flow and the equivariant Conley index is well defined. Let $M_2 = (0, 0)$ (red dot) and $M_1 = (\pi, 0)$ (green dot). The equilibria $(k\pi, 0)$ for $k$ even (resp. odd) are qualitatively the same. The index pairs for $M_2$ and $M_1$ are shown in the Figure 1(b). The indices of $M_2$ and $M_1$ are $[S^{\mathbb{R}_t}]_{\mathbb{Z}_2}$ and $[S^{\mathbb{R}_a}]_{\mathbb{Z}_2}$ respectively, where $S^{\mathbb{R}_t}$ (resp. $S^{\mathbb{R}_a}$) is a compactification of the real line with the trivial (resp. antipodal) action. Since $\mathfrak{u}(S^{\mathbb{R}_a}) = \mathfrak{u}_{\mathbb{Z}_2}^{\mathbb{Z}_2} - \mathfrak{u}_E^{\mathbb{Z}_2}$ and $\mathfrak{u}(S^{\mathbb{R}_t}) = -\mathfrak{u}_{\mathbb{Z}_2}^{\mathbb{Z}_2}$, these indices turns out to be different.

## 5. Equivariant Morse equation

Now and subsequently let $H^*$ denote the Alexander–Spanier cohomology with coefficients in some principal ideal domain $R$. This particular cohomology theory is chosen because it satisfies the following *strong excision property*: Given two closed pairs $(X, A)$ and $(Y, B)$ in $V$ and a closed continuous map $f\colon (X, A) \to (Y, B)$ such that $f$ induces a bijection of $X \setminus A$ onto $Y \setminus B$, one has an isomorphism $f^*\colon H^q(Y, B; R) \to H^q(X, A; R)$ for all $q \geq 0$. For a more general statement of this fact we refer to the book by Spanier [21, Theorem 6.6.5].

If $E$ is an $R$-module then we set $\operatorname{rank} E = \dim(E \otimes_R \mathfrak{Q}_R)$, whenever $\dim(E \otimes_R \mathfrak{Q}_R)$ is finite. Otherwise $\operatorname{rank} E = \infty$. Here $\mathfrak{Q}_R$ stands for the field of quotients of the ring $R$. The comparison of the classical Euler characteristic with its equivariant analogue $\mathfrak{u}(X)$ (defined merely for a homotopy type of $G$-complexes, see Proposition 2.7), being an element of the Euler ring $U(G)$, leads us to the conclusion that the $k$-th Betti numbers of $X \in \mathcal{F}(G)$ should be the collection of the numbers $\operatorname{rank} H^k\big(X^{(H)}/G, X^{>(H)}/G\big)$, where $(H) \in \Phi(G)$. Since we are concerned with the $G$-index which is determined by an arbitrary $G$-invariant index pair, the following definition seems to be reasonable.

### *Definition 5.1.*

Let $(X, A)$ be a compact pair of $G$-invariant subsets of $V$. The numbers

$$\beta^q_{(H)}(X, A) = \operatorname{rank} H^q\big(X^{(H)}/G,\ \big(X^{>(H)} \cup A^{(H)}\big)/G\big), \qquad (H) \in \Phi(G),$$

are called the *$q$-th Betti numbers of the pair* $(X, A)$.

For an abbreviation we put $(X_H A) = \big(X^{(H)}/G,\ \big(X^{>(H)} \cup A^{(H)}\big)/G\big)$. Assuming that the modules $H^q(X_H A)$ are of finite rank, we define the formal power series taking values in $U(G)$:

$$\mathcal{P}_G(t, X, A) = \sum_{(H)\in\Phi(G)} \left( \sum_{q=0}^{\infty} \beta^q_{(H)}(X, A)\, t^q \right) \mathfrak{u}^G_{(H)}.$$

If $\beta^q_{(H)}(X, A) = 0$ for $q$ sufficiently large and for all $(H) \in \Phi(G)$, then we call it the *Poincaré polynomial* of the pair $(X, A)$. Notice that $\mathcal{P}_G(\cdot\,, X, A)$ can be viewed as an element of the polynomial ring $U(G)[t]$.

### *Definition 5.2.*

Let $S$ be an isolated invariant set of a $G$-equivariant flow $\phi$. Define the *Poincaré polynomial* of the $G$-index of $S$ as $\mathcal{P}_G(t, h_G(S)) = \mathcal{P}_G(t, N, L)$, where $(N, L)$ is an arbitrary index pair for $S$.

Let $\phi^t$ be a flow on $X$. Recall that the $\alpha$-limit and the $\omega$-limit sets of a point $x \in X$ are defined as follows:

$$\alpha(x) = \bigcap_{t\geq 0} \overline{\phi^{(-\infty,-t]}(x)} \qquad \text{and} \qquad \omega(x) = \bigcap_{t\geq 0} \overline{\phi^{[t,+\infty)}(x)}.$$

### *Definition 5.3.*

A *Morse decomposition* of an isolated invariant set $S$ is a finite collection $\mathcal{M}(S) = \{M_i : 1 \leq i \leq l\}$ of subsets $M_i \subset S$, which are disjoint, compact and invariant, and which can be ordered, $(M_1, M_2, \ldots, M_l)$, so that for every $x \in S \setminus \bigcup_{1\leq j\leq l} M_j$ there are indices $i < j$ such that $\omega(x) \subset M_i$ and $\alpha(x) \subset M_j$.

Recall that for compact sets $X \supset Y \supset Z$ there exists a connecting homomorphism $\delta^q\colon H^q(Y, Z) \to H^{q+1}(X, Y)$, and a long exact sequence of the triple $(X, Y, Z)$:

$$\ldots \xrightarrow{\delta^{q-1}} H^q(X, Y) \xrightarrow{\iota^q} H^q(X, Z) \xrightarrow{\jmath^q} H^q(Y, Z) \xrightarrow{\delta^q} \ldots,$$

where $\iota^q$ and $\jmath^q$ are homomorphisms induced by inclusions $\iota\colon (X, Z) \hookrightarrow (X, Y)$ and $\jmath\colon (Y, Z) \hookrightarrow (X, Z)$, respectively.

The remainder of this section is devoted to the proof of the Equivariant Morse Equation, Theorem 1.1. In order to overcome difficulties connected with definition of the Betti numbers, exposition is divided into several lemmas and propositions.

### *Lemma 5.4.*

*If $N_2 \supset N_1 \supset N_0$ is a triple of compact $G$-sets, $(H) \in \Phi(G)$, then*

$$H^*\left( N_1^{(H)}/G,\ \frac{N_1^{>(H)} \cup N_0^{(H)}}{G} \right) \cong H^*\left( \frac{N_2^{>(H)} \cup N_1^{(H)}}{G},\ \frac{N_2^{>(H)} \cup N_0^{(H)}}{G} \right).$$

***Proof.*** We are going to use the strong excision property of the Alexander–Spanier cohomology. Firstly we check that the pairs in question are closed. Indeed, for a closed $G$-subset $N \subset V$ and a closed subgroup $H \subset G$ one has $N^H = N \cap V^H$. Since $V^H$ is a linear subspace of $V$, $V^H$ is closed and so is $N^H$. Further $N^{(H)} = GN^H$ is closed, because the action of a compact Lie group is a closed map [1, Theorem 1.1.2]. The set of orbit types of a finite dimensional representation is always finite, hence $N^{>(H)}$ is closed as a finite sum of closed sets. Lastly, the set of orbits $N/G$ endowed with the quotient topology is closed since the projection $N \to N/G$ taking $x$ into its orbit is closed [1, Theorem 1.3.1]. Clearly, the inclusion

$$e\colon \left(N_1^{(H)}/G, \frac{N_1^{>(H)} \cup N_0^{(H)}}{G}\right) \hookrightarrow \left(\frac{N_2^{>(H)} \cup N_1^{(H)}}{G}, \frac{N_2^{>(H)} \cup N_0^{(H)}}{G}\right)$$

is continuous and closed. Moreover, for each $x \in (N_1^{(H)}/G) \setminus \left(\left(N_1^{>(H)} \cup N_0^{(H)}\right)/G\right)$ one has $e(x) = x$. So the strong excision property applies and the result follows. □

### Lemma 5.5.

*Assume that the bottom row of the diagram*

$$\begin{array}{ccccccccc}
\xrightarrow{\iota^*} & H^*(X,Z) & \xrightarrow{\xi^* \jmath^*} & H^*(A,B) & \xrightarrow{\delta^* \eta^*} & H^{*+1}(X,Y) & \xrightarrow{\iota^{*+1}} \\
 & \| & & \xi^* \uparrow \cong & & \| & \\
\xrightarrow{\iota^*} & H^*(X,Z) & \xrightarrow{\jmath^*} & H^*(Y,Z) & \xrightarrow{\delta^*} & H^{*+1}(X,Y) & \xrightarrow{\iota^{*+1}}
\end{array}$$

*is exact, $\xi^*\colon H^*(Y,Z) \to H^*(A,B)$ is an isomorphism and $\eta^* = (\xi^*)^{-1}$. Then the upper row is exact.*

***Proof.*** Let $a \in \operatorname{Im} \iota^*$. Then $a \in \ker \jmath^*$ is equivalent to $a \in \ker(\xi^* \jmath^*)$ since $\xi^*$ is an isomorphism. If $b$ lies in $\operatorname{Im}(\xi^* \jmath^*)$ then $b = \xi^* \jmath^* a$ for some $a \in H^*(X,Z)$ and $\jmath^* a = \eta^* b$ which means that $\eta^* b \in \ker \delta^*$ and $b \in \ker \delta^* \eta^*$. This reasoning can be reverted. And at last, if $c = \delta^* \eta^* b$ for some $b$, then $c \in \operatorname{Im} \delta^*$ so $c \in \ker \iota^{*+1}$. □

The following lemma is a consequence of a well-known theorem from linear algebra. The proof can be found for instance in [18].

### Lemma 5.6.

*If $E \xrightarrow{f} F \xrightarrow{g} G$ is an exact sequence of homomorphisms of $R$-modules, then* $\operatorname{rank} F = \operatorname{rank} \operatorname{Im} f + \operatorname{rank} \operatorname{Im} g$.

### Proposition 5.7.

*If $X_0 \subset X_1 \subset \ldots \subset X_m$ is a filtration of compact $G$-sets, then there exists $\mathcal{Q}_G(t)$ of the form (1), with all $\rho^q_{(H)} \geq 0$ and such that*

$$\sum_{j=1}^{m} \mathcal{P}_G(t, X_j, X_{j-1}) = \mathcal{P}_G(t, X_m, X_0) + (1+t)\mathcal{Q}_G(t).$$

***Proof.*** Fix $(H) \in \Phi(G)$. By Lemmas 5.4 and 5.5 we have a long exact sequence

$$\ldots \xrightarrow{\delta^{q-1}_{(H)} \eta^{q-1}_{(H)}} H^q(X_{j\,H} X_{j-1}) \xrightarrow{\iota^q_{(H)}} H^q(X_{j\,H} X_0) \xrightarrow{\xi^q_{(H)} \jmath^q_{(H)}} H^q(X_{j-1\,H} X_0) \to \ldots \tag{5}$$

Here $\iota_{(H)}$ and $\jmath_{(H)}$ are suitable inclusions, $\xi^q_{(H)}$ stands for the isomorphism

$$H^q(X_{j-1\,H} X_0) \cong H^q\left(\frac{X_j^{>(H)} \cup X_{j-1}^{(H)}}{G}, \frac{X_j^{>(H)} \cup X_0^{(H)}}{G}\right),$$

and $\eta^q_{(H)}$ is its inverse. Set $\rho^q_{(H)}(X_j, X_{j-1}, X_0) = \operatorname{rank}\operatorname{Im}\big(\delta^q_{(H)}\eta^q_{(H)}\big)$. The exactness of (5) and Lemma 5.6 imply that

$$\begin{aligned}\beta^q_{(H)}(X_{j-1}, X_0) &= \rho^q_{(H)}(X_j, X_{j-1}, X_0) + \operatorname{rank}\operatorname{Im}\xi^q_{(H)}J^q_{(H)} = \rho^q_{(H)}(X_j, X_{j-1}, X_0) + \beta^q_{(H)}(X_j, X_0) - \operatorname{rank}\operatorname{Im}\iota^q_{(H)} \\ &= \rho^q_{(H)}(X_j, X_{j-1}, X_0) + \beta^q_{(H)}(X_j, X_0) - \beta^q_{(H)}(X_j, X_{j-1}) + \rho^{q-1}_{(H)}(X_j, X_{j-1}, X_0).\end{aligned}$$

Consequently, $\beta^q_{(H)}(X_j, X_{j-1}) + \beta^q_{(H)}(X_{j-1}, X_0) = \beta^q_{(H)}(X_j, X_0) + \rho^q_{(H)} + \rho^{q-1}_{(H)}$. Multiplying this equality by $t^q$ and summing over $q \geq 0$ and $(H) \in \Phi(G)$ one has

$$\mathcal{P}_G(t, X_j, X_{j-1}) + \mathcal{P}_G(t, X_{j-1}, X_0) = \mathcal{P}_G(t, X_j, X_0) + (1+t)\widehat{\mathcal{Q}}_G(t, X_j, X_{j-1}, X_0), \tag{6}$$

where

$$\widehat{\mathcal{Q}}_G(t, X_j, X_{j-1}, X_0) = \sum_{(H)\in\Phi(G)} \left(\sum_{q=0}^{\infty} \rho^q_{(H)}(X_j, X_{j-1}, X_0)\, t^q\right) \mathfrak{u}^G_{(H)}.$$

Summing (6) over $2 \leq j \leq m$ and setting $\mathcal{Q}_G(t) = \sum_{j=2}^{m} \widehat{\mathcal{Q}}_G(t, X_j, X_{j-1}, X_0)$ we obtain the desired result. □

### *Definition 5.8.*

Let $X$ be an isolating neighbourhood of a $G$-flow on $V$ and $(M_1, \ldots, M_m)$ be a $G$-invariant Morse decomposition of $S = \operatorname{inv}(X)$. A *$G$-invariant index filtration* is a sequence $N_0 \subset N_1 \subset \ldots \subset N_m$ of compact $G$-invariant subsets of $V$ such that $(N_k, N_{k-1})$ is a $G$-index pair for $M_k$ and $(N_m, N_0)$ is an index pair for $S$.

### *Proposition 5.9.*

*Every $G$-invariant Morse decomposition admits a $G$-invariant index filtration.*

***Proof.*** Let us forget for a while that a Morse decomposition has a group symmetry. It is well known that every Morse decomposition admits an index filtration $N_0 \subset \ldots \subset N_m$, cf. for instance [16]. Averaging a given filtration over group $G$ we obtain a $G$-invariant index filtration for a $G$-invariant Morse decomposition. The compactness of $N_i$, $0 \leq i \leq m$, survives since $G$ is assumed to be compact [1, Corollary 1.1.3]. □

The proof of Morse Equation (Theorem 1.1) is a straightforward consequence of Propositions 5.7 and 5.9.

### *Example 5.10.*

Consider again the Hamiltonian flow from Example 4.5. Let $S$ be the set consisting of equilibria $M_1, M_2$ and connecting the orbit between them. The sets $M_1$ and $M_2$ form the Morse decomposition of $S$. The corresponding Poincaré polynomials are of the form

$$\mathcal{P}_{\mathbb{Z}_2}(t, h_{\mathbb{Z}_2}(S)) = t\mathfrak{u}^{\mathbb{Z}_2}_E, \qquad \mathcal{P}_{\mathbb{Z}_2}(t, h_{\mathbb{Z}_2}(M_1)) = \mathfrak{u}^{\mathbb{Z}_2}_{\mathbb{Z}_2} + t\mathfrak{u}^{\mathbb{Z}_2}_E, \qquad \mathcal{P}_{\mathbb{Z}_2}(t, h_{\mathbb{Z}_2}(M_1)) = t\mathfrak{u}^{\mathbb{Z}_2}_{\mathbb{Z}_2},$$

and one can see the relation

$$\mathcal{P}_{\mathbb{Z}_2}(t, h_{\mathbb{Z}_2}(M_1)) + \mathcal{P}_{\mathbb{Z}_2}(t, h_{\mathbb{Z}_2}(M_2)) = \mathcal{P}_{\mathbb{Z}_2}(t, h_{\mathbb{Z}_2}(S)) + (1+t)\mathfrak{u}^{\mathbb{Z}_2}_{\mathbb{Z}_2}.$$

## 5.1. Poincaré polynomial of a critical orbit

Recall that if $\varphi\colon M \to \mathbb{R}$ is a smooth function defined on compact closed Riemannian manifold, then the point $p \in M$ is called *critical* if $\nabla\varphi(p) = 0$. We say that $p \in M$ is a *nondegenerate* critical point if the Hessian of $\varphi$ at $p$ is nonsingular. In this case the *index* of $\varphi$ at $p$, denoted by $\operatorname{ind}_\varphi(p)$ is the dimension of the maximal subspace of $T_pM$ on which the Hessian is negative definite. In other words, this is the number of negative eigenvalues of the Hessian, counting with multiplicity. It is easy to see that $\{p\}$ is an isolated invariant set of a flow given by $\dot{x} = -\nabla\varphi(x)$ and the Conley index of $\{p\}$ is a homotopy type of pointed sphere of dimension $\mathrm{ind}_\varphi(p)$. In consequence $\mathcal{P}(t, h(\{p\})) = t^{\mathrm{ind}_\varphi(p)}$, cf. [14]. We expect a similar result in the equivariant case.

In this section we are going to calculate the Poincaré polynomial of a critical orbit of a $G$-invariant function $\varphi\colon V \to \mathbb{R}$, see Proposition 5.16. In order to do this, we need to impose some nondegeneracy condition.

### Definition 5.11.

Let $\varphi\colon V \to \mathbb{R}$ be a smooth $G$-invariant function.

- The orbit $Gx$ is called a *critical orbit of* $\varphi$, if $\nabla\varphi(x) = 0$ (and consequently, for each $y \in Gx$, $\nabla\varphi(y) = 0$).
- The critical orbit $Gx$ of $\varphi$ is said to be *hyperbolic*, if $Gx$ is a regular zero orbit of $\nabla\varphi$, cf. Definition 3.2.

### Proposition 5.12 ([1, p. 4]).

*Let $G$ be a compact group and $H$ a closed subgroup of $G$. Then $gHg^{-1} = H$ iff $gHg^{-1} \subset H$.*

### Corollary 5.13.

*Let $x \in V$, $H = G_x$ and $S_x$ be a slice at $x$. If $Q \subset S_x \cap V_{(H)}$, then each point of $Q$ is stationary under $H$.*

**Proof.** Let $y \in Q$. By the slice theorem [15, p. 184] we see that $G_y \subset H$. Since $G_y$ and $H$ are conjugate, there exists $g \in G$ such that $gHg^{-1} = G_y \subset H$. Proposition 5.12 implies that $G_y = H$. □

### Lemma 5.14.

*Let $x \in V$, $H = G_x$ and $S_x$ be a slice at $x$. If $Q \subset S_x \cap V_{(H)}$, then $(G\times_H Q)^{(H)} = (G/H)\times Q$ and $(G\times_H Q)^{>(H)} = \emptyset$.*

**Proof.** By the definition $G\times_H Q$ is a homogenous space of an action of the group $H$ on $G\times Q$ defined by $h(g,x) = (gh^{-1}, hx)$. Since $Q \subset S_x \cap V_{(H)}$ we have $h(g,x) = (gh^{-1}, x)$ so the quotient space is $(G/H)\times Q$. We claim that $(G/H)\times Q \subset V_{(H)}$. Indeed, both $Q$ and $G/H \approx Gx$ are contained in $V_{(H)}$, hence $h(Hg, q) = (Hgh^{-1}, hq) = (Hg, q)$ for $h \in H$, that is $(G/H)\times Q \subset V^{(H)}$. If there existed $K \supsetneq H$ such that $k(Hg, q) = (Hg, q)$ for all $k \in K$, then $Hg, q \in V^{(K)}$ and it would be a contradiction, since $G/H$ and $Q \subset V_{(H)}$. The result follows. □

### Definition 5.15.

We say that a $G$-invariant subset $X_0$ of a $G$-set $X$ is a *strong $G$-deformation retract of $X$* if there exists a $G$-homotopy $r\colon X\times[0,1] \to X$ such that the following properties hold true:

- $r(x, 0) = x$ for all $x \in X$;
- $r(x, t) = x$ for all $(x, t) \in X_0\times[0,1]$;
- $r(x, 1) \in X_0$.

If $\pi\colon E \to M$ is a $G$-vector bundle, then $M$ is a strong $G$-deformation retract of $E$. Indeed, we identify $M$ with a zero section of a bundle $\pi\colon E \to M$, that is $M = \{(x, v) \in E : v = 0 \in E_x\}$. The homotopy is given by the formula $r((x, v), t) = (x, (1-t)v)$.

### Proposition 5.16.

*Suppose that $(f, \Omega)$ is a $\nabla_G$-admissible and generic pair. Let $\phi$ denotes the flow generated by $\dot{x} = -f(x)$ and $Gx_0$ is an isolated zero orbit of $f$ such that $Gx_0 = \mathrm{inv}_\phi(\overline{\Omega})$. Then*

$$\mathcal{P}_G(t, h_G(Gx_0)) = t^{\dim W^-_{x_0}}\mathfrak{u}^G_{(H)},$$

*where $(H) = (G_{x_0})$. If $Gx_0$ is a principal regular zero orbit then the assumption about genericity can be removed. Recall that $W^-_{x_0}$ stands for unstable subspace of $-Df(x_0)$.*

**Proof.** **(a)** Suppose that $(f,\Omega)$ is a $\nabla_G$-admissible and generic. We shall construct an index pair for $Gx_0$ via suitable choice of the index pair in the fiber of the bundle over an orbit $Gx_0$. Let $E_y = (T_y Gx_0)^\perp$. By the slice theorem [15, p. 184] the projection $p\colon E \to Gx_0$, where $E = \{(x,v) \in Gx_0 \times V : v \in E_x\}$, is a smooth vector bundle isomorphic to $\pi\colon G\times_H E_{x_0} \to Gx_0$. Recall that $E_{x_0}$ is an $H$-representation space. The subspace $E_{x_0} \subset V$ splits into $E^+_{x_0} \oplus E^-_{x_0}$, the stable and unstable subspaces corresponding to positive and negative spectrum of $Df(x_0)$. Since $x_0$ is a nondegenerate critical point, there exists an open $H$-neighbourhood $U$ of zero in $E_{x_0}$ such that the flow is given in local $H$-coordinates $\psi\colon U \to E_{x_0}$ by the system of equations

$$\dot{x}_+ = A_1 x_+ + g_1(x) \quad \text{and} \quad \dot{x}_- = A_2 x_- + g_2(x), \qquad x = (x_+, x_-) \in E^+_{x_0} \oplus E^-_{x_0},$$

for $|x| = \max\{|x_+|, |x_-|\} \leq 2$ with $g_{1,2}$ and $Dg_{1,2}$ vanishing at zero, i.e. $|g_{1,2}(x)| = o(|x|)$ as $|x| \to 0$. Moreover, one can choose the coordinates so that $|g_{1,2}(x)|$ and $\|Dg_{1,2}(x)\|$ are as small as we want, cf. Appendix. The linear parts are chosen so that there exists $\lambda > 0$ for which the following estimations hold:

$$\langle A_1 x_+, x_+\rangle \leq -\lambda |x_+|^2, \qquad \langle A_2 x_-, x_-\rangle \geq \lambda |x_-|^2.$$

If so, let $B = \{x \in E_{x_0} : |x| \leq 1\}$ and $B^- = \{x \in B : |x_-| = 1\}$. Then $N = \psi^{-1}(B)$ and $L = \psi^{-1}(B^-)$ is an index pair for the system on $E_{x_0}$. Finally, the index pair $(X,A)$ for $Gx_0$ is given by $X = G\times_H N$ and $A = G\times_H L$. We shall use the assumption that $(f,\Omega)$ is generic. It implies that the normal direction for $V_{(H)}$ is attracting. Set $B_0 = \{(x_+,x_-) \in B : |x_+| = 0\}$ and $B_0^- = \{(x_+,x_-) \in B^- : |x_+| = 0\}$ and next $N_0 = \psi^{-1}(B_0)$ and $L_0 = \psi^{-1}(B_0^-)$. There is a strong $H$-deformation retract of $(N,L)$ onto $(N_0,L_0)$, hence by the functoriality property of the twisted product the pair $X_0 = G\times_H N_0$ and $A_0 = G\times_H L_0$ is a strong $G$-deformation retract of $(X,A)$. The sets $N_0$ and $L_0$ are contained in $V_{(H)}$ and by Lemma 5.14 we have

$$H^q(X_H A) \cong H^q(X_{0H} A_0) \cong H^q\big(((G/H)\times N_0)/G,\, ((G/H)\times L_0)/G\big) \cong H^q(N_0, L_0) = \begin{cases} R & \text{for } q = k, \\ 0 & \text{otherwise,}\end{cases}$$

since the pair $(N_0,L_0)$ is a homological pointed $k$-sphere, where $k = \dim W^-_{x_0}$ a dimension of a subspace composed by the repelling directions, that is the number of negative eigenvalues of $Df(x_0)$. Hence $\mathcal{P}_G(t, h_G(Gx_0)) = t^k \mathfrak{u}^G_{(H)}$.

**(b)** Suppose now that $(f,\Omega)$ is not a generic pair, but $Gx_0$ is a regular zero orbit and $(H)$ is a principal orbit type, $H = G_{x_0}$. Then one can find an open $G$-subset $\Omega_0 \subset \overline{\Omega}_0 \subset \Omega$ such that $Gx_0 = \mathrm{inv}_\phi(\Omega_0)$ and $\overline{\Omega}_0 \subset V_{(H)}$. The result follows by using the same arguments as above. □

**Corollary 5.17.**
*Let $(f,\Omega)$ be a $\nabla_G$-admissible and generic pair, $\phi$ a flow generated by $\dot{x} = -f(x)$ and $Gx_0$ an isolated zero orbit of $f$ such that $Gx_0 = \mathrm{inv}_\phi(\overline{\Omega})$. Then*

$$\mathfrak{u}(h_G(Gx_0)) = \sigma(Gx_0)\mathfrak{u}^G_{(H)},$$

*where $(H) = (G_{x_0})$. The formula remains valid if $(f,\Omega)$ is a $\nabla_G$-admissible pair and $Gx_0$ is a principal regular zero orbit.*

**Proof.** By the above proposition, $\mathfrak{u}(h_G(Gx_0)) = \mathcal{P}(-1, X, A) = (-1)^{\dim W^-_{x_0}} \mathfrak{u}^G_{(H)} = \sigma(Gx_0)\mathfrak{u}^G_{(H)}$. □

**Corollary 5.18.**
*Let $(f,\Omega)$ be a $\nabla_G$-admissible generic pair, $\phi$ a flow generated by $\dot{x} = -f(x)$, and $Gx_0$ an isolated zero orbit of $f$ such that $Gx_0 = \mathrm{inv}_\phi(\overline{\Omega})$. Then $\mathfrak{u}(h_G(Gx_0)) = \deg^\nabla_G(f,\Omega)$. The formula remains valid if $(f,\Omega)$ is a $\nabla_G$-admissible pair and $Gx_0$ is a principal regular zero orbit.*

The $G$-index of Conley is additive in the following sense.

***Proposition 5.19.***
*If $S$ is an isolated invariant $G$-set, and $S$ is a disjoint union $S_1 \cup S_2$ of isolated invariant $G$-sets, then*

$$\mathcal{P}_G(t, h_G(S)) = \mathcal{P}_G(t, h_G(S_1)) + \mathcal{P}_G(t, h_G(S_2)).$$

***Proof.*** Let $(X, A)$ (resp. $(Y, B)$) be a $G$-index pair for $S_1$ (resp. $S_2$). Since $S_1$ and $S_2$ are isolated one can chose those pair to be disjoint. It is clear that $(X \cup Y, A \cup B)$ is a $G$-index pair for $S$. Since the pairs in question are disjoint and $G$-invariant, one has

$$H^q((X \cup Y)_H(A \cup B)) \;=\; H^q\left(X^{(H)}/G \cup Y^{(H)}/G,\ \left(X^{>(H)} \cup A^{(H)}\right)/G \cup \left(Y^{>(H)} \cup B^{(H)}\right)/G\right).$$

Therefore, by the fact that pairs $(X_H A)$ and $(Y_H B)$ are disjoint, we conclude that

$$H^q((X \cup Y)_H(A \cup B)) \;\cong\; H^q(X_H A) \oplus H^q(Y_H B).$$

The above isomorphism implies that $\operatorname{rank} H^q((X \cup Y)_H(A \cup B)) = \operatorname{rank} H^q(X_H A) + \operatorname{rank} H^q(Y_H B)$ and the result follows (according to the fact that addition in $U(G)$ is coordinatewise, cf. Proposition 2.7). □

## 5.2. Comparison of the equaivariant gradient degree and the equivariant Conley index

In [8], Gęba shows that the gradient equivariant degree of a $\nabla_G$-admissible pair $(f, \Omega)$ is equal to the class in $U(G)$ representing the homotopy type of the equivariant Conley index $h_G(\Omega, \phi_f)$, where $\phi_f$ stands for the flow generated by $\dot{x} = -f(x)$. This result is interesting from a theoretical point of view as well as applications, see [10]. We are going to obtain this equality using the equivariant Morse equation.

***Theorem 5.20 (Gęba [8]).***
*Let $(f, \Omega)$ be a $\nabla_G$-admissible pair and let $\overline{\Omega}$ be an isolating $G$-invariant neighbourhood of a flow $\phi_f$ generated by the equation $\dot{x} = -f(x)$, $S = \operatorname{inv}(\Omega)$. Then $\mathfrak{u}(h_G(S)) = \deg_G^{\nabla}(f, \Omega)$.*

***Proof.*** The proof of the theorem is divided into two parts. The first one is almost word for word rewritten from [8].

**(a)** We will show that $S$ can be continued to an isolated invariant $G$-set of a flow given by generic function. By the compactness of $\partial\Omega$ one can choose $T > 0$ so that for any $x \in \partial\Omega$ there is $t \in [-T, T]$ and $\phi(t, x) \notin \overline{\Omega}$. Define $\Omega_1 = \Omega \setminus \phi(\partial\Omega \times [-T, T])$. It is clear that $(f, \Omega_1)$ is $\nabla_G$-admissible. By Lemma 3.5 there is a gradient $G$-map $f_1 \colon V \to V$ satisfying $f_1(x) = f(x)$ for all $x \in V \setminus \Omega_1$ with the pair $(f_1, \Omega_1)$ being generic. Define the homotopy $h \colon V \times [0, 1] \to V$ by the formula $h(x, \lambda) = (1 - \lambda) f(x) + \lambda f_1(x)$ and let $\phi^\lambda$ stand for the flow generated by $-h(\cdot, \lambda)$. Notice that $h(x, \lambda) = f(x)$ for all $x \in V \setminus \Omega_1$, that is $\phi^\lambda = \phi$ on the set $\partial\Omega \times [-T, T]$. Therefore $\overline{\Omega}$ is an isolating neighbourhood for the flow $\phi^\lambda$ for $\lambda \in [0, 1]$. The continuation property of the equivariant Conley index applies and one has $h_G(S_1) = h_G(S)$, where $S_1 = \operatorname{inv}_{\phi_{f_1}}(\Omega)$.

**(b)** Since the pair $(f_1, \Omega)$ is generic, the set $S_1$ is composed of regular zero orbits $Gx_1, \dots, Gx_m$ of the function $f_1$ and flow lines between them. Moreover, the collection of orbits $\mathcal{M} = (Gx_1, \dots, Gx_m)$ forms a Morse decomposition of $S_1$. We choose an ordering of $\mathcal{M}$ given by the potential $\varphi_1 \colon \Omega \to \mathbb{R}$, $f_1 = \nabla\varphi_1$, i.e. we order the critical orbits in such a manner that $\varphi_1(Gx_i) < \varphi_1(Gx_j)$ whenever $i > j$. By the equivariant Morse equation,

$$\mathfrak{u}(h_G(S_1)) = \mathcal{P}_G(-1, h_G(S_1)) = \sum_{k=1}^{m} \mathcal{P}_G(-1, h_G(Gx_k, \phi_{f_1}))$$

For $1 \le k \le m$ take open $G$-subsets $\Omega_k \subset \Omega$ such that $\Omega_i \cap \Omega_j = \emptyset$ and $\overline{\Omega}_k$ is an isolating neighbourhood for an isolated critical zero orbit $Gx_k$. By Corollary 5.17 one has $\mathcal{P}_G(-1, h_G(Gx_k, \phi_{f_1})) = \deg_G^{\nabla}(f_1, \Omega_k)$. Hence

$$\mathfrak{u}(h_G(S)) = \mathfrak{u}(h_G(S_1)) = \sum_{k=1}^{m} \deg_G^{\nabla}(f_1, \Omega_k) = \deg_G^{\nabla}(f_1, \Omega) = \deg_G^{\nabla}(f, \Omega)$$

by the additivity property and the homotopy invariance of the gradient equivariant degree, cf. [17, Theorem 3.2]. □

***Example 5.21.***
As an example we compute the gradient equivariant degree of the pair $(-\mathrm{id}, B)$, where $-\mathrm{id}\colon V \to V$, $V$ is an orthogonal finite dimensional representation of $G = SO(2) = \{\gamma_\theta : 0 \le \theta < 2\pi\}$ ($\gamma_\theta$ is given by (4)) and $B$ stands for the unit ball in $V$. The result coincides with the calculation made by Rybicki [20, Lemma 4.1]. Let us introduce the following irreducible representation of $G$. The notation is borrowed from [20]. For $m \in \mathbb{N}$ let $\mathbb{R}[1,m] = (\mathbb{R}^2, \rho_m)$, where $\rho_m\colon G \to O(2)$ is given by

$$\rho_m(\gamma_\theta)(x,y) = (x\cos m\theta - y\sin m\theta,\ x\sin m\theta + y\cos m\theta).$$

For $k \in \mathbb{N}$ set $\mathbb{R}[k,m] = \bigoplus_{i=1}^{k} \mathbb{R}[1,m]$. Similarly we define $\mathbb{R}[k,0] = \bigoplus_{i=1}^{k} \mathbb{R}[1,0]$, where $\mathbb{R}[1,0]$ stands for trivial representation on the real line. Each orthogonal finite dimensional representation of $G$ can be represented, up to equivalence, as $V = \bigoplus_{i=0}^{p} \mathbb{R}[k_i, m_i]$, where $k_i, m_i \in \mathbb{N}$ for $1 \le i \le p$, $k_0 \in \mathbb{N} \cup \{0\}$ and $0 = m_0 < m_1 < \ldots < m_p$.

The multiplicative structure of $U(G)$ is well known and can be expressed explicitly, cf. [20]. For convenience we denote the trivial subgroup of $G$ as $\mathbb{Z}_1$. If $a = a_0\mathfrak{u}_G^G + \sum_{j=1}^{\infty} a_j \mathfrak{u}_{\mathbb{Z}_j}^G$ and $b = b_0\mathfrak{u}_G^G + \sum_{j=1}^{\infty} b_j \mathfrak{u}_{\mathbb{Z}_j}^G$ then

$$ab = a_0 b_0 \mathfrak{u}_G^G + \sum_{j=1}^{\infty} (a_0 b_j + a_j b_0)\mathfrak{u}_{\mathbb{Z}_j}^G. \tag{7}$$

Notice that $\overline{B}$ is an isolating $G$-neighbourhood for the flow defined by the identity vector field and the sphere $S = \partial B$ is an exit set. Let $S^V = \overline{B}/S$ with the $G$-action induced from $V$. Hence, one has to compute $\mathfrak{u}(S^V)$. According to the $G$-homeomorphism $S^{V\oplus W} \approx S^V \wedge S^W$ and the formula (7),

$$\mathfrak{u}(S^V) = \mathfrak{u}\big(S^{\oplus_{i=0}^{p}\mathbb{R}[k_i,m_i]}\big) = \prod_{i=0}^{p} \mathfrak{u}(S^{\mathbb{R}[k_i,m_i]}) = \prod_{i=0}^{p} \mathfrak{u}(S^{\mathbb{R}[1,m_i]})^{k_i}.$$

Since $S^{\mathbb{R}[1,m_i]}$ is composed of, for instance, one 0-cell of orbit type $G$ and one 1-cell of orbit type $\mathbb{Z}_{m_i}$, the equality $\mathfrak{u}(S^{\mathbb{R}[1,m_i]}) = \mathfrak{u}_G^G - \mathfrak{u}_{\mathbb{Z}_{m_i}}^G$ holds. Also $\mathfrak{u}(S^{\mathbb{R}[1,0]}) = -\mathfrak{u}_G^G$, therefore

$$\mathfrak{u}(S^V) = (-1)^{k_0}\mathfrak{u}_G^G \prod_{i=1}^{p} \big(\mathfrak{u}_G^G - \mathfrak{u}_{\mathbb{Z}_{m_i}}^G\big)^{k_i} = (-1)^{k_0}\mathfrak{u}_G^G \prod_{i=1}^{p} \big(\mathfrak{u}_G^G - k_i\mathfrak{u}_{\mathbb{Z}_{m_i}}^G\big) = (-1)^{k_0}\left(\mathfrak{u}_G^G + \sum_{i=1}^{p} k_i \mathfrak{u}_{\mathbb{Z}_{m_i}}^G\right).$$

By Theorem 5.20 we obtain $\deg_G^{\nabla}(-\mathrm{id}, B) = (-1)^{k_0}\big(\mathfrak{u}_G^G + \sum_{i=1}^{p} k_i \mathfrak{u}_{\mathbb{Z}_{m_i}}^G\big)$.

## 6. Multiplicity results

As an application of the equivariant Morse equation we shall prove a simple multiplicity result in the critical point problem. Before we proceed to the result's statement we briefly describe some special action of the cyclic group.

Let $p$ be a prime number and $k_1, \ldots, k_n$ integers relatively prime to $p$. Consider an action of $\mathbb{Z}_p$ on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ generated by the rotation

$$\rho(z_1, \ldots, z_n) = \big(e^{2\pi i k_1/p} z_1, \ldots, e^{2\pi i k_n/p} z_n\big). \tag{8}$$

This action is free. Any nonzero $z \in \mathbb{C}^n$ has a nonzero coordinate $z_j$ and then $e^{2\pi i s k_j/p} z_j \ne z_j$ for $0 < s < p$ since $k_j$ is relatively prime to $p$. The group acts via isometries hence the sphere $S^{2n-1}$ is a $\mathbb{Z}_p$-invariant set. The orbit space $S^{2n-1}/\mathbb{Z}_p$ is called the *lens space*, denoted by $L_p = L_p(k_1, \ldots, k_n)$. In particular for $p = 2$ we have $L = \mathbb{R}P^{2n-1}$. The above construction can be performed for an arbitrary integer $p > 1$. We choose $p$ prime to have a structure of a field in the set $\mathbb{Z}_p$, as a set of coefficients for a cohomology theory. The cohomology groups of $L$ with $\mathbb{Z}_p$ coefficients are known and they are [11, Example 3.41, p. 251]

$$\widetilde{H}^q(L_p; \mathbb{Z}_p) \cong \begin{cases} 0 & \text{for } q = 0, \\ \mathbb{Z}_p & \text{for } 1 \le q \le 2n-1. \end{cases}$$

### Proposition 6.1.

*Assume that $V = \mathbb{R}^{2n}$ is a $\mathbb{Z}_p$-representation with the action given by (8) and $f\colon V \to \mathbb{R}$ is a smooth $\mathbb{Z}_p$-invariant function. Suppose that*

(i) *there exists a $\mathbb{Z}_p$-isolating neighbourhood $X_0$ such that $0 \in S_0 = \operatorname{inv}(X_0)$ and $\mathcal{P}_{\mathbb{Z}_p}\big(t, h_{\mathbb{Z}_p}(S_0)\big) = \mathfrak{u}_{\mathbb{Z}_p}^{\mathbb{Z}_p}$;*

(ii) *$f(x) = -|x|^2/2 + \varphi_\infty(x)$, in a neighbourhood of the infinity and $\nabla\varphi_\infty$ is bounded.*

*If $f$ has only a finite number of critical orbits, say $\{\mathbb{Z}_p x_0, \dots, \mathbb{Z}_p x_m\}$, and all of them are hyperbolic, then there are at least $2n$ of them. Moreover, $(\mathbb{Z}_p)_{x_k} = E$ ($E$ stands for the trivial subgroup) for $1 \le k \le 2n$ ($2np$ critical points) and each number in the set $\{1, \dots, 2n\}$ is the Morse index of some critical point.*

***Proof.*** Consider the negative gradient $\mathbb{Z}_p$-flow $\phi_f$ of $\dot{x} = -f(x)$. Let $D_\rho(V)$ (resp. $S_\rho(V)$) stand for the disk (sphere) in $V$ of radius $\rho > 0$. It follows from (ii) that $D_R(V)$ is a $\mathbb{Z}_p$-isolating neighbourhood for sufficiently large $R$, and $(X, A) = (D_R(V), S_R(V))$ is a $\mathbb{Z}_p$-index pair for this flow. Notice that $X^E/\mathbb{Z}_p \approx (L_p \times [0,1])/(L_p \times \{1\}) = CL_p$ is a cone over $L_p$ and $(X^{\mathbb{Z}_p} \cup A^E)/\mathbb{Z}_p \approx (L_p \times \{0\}) \dot{\cup} (L_p \times \{1\}) = \widehat{C}L_p$ is a disjoint union of the bottom and the top of the cone. One has

$$H^q(CL_p, \widehat{C}L_p; \mathbb{Z}_p) \;\cong\; H^q\big(S(L_p) \vee S^1, \{\mathrm{pt}\}; \mathbb{Z}_p\big) \;\cong\; \begin{cases} 0 & \text{for } q = 0, \\ \mathbb{Z}_p & \text{for } 1 \le q \le 2n. \end{cases}$$

It is easily seen that $H^q\big(X^{\mathbb{Z}_p}/\mathbb{Z}_p, (X^{>\mathbb{Z}_p} \cup A^{\mathbb{Z}_p})/\mathbb{Z}_p\big) \cong H^q(S^0, \{\mathrm{pt}\}) \cong \mathbb{Z}_p$ for $q = 0$ and is zero otherwise. Thus the Poincaré polynomial of the $\mathbb{Z}_p$-Conley index of $S = \operatorname{inv}(X)$ is

$$\mathcal{P}_{\mathbb{Z}_p}(t, h_{\mathbb{Z}_p}(S)) = \mathfrak{u}_{\mathbb{Z}_p}^{\mathbb{Z}_p} + \big(t^{2n} + t^{2n-1} + \cdots + t\big)\,\mathfrak{u}_E^{\mathbb{Z}_p}.$$

Since all equilibria are hyperbolic they form together with $S_0$ a Morse decomposition $(S_0, M_1, \dots, M_m)$ of $S$. All nonzero orbits are principal, hence by Proposition 5.16, the Poincaré polynomial of $h_{\mathbb{Z}_p}(M_i)$ is $t^q \mathfrak{u}_E^{\mathbb{Z}_p}$ provided that $q$ is the Morse index of $M_i$. Denote by $c_k$ the number of critical orbits of index $k$. By the equation (EME) there are nonnegative integers $a_0, a_1, \dots$ such that

$$\sum_{k=0}^{2n} c_k t^k = \sum_{k=1}^{2n} t^k + a_0 + \sum_{k=1}^{2n} (a_{k-1} + a_k)\, t^k.$$

That is,

$$c_0 + \sum_{k=1}^{2n} c_k t^k = a_0 + \sum_{k=1}^{2n} (a_{k-1} + a_k + 1)\, t^k.$$

Since $a_0$ might be zero we have no information about $c_0$, but $c_k \ge 1$ for $k = 1, \dots, 2n$. □

### Remark 6.2.

The assumption (i) of the above proposition can be achieved by the following: $f(x) = |x|^2/2 + \varphi_0(x)$, in a neighbourhood of zero, $|\nabla\varphi_0(x)| = o(|x|)$ as $x \to 0$. Indeed, such condition implies that the origin is a critical point of $f$, and $S_0 = \{0\}$ is an isolated invariant set. The $\mathbb{Z}_p$-index pair for $S_0$ is given by $(D_r(V), \emptyset)$, where $r$ is sufficiently small. The pair $\big(D_r^E/\mathbb{Z}_p, D_r^{\mathbb{Z}_p}/\mathbb{Z}_p\big)$ is homotopy equivalent to the pointed one point space and $H^q\big(D_r^{\mathbb{Z}_p}/\mathbb{Z}_p, \emptyset\big) \cong \mathbb{Z}_p$ only for $q = 0$. Hence $\mathcal{P}_{\mathbb{Z}_p}(t, h_{\mathbb{Z}_p}(S_0)) = \mathfrak{u}_{\mathbb{Z}_p}^{\mathbb{Z}_p}$.

In the next proposition, let $G = SO(2)$.

### Proposition 6.3.

*Let $V \cong \mathbb{R}[n+1, 1]$ be a $G$-representation. Assume that $f\colon V \to \mathbb{R}$ is a smooth $G$-invariant function and*

(i) *there exists a $G$-isolating neighbourhood $X_0$ such that $0 \in S_0 = \operatorname{inv}(X_0)$ and $\mathcal{P}_G(t, h_G(S_0)) = \mathfrak{u}_G^G$;*

(ii) *$f(x) = -|x|^2/2 + \varphi_\infty(x)$ in a neighbourhood of infinity and $\nabla\varphi_\infty$ is bounded.*

*If $f$ has only a finite number of critical orbits, say $\{Gx_0, \dots, Gx_m\}$, and all of them are hyperbolic, then there is at least $n+1$ of them. Moreover, $G_{x_k} = E$ for $1 \le k \le n+1$ and each number in the set $\{2k-1 : 1 \le k \le n+1\}$ is a Morse index of some critical orbit.*

***Proof.*** As in the preceding proof we take the pair $(X, A) = (D_R(V), S_R(V))$ as a $G$-index pair for the $G$ flow of $-\nabla f$. Here we have $X^E/G \approx (\mathbb{C}P^n \times [0,1])/(\mathbb{C}P^n \times \{1\}) = C\mathbb{C}P^n$, a cone over the complex projective space $\mathbb{C}P^n$ and $(X^G \cup A^E)/G \approx (\mathbb{C}P^n \times \{0\}) \dot\cup (\mathbb{C}P^n \times \{1\}) = \widehat{C}\mathbb{C}P^n$ is a disjoint union of the bottom and the top of the cone. Now, we are going to use the cohomology with integer coefficient. Thus

$$H^q(C\mathbb{C}P^n, \widehat{C}\mathbb{C}P^n; \mathbb{Z}) \cong H^q\big(S(\mathbb{C}P^n) \vee S^1, \{\mathrm{pt}\}; \mathbb{Z}\big) \cong \begin{cases} \mathbb{Z} & \text{for } q \le 2n+1 \text{ odd}, \\ 0 & \text{for } q \text{ even}. \end{cases}$$

Moreover, $H^q\big(X^G/G, (X^{>G} \cup A^G)/G\big) \cong \mathbb{Z}$ for $q = 0$ and is zero otherwise. Therefore

$$\mathcal{P}_G(t, h_G(X)) = \mathfrak{u}_G^G + \big(t^{2n+1} + t^{2n-1} + \dots + t^3 + t\big)\mathfrak{u}_E^G.$$

Applying the Morse equation one obtains the equality

$$c_0 + \sum_{k=1}^{n+1} c_{2k-1} t^{2k-1} + \sum_{k=1}^{n+1} c_{2k} t^{2k} = a_0 + \sum_{k=1}^{n+1} (a_{2k-2} + a_{2k-1} + 1) t^{2k-1} + \sum_{k=1}^{n+1} (a_{2k-1} + a_{2k}) t^{2k}, \tag{9}$$

where $c_j$ is the number of critical orbits of index $j$ and $a_0, a_1, \dots$ are nonnegative integers. From (9) we read off that $c_0 \ge 0$, $c_{2k} \ge 0$ and $c_{2k-1} \ge 1$ for $1 \le k \le n+1$. □

## 6.1. The general case of $\mathbb{Z}_2$-action

We turn now to the case of the most general $\mathbb{Z}_2$-representation. Let $\mathbb{R}_t$ (resp. $\mathbb{R}_a$) be a one-dimensional $\mathbb{Z}_2$-representation with the trivial (resp. antipodal) action. Let $V$ be an orthogonal representation of a group $\mathbb{Z}_2$ isomorphic to $\mathbb{R}_t^\ell \oplus \mathbb{R}_a^k$ for $k \ge 1$. Notice that a $\mathbb{Z}_2$-equivariant isomorphism $A : V \to V$ is of the form $A_t \oplus A_a$, where $A_t \colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ and $A_a \colon \mathbb{R}^k \to \mathbb{R}^k$. Assume that $f \colon V \to \mathbb{R}$ is an asymptotically quadratic $\mathbb{Z}_2$-invariant smooth function, i.e., there exist two symmetric linear $\mathbb{Z}_2$-maps $A_0, A_\infty \colon V \to V$ such that

- $(1_f)$ $f(x) = -\langle A_0 x, x\rangle/2 + \varphi_0(x)$, where $\nabla\varphi_0(x) = o(|x|)$ as $x \to 0$;
- $(2_f)$ $f(x) = -\langle A_\infty x, x\rangle/2 + \varphi_\infty(x)$, where $\nabla\varphi_\infty(x) = o(|x|)$ as $x \to \infty$.

Clearly, if $f$ is asymptotically quadratic, then the map $\nabla f$ is asymptotically linear. Moreover, assume that

- $(3_f)$ $f$ is *nonresonance* at zero and infinity, i.e., both maps $A_0$ and $A_\infty$ are isomorphisms, and
- $(4_f)$ $f$ has only a finite number of critical $\mathbb{Z}_2$-orbits, $\{x_1, gx_1, \dots, x_n, gx_n\}$, and all of them are hyperbolic.

Consider a $\mathbb{Z}_2$-flow $\phi_f$ generated by $-\nabla f$. It follows, from the assumptions above, that the origin is an isolated invariant set for $\phi_f$ and there is another, maximal isolated invariant $\mathbb{Z}_2$-set $T$ such that $\{0\} \in T$. There is a decomposition $V = V_0^+ \oplus V_0^-$ (resp. $V = V_\infty^+ \oplus V_\infty^-$) corresponding to the positive and negative spectrum of $A_0$ (resp. $A_\infty$). Denote by $D_\rho(V)$ (resp. $S_\rho(V)$) the disk (resp. sphere) in $V$ of radius $\rho$. It is clear that the pair $(D_r(V), S_r(V_0^+))$ is a $\mathbb{Z}_2$-index pair for $\{0\}$ for $r$ sufficiently small. Similarly, the $\mathbb{Z}_2$-pair $(D_R(V), S_R(V_\infty^+))$, for $R$ sufficiently large, is an index pair for $T$.

To proceed further we will calculate the cohomology groups of the pair $(D(V)_E S(V))$ using $\mathbb{Z}_2$ coefficients, that is the groups

$$H^q\big(D(V)^E/\mathbb{Z}_2, \big(D(V)^{\mathbb{Z}_2} \cup S(V)^E\big)/\mathbb{Z}_2; \mathbb{Z}_2\big), \qquad q \ge 0.$$

In order to visualize the geometry we need the concept of the join of two topological spaces. Since we are dealing with quite friendly spaces, as disks and spheres, the task is much simpler than it might be possible in general. Given two topological spaces $X$ and $Y$, the *join* $X * Y$ is the quotient space $X \times Y \times [0,1]/\sim$, where the equivalence $\sim$ is given by $(x_1, y, 0) \sim (x_2, y, 0)$ for $x_1, x_2 \in X$ and $y \in Y$ and $(x, y_1, 1) \sim (x, y_2, 1)$ for all $x \in X$ and $y_1, y_2 \in Y$. We shall list some properties of the join which will be needed later. **(i)** The join of $X$ and a 0-sphere is homeomorphic to the (unreduced) suspension of $X$: $S^0 * X \simeq SX$; **(ii)** $S^k * S^\ell \simeq S^{k+\ell+1}$. Since the join is associative it follows by induction that $S^k * X \simeq S^{k+1}X$, the $(k+1)$-folded suspension of $X$. The property **(ii)** implies in particular that if $V$ and $W$ are two finite dimensional orthogonal $G$-representations, and $S(V)$ denotes the sphere $\{x \in V : |x| = 1\}$, then

$$S(V \oplus W) \simeq S(V) * S(W).$$

Notice that $S^V = D(V)/S(V) \simeq S(V \oplus \mathbb{R}_t)$, cf. [22, Lemma 4.2]. The disk $D(V)^E$ is $(k+\ell)$-dimensional and contains the $\ell$-disk $D(V)^{\mathbb{Z}_2}$ on which the group acts trivially. After collapsing the sphere $S(V)$ in $D(V)$ the $\ell$-disk $D(V)^{\mathbb{Z}_2}$ becomes an $\ell$-sphere contained as a meridian in a sphere $S^V \simeq S(V \oplus \mathbb{R}_t) = S(\mathbb{R}_t^{\ell+1} \oplus \mathbb{R}_a^k) \simeq S(\mathbb{R}_t^{\ell+1}) * S(\mathbb{R}_a^k)$. The group acts on the join $S(\mathbb{R}_t^{\ell+1}) * S(\mathbb{R}_a^k)$ as follows: $g(x, y, t) = (gx, gy, t) = (x, -y, t)$ for all $x \in S(\mathbb{R}_t^{\ell+1})$, $y \in S(\mathbb{R}_a^k)$ and $t \in [0,1]$. Factoring out by the action of $\mathbb{Z}_2$ we obtain $S^\ell * \mathbb{R}P^{k-1}$. Collapsing away the circle $S^\ell$ (coming from the disk $D(V)^{\mathbb{Z}_2}$) one can see that the pair $(D(V)_E S(V))$ is equivalent, up to a homotopy type, to the pair $(S^\ell * \mathbb{R}P^{k-1}, S^\ell)$. We will examine the groups $H^q(S^\ell * \mathbb{R}P^{k-1}, S^\ell; \mathbb{Z}_2)$ using the long exact sequence of a pair. One has an exact sequence of reduced cohomology groups

$$\ldots \to H^{q-1}(S^\ell) \to H^q(S^\ell * \mathbb{R}P^{k-1}, S^\ell) \to H^q(\mathbb{R}P^{k-1} * S^\ell) \to H^q(S^\ell) \to \ldots$$

for $q \geq 0$. By the suspension isomorphism one obtains $H^q(S^\ell * \mathbb{R}P^{k-1}) \cong \mathbb{Z}_2$ for $\ell + 2 \leq q \leq \ell + k$. Substituting in the above sequence $q = \ell + i$ for $i = 2, \ldots, k$ we obtain a short exact sequence

$$0 \to H^{\ell+i}(S^\ell * \mathbb{R}P^{k-1}, S^\ell) \xrightarrow{\cong} \mathbb{Z}_2 \to 0.$$

If $q = \ell + 1$, then

$$0 \to \mathbb{Z}_2 \xrightarrow{\cong} H^{\ell+1}(S^\ell * \mathbb{R}P^{k-1}, S^\ell) \to 0.$$

Hence, for $k \geq 1$ and $\ell \geq 0$ one has

$$H^q(D(V)_E S(V)) \cong \begin{cases} \mathbb{Z}_2 & q = \ell + i \ \text{ for } \ i = 1, 2, \ldots, k, \\ 0 & \text{otherwise.} \end{cases}$$

We also are interested in the cohomology of the pair $(D(V \oplus U)_E S(V))$, where $V$ is as above and $U$ is an arbitrary $\mathbb{Z}_2$-representation. By the following lemma one can reduce the task to the previous situation.

**Lemma 6.4.**
*The pair $(D(V \oplus U)_E S(V))$ is homotopy equivalent to the pair $(D(V)_E S(V))$.*

***Proof.*** It suffices to show that the pairs $(D(V \oplus U), S(V))$ and $(D(V), S(V))$ are $\mathbb{Z}_2$-homotopy equivalent. Identify $D(V \oplus U)$ with $D(V) \times D(U)$ via the natural $\mathbb{Z}_2$-homeomorphism. Define

$$p\colon (D(V) \times D(U), S(V)) \to (D(V), S(V)) \qquad \text{and} \qquad q\colon (D(V), S(V)) \to (D(V) \times D(U), S(V))$$

by setting $p(x, y) = x$ and $q(x) = (x, 0)$. Clearly both $p$ and $q$ are $\mathbb{Z}_2$-equivariant, $pq = \mathrm{id}_{(D(V),S(V))}$ and $qp$ is homotopic with $\mathrm{id}_{(D(V) \times D(U),S(V))}$ via $\mathbb{Z}_2$-homotopy $h(x, y, t) = (x, ty)$. □

Let us now go back to the computations of the indices of $\{0\}$ and $T$. Suppose that $V_0^+ = \mathbb{R}_t^{\ell_0} \oplus \mathbb{R}_a^{k_0}$ and $V_\infty^+ = \mathbb{R}_t^{\ell_\infty} \oplus \mathbb{R}_a^{k_\infty}$. The above considerations show that the Poincaré polynomials of the indices of $\{0\}$ and $T$ are of the form

$$\begin{aligned} \mathcal{P}_{\mathbb{Z}_2}(t, h_{\mathbb{Z}_2}(\{0\}, \phi_f)) &= t^{\ell_0} \mathfrak{u}_{\mathbb{Z}_2}^{\mathbb{Z}_2} + \left(t^{\ell_0+1} + \cdots + t^{\ell_0+k_0}\right) \mathfrak{u}_E^{\mathbb{Z}_2}, \\ \mathcal{P}_{\mathbb{Z}_2}(t, h_{\mathbb{Z}_2}(T, \phi_f)) &= t^{\ell_\infty} \mathfrak{u}_{\mathbb{Z}_2}^{\mathbb{Z}_2} + \left(t^{\ell_\infty+1} + \cdots + t^{\ell_\infty+k_\infty}\right) \mathfrak{u}_E^{\mathbb{Z}_2}. \end{aligned}$$

Notice that if $x \in V$ is a nondegenerate critical orbit with isotropy group $\mathbb{Z}_2$ (i.e., in fact, is a critical point), then

$$\mathcal{P}_{\mathbb{Z}_2}(t, h_{\mathbb{Z}_2}(\{x\})) = t^{\ell_x} \mathfrak{u}_{\mathbb{Z}_2}^{\mathbb{Z}_2} + \left(t^{\ell_x+1} + \cdots + t^{\ell_x+k_x}\right) \mathfrak{u}_E^{\mathbb{Z}_2},$$

where the numbers $\ell_x$ and $k_x$ are defined via the equality $V_x^+ = \mathbb{R}_t^{\ell_x} \oplus \mathbb{R}_a^{k_x}$. On the other hand, for a nondegenerate critical orbit $\{y, gy\}$ with the isotropy group $E$ one has $\mathcal{P}_{\mathbb{Z}_2}(t, h_{\mathbb{Z}_2}(\{y, gy\})) = t^{\dim V_y^+} \mathfrak{u}_E^{\mathbb{Z}_2}$, cf. Proposition 5.16. Here $V_x^+$ (resp. $V_y^+$) is the unstable subspace of a linear map $-\nabla^2 f(x)$ (resp. $-\nabla^2 f(y)$). If ($1_f$)–($4_f$) are satisfied, then combining all these data with the equation (EME) one obtains the following equalities:

$$\begin{aligned} t^{\ell_0} + \sum_{i=1}^{n} a_i t^{\ell_{x_i}} &= t^{\ell_\infty} + (1+t)\mathcal{Q}_1(t), \\ \sum_{i=1}^{k_0} t^{\ell_0+i} + \mathcal{Z}(t) &= \sum_{i=1}^{k_\infty} t^{\ell_\infty+i} + (1+t)\mathcal{Q}_2(t), \end{aligned} \tag{10}$$

where $\mathcal{Z}, \mathcal{Q}_1, \mathcal{Q}_2$ are some unknown polynomials with nonnegative integer coefficients. The numbers $a_i$ for $1 \le i \le n$ may be one or zero. Notice that $a_i = 1$ if $x_i$ is a critical orbit of orbit type $\mathbb{Z}_2$. It may happen that $\ell_{x_i} = \ell_{x_j}$ for $i \neq j$.

***Proposition 6.5.***
*Suppose that $f\colon V \to \mathbb{R}$ is a smooth $\mathbb{Z}_2$-invariant function satisfying conditions ($1_f$)–($4_f$). If $\ell_\infty \neq \ell_0$, then $f$ has at least two nonzero critical points $x, y \in V$ (two orbits of orbit type $\mathbb{Z}_2$). Additionally, one has estimations on the Morse indices: $\dim V_x^+ \ge \ell_\infty$ and $\dim V_y^+ \ge \ell_0 - 1$.*

***Proof.*** We will examine the equation (10). The right-hand side of (10) contains the exponent $\ell_\infty$. Therefore there exists $1 \le i \le n$ such that $\ell_{x_i} = \ell_\infty$. On the left-hand side of (10) there is the exponent $\ell_0$, hence the polynomial $(1+t)\mathcal{Q}_1$ contains two nonzero terms with exponents $\ell_0$ and $\ell_0 + 1$ or $\ell_0 - 1$ and $\ell_0$. Therefore, there exists $1 \le j \le n$, such that $\ell_{x_j} = \ell_0 - 1$ or $\ell_{x_j} = \ell_0 + 1$. Consequently $x = x_i$ and $y = x_j$ are critical points of $f$. The inclusions $\mathbb{R}_t^{\ell_\infty} \subset V_x^+$ and $\mathbb{R}_t^{\ell_0-1} \subset V_y^+$ give us the estimations on dimension of $V_x^+$ and $V_y^+$. □

# 7. Appendix

Let $V$ be an orthogonal finite dimensional representation of a compact Lie group $G$. Assume that $\Phi\colon V \to \mathbb{R}$ is a smooth $G$-invariant function and the origin is a nondegenerate critical point of $\Phi$. It is a rather standard fact, that near the origin the $G$-flow given by an equation $\dot{x} = -\nabla\Phi(x)$ is equivalent to the $G$-flow given by

$$\dot{x}_+ = A_1 x_+ + g_1(x) \quad \text{and} \quad \dot{x}_- = A_2 x_- + g_2(x), \qquad x = (x_+, x_-) \in V^+ \oplus V^-, \tag{11}$$

where $|g_{1,2}(x)| = o(|x|)$, the norms $|g_{1,2}(x)| < \tau$ and $\|Dg_{1,2}(x)\| < \tau$, where $\tau$ is arbitrary small. The linear maps $A_{1,2}\colon V \to V$ are such that

$$\langle A_1 x_+, x_+\rangle \le -\lambda|x_+|^2, \qquad \langle A_2 x_-, x_-\rangle \ge \lambda|x_-|^2, \tag{12}$$

for some $\lambda > 0$. Here $V^+$ (resp. $V^-$) denotes the eigenspace of the Hessian $\nabla^2\Phi(0)$ corresponding to the positive (resp. negative) eigenvalues.

For the sake of completeness we include the proof and next we will show how to find the $G$-index pair for an isolated zero.

The equivalence above means that there is a $G$-neighbourhood $U \ni 0$ and a $G$-homeomorphism $h\colon U \to h(U)$ such that $h(0) = 0$ and $h$ maps orbits in $U$ of the first system onto orbits of the second one preserving the direction in time. In particular, such an equivalence takes place when the second system is obtained by the smooth (diffeomorphic) $G$-equivariant change of coordinates $y = h(x)$, i.e., the flows defined by $\dot{x} = f(x)$ and $\dot{y} = g(y)$ are equivalent provided that $f(x) = (Dh(x))^{-1}g(h(x))$.

Let $A = \nabla^2\Phi(0)$. Then $\nabla\Phi(x) = Ax + \phi(x)$, where $|\phi(x)| = o(|x|)$ as $|x| \to 0$. Choose a Jordan basis $\{v_i\}$ of $V$ such that $A$ with respect to $\{v_i\}$ has a matrix representation

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where $A_1 = A|_{V^+}$ and $A_2 = A|_{V^-}$ are diagonal matrices. The inequalities (12) are clear since $A_1$ (resp. $A_2$) has only negative (resp. positive) entries on the main diagonal. For an element $x = (x_+, x_-) \in V^+ \oplus V^-$ define its norm $|x| = \max\{|x_+|, |x_-|\}$. The linear change of coordinates $x \mapsto \varepsilon x$ gives us an equivariant map $F_\varepsilon(x) = Ax + \phi_\varepsilon(x)$, where $\phi_\varepsilon(x) = \phi(\varepsilon x)/\varepsilon$.

***Lemma 7.1.***
*For any $\tau > 0$ there exists an $\varepsilon > 0$ such that $|\phi_\varepsilon(x)| < \tau$ and $\|D\phi_\varepsilon(x)\| < \tau$ uniformly for $x \in B_2(0)$.*

***Proof.*** Fix $\tau > 0$. Since $|\phi(x)| = o(\|x\|)$ as $x$ tends to 0, there is a $\delta > 0$ such that $|\phi(\varepsilon x)|/|\varepsilon x| \leq \tau/2$ provided that $|\varepsilon x| < \delta$. Let $\varepsilon$ be chosen such that $|\varepsilon x| < \delta$. Then

$$|\phi_\varepsilon(x)| = \frac{1}{\varepsilon}|\phi(\varepsilon x)| = |x|\,\frac{|\phi(\varepsilon x)|}{|\varepsilon x|} \leq 2\,\frac{\tau}{2} = \tau.$$

The derivative of $\phi_\varepsilon(x)$ is $D\phi_\varepsilon(x) = D\phi(\varepsilon x)$. Since $D\phi(x)$ is continuous and $D\phi(0) = 0$, for any $\tau > 0$ one can take $\delta_1 > 0$ such that $\|D\phi(\varepsilon x)\| \leq \tau$ if only $|\varepsilon x| < \delta_1$. Taking $\varepsilon$ small enough we are done. □

In order to find the index pair for the isolated invariant set $\{0\}$ for the flow given by (11) we proceed as follows. Let $N$ be the square $\{|x| \leq 1\}$. It is easily seen that $N$ is a $G$-set since the action is orthogonal. If $|x_+| \geq |x_-|$ then $d/dt\,|x_+|^2 = 2\langle \dot{x}_+, x_+\rangle = 2\langle A_1 x_+, x_+\rangle + 2\langle g_1(x), x_+\rangle \leq -\lambda|x_+|^2$ provided that $2\tau \leq \lambda$. The same argument shows that if $|x_+| \leq |x_-|$ then $d/dt\,|x_-|^2 \geq \lambda|x_-|^2$. Therefore, the flow of (11) leaves the square $N$ via the set $N^- = \{x \in N : |x_-| = 1\}$ while the entrance set is $N^+ = \{x \in N : |x_+| = 1\}$. That is, the pair $(N, N^-)$ is a $G$-index pair for $\{0\}$. To see that $N^-$ is a $G$-set suppose, to the contrary, that $x \in N^-$ and $gx \in \{|x| = 1\} \setminus N^-$ for some $g \in G$ (the sphere $\{|x| = 1\}$ is obviously a $G$-set). If so, there exists sufficiently small $t > 0$ such that $\phi^{(0,t)}(gx) \subset N$ while $\phi^{(0,t)}(x) \not\subset N$ and by the $G$-invariance of $N$ one has $g\phi^{(0,t)}(x) \not\subset N$. But this contradicts the fact that $\phi^t$ is a $G$-map.

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