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Fractional equations of Volterra type involving a Riemann–Liouville derivative

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ABSTRACT

In this paper, we will discuss the existence of solutions of fractional equations of Volterra type with the Riemann–Liouville derivative. Existence results are obtained by using a Banach fixed point theorem with weighted norms and by a monotone iterative method too. An example illustrates the results.

1. Introduction

This paper discusses the existence of solutions of problems:

$$\begin{cases} D^q x(t) = f\left(t, x(t), \int_0^t k(t,s)x(s)ds\right) \equiv \mathcal{F}x(t), \quad t \in J_0 = (0,T],\ T > 0, \\ \tilde{x}(0) = r, \end{cases} \tag{1}$$

where $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $J = [0,T]$, $\tilde{x}(0) = t^{1-q}x(t)_{|t=0}$, and $D^q x$ denotes a Riemann–Liouville fractional derivative of x with $q \in (0,1)$.

Recently, much attention has been paid to study fractional problems, see for example [1–10]. The monotone iterative technique can be successfully applied to obtain existence results for fractional differential problems, see book [2], and for example papers [1,3,4,6,8–10]. Authors of papers, [3,4,6,7,9,10], obtained their existence results under the assumption that function f satisfies a one-sided Lipschitz condition with respect to the second variable with a corresponding constant coefficient M. In our paper, we consider a more general case when constant M is replaced by a function $M \in C(J, \mathbb{R})$. We also obtained existence results by using the Banach fixed point theorem with the corresponding weighted norms.

The organization of this paper is as follows. In Section 2, Theorem 1 presents the existence result giving sufficient conditions under which problem (1) has a unique solution. To achieve this we apply a Banach fixed point theorem with a corresponding weighted norm (Bielecki norm) assuming the Lipschitz condition of f with respect to the last two arguments with nonnegative coefficients. It is important to indicate that in the case when $\frac{1}{2} < q < 1$, we do not need any conditions on the coefficients. In Section 3, we use the monotone iterative method. First we discuss a comparison result. Theorem 2 presents the existence result for problems of type (1), by using the monotone iterative method. An example is given to illustrate the results.

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2. Existence results for problem (1), by a Banach fixed point theorem

Let $C_{1-q}(J,\mathbb{R}) = \{u \in C((0,T],\mathbb{R}) : t^{1-q}u \in C(J,\mathbb{R})\}$. For $u \in C_{1-q}(J,\mathbb{R})$ we define two weighted norms:

$$\|u\|^* = \max_{[0,T]} t^{1-q}|x(t)| \quad \text{or} \quad \|u\|_* = \max_{[0,T]} t^{1-q}e^{-\lambda t}|x(t)|$$

with a fixed positive constant λ.

Theorem 1. *Let $q \in (0,1), f \in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R}),\ k \in C(J\times J,\mathbb{R})$. In addition, we assume that:*

H_1 : *there exist nonnegative constants K, L, W such that: $|k(t,s)| \le W$ and*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \le K|v_1-u_1| + L|v_2-u_2|,$$

H_1 : $\rho \equiv \frac{T^q\Gamma(q)}{\Gamma(2q)}\left(K + \frac{WLT}{2q}\right) < 1$ *if* $0 < q \le \frac{1}{2}$.

Then problem (1) has a unique solution.

Proof. Consider the problem $x = \mathcal{N}x$, where operator $\mathcal{N}$ is defined by

$$\mathcal{N}x(t) = rt^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\mathcal{F}x(s)ds.$$

Now, we have to show that operator $\mathcal{N}$ has a fixed point. To do it we shall show that $\mathcal{N}$ is a contraction map. Let $x, y \in C_{1-q}(J,\mathbb{R})$. We consider two cases.

Case 1. Let $0 < q \le \frac{1}{2}$. Then, in view of assumption H_1, we have

$$\begin{aligned}
\|\mathcal{N}x - \mathcal{N}y\|^* &\le \frac{1}{\Gamma(q)} \max_{t\in J} t^{1-q}\int_0^t (t-s)^{q-1}|\mathcal{F}x(s) - \mathcal{F}y(s)|ds \\
&\le \frac{1}{\Gamma(q)} \max_{t\in J} t^{1-q}\int_0^t (t-s)^{q-1}\left[K|x(s)-y(s)| + L\int_0^s |k(s,\tau)|\,|x(\tau)-y(\tau)|d\tau\right]ds \\
&\le \frac{1}{\Gamma(q)}\|x-y\|^* \max_{t\in J} t^{1-q}\int_0^t (t-s)^{q-1}\left[Ks^{q-1} + LW\int_0^s \tau^{q-1}d\tau\right]ds \\
&= \frac{1}{\Gamma(q)}\|x-y\|^* \max_{t\in J} t^{1-q}\int_0^t (t-s)^{q-1}\left(Ks^{q-1} + \frac{LW}{q}s^q\right)ds \\
&= \rho\|x-y\|^*.
\end{aligned}$$

Hence, operator $\mathcal{N}$ has a unique fixed point, by the Banach fixed point theorem.

Case 2. Assume that $\frac{1}{2} < q < 1$. Now, we use the norm $\|\cdot\|_*$ with a positive λ such that:

$$\sqrt{\lambda} > \rho_1 \equiv \frac{Kq + LWT}{q\Gamma(q)}\frac{\Gamma(2q-1)}{\sqrt{2\Gamma(2(2q-1))}}\sqrt{T^{2q-1}}.$$

Note that

$$\begin{cases} \displaystyle\int_0^t e^{2\lambda s}ds \le \frac{1}{2\lambda}e^{2\lambda t}, \\ \displaystyle t^{1-q}\sqrt{\int_0^t (t-s)^{2(q-1)}s^{2(q-1)}ds} = \frac{\Gamma(2q-1)}{\sqrt{\Gamma(2(2q-1))}}\sqrt{t^{2q-1}}. \end{cases} \tag{2}$$

We will use the Schwarz inequality for integrals

$$\int_0^t |a(s)|\,|b(s)|ds \le \sqrt{\int_0^t a^2(s)ds}\sqrt{\int_0^t b^2(s)ds}.$$

Using assumption H_1, the Schwarz inequality and (2), we have

$$\begin{aligned}
\|\mathcal{N}x - \mathcal{N}y\|_* &\le \frac{1}{\Gamma(q)} \max_{t\in J} t^{1-q}e^{-\lambda t}\int_0^t (t-s)^{q-1}|\mathcal{F}x(s) - \mathcal{F}y(s)|ds \\
&\le \frac{1}{\Gamma(q)}\|x-y\|_* \max_{t\in J} t^{1-q}e^{-\lambda t}\int_0^t (t-s)^{q-1}\left[Ks^{q-1}e^{\lambda s} + \frac{LW}{q}s^q e^{\lambda s}\right]ds
\end{aligned}$$

$$\leq \frac{(Kq+LWT)}{q\Gamma(q)}\|x-y\|_* \max_{t\in J} t^{1-q}e^{-\lambda t}\int_0^t (t-s)^{q-1}s^{q-1}e^{\lambda s}ds$$
$$\leq \frac{(Kq+LWT)}{q\Gamma(q)}\|x-y\|_* \max_{t\in J} t^{1-q}e^{-\lambda t}\sqrt{\int_0^t (t-s)^{2(q-1)}s^{2(q-1)}ds}\sqrt{\int_0^t e^{2\lambda s}ds}$$
$$\leq \frac{\rho_1}{\sqrt{\lambda}}\|x-y\|_*.$$

It proves that problem (1) has a unique solution. This ends the proof. □

Consider the linear problem:

$$\begin{cases} D^q u(t) = -M(t)u(t)+\sigma(t), & t\in J_0,\\ \tilde{u}(0) = r. \end{cases} \tag{3}$$

Lemma 1. *Let $q\in(0,1)$, $M\in C(J,\mathbb{R})$, $\sigma\in C_{1-q}(J,\mathbb{R})$. Moreover, we assume that Assumption H_3 holds with:*

H_3 : (i) *$M(t)=M$, $t\in J$,*
or
(ii) *function M is not a constant on J and*

$$\frac{T^q\Gamma(q)}{\Gamma(2q)}\max_{t\in J}|M(t)| < 1 \quad \textit{only in the case when } 0<q\leq\frac{1}{2}.$$

Then problem (3) has a unique solution.

Proof. In case (i), problem (3) has a unique solution in terms of Mittag-Leffler's function, see for example [2].
In case (ii), the assertion results from Theorem 1. □

Remark 1. Note that if $\frac{1}{2}<q<1$, then problem (1) has a unique solution for arbitrary $M\in C(J,\mathbb{R})$.

3. Existence results for problem (1), by a monotone iterative method

To apply the monotone iterative method we have to introduce the notation of lower and upper solution for (1) and discuss corresponding fractional inequality. Comparison results will play a very important role in our research. First, we discuss fractional differential inequalities.

Let us introduce the following assumption:

H_4:(i) $M(t)=M$, $t\in J$,
or
(ii) function M is not a constant on J and if $M(t)\leq 0$ on J, we extra assume that: $-M(t)\leq\bar{M}(t)$ on J, $\bar{M}$ is nondecreasing and

$$\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}\bar{M}(s)ds < 1. \tag{4}$$

Lemma 2. *Let $q\in(0,1)$ and $M\in C(J,[0,\infty))$ or $M\in C(J,(-\infty,0])$. Suppose that $p\in C_{1-q}(J,\mathbb{R})$ satisfies the problem:*

$$\begin{cases} D^q p(t)\leq -M(t)p(t), & t\in J_0,\\ \tilde{p}(0)\leq 0. \end{cases} \tag{5}$$

Let Assumption H_4 hold.
Then $p(t)\leq 0$ on J.

Proof. We consider only the case when function M is not a constant on J. Assume that the assertion is not true. It means that there exist points $t_2, t^*\in(0,T]$ such that $p(t_2)=0$, $p(t^*)>0$ and $p(t)\leq 0$, $t\in(0,t_2]$; $p(t)>0$, $t\in(t_2,t^*]$. Let t_0 be the first maximal point of p on $[t_2,t^*]$. Some ideas in the proof are taken from paper [10].

Case 1. Let $M(t)\geq 0$ on J. Then

$$D^q p(t)\leq 0, \quad t\in[t_2,t^*],$$

so

$$\int_{t_2}^{t_0} D^q p(s)ds\leq 0.$$

Hence, from the definition of Riemann–Liouville fractional derivative, we have

$$0 \geq I^{1-q}p(t_0) - I^{1-q}p(t_2) \equiv A. \tag{6}$$

On the other hand, we have

$$\begin{aligned} A &= \frac{1}{\Gamma(1-q)}\left[\int_0^{t_0}(t_0-s)^{-q}p(s)ds - \int_0^{t_2}(t_2-s)^{-q}p(s)ds\right] \\ &= \frac{1}{\Gamma(1-q)}\left\{\int_0^{t_2}\left[(t_0-s)^{-q}-(t_2-s)^{-q}\right]p(s)ds + \int_{t_2}^{t_0}(t_0-s)^{-q}p(s)ds\right\} \\ &> \frac{1}{\Gamma(1-q)}\int_{t_2}^{t_0}(t_0-s)^{-q}p(s)ds > 0. \end{aligned}$$

It contradicts relation (6), so the assertion holds in this case.

Case 2. Let $M(t) \leq 0$ on J and let $\bar{M}$ be nondecreasing on J. Note that Riemann–Liouville fractional integral I^q is a monotone operator. Now, using the fractional integral I^q to the both sides of (5) we obtain

$$p(t) - \tilde{p}(0)t^{q-1} \leq -I^q[M(t)p(t)], \quad t \in [t_2, t^*].$$

Note that $\tilde{p}(0)t^{q-1} \leq 0$, so in view of the fact that $\bar{M}$ is nondecreasing we obtain

$$\begin{aligned} p(t_0) &\leq -\frac{1}{\Gamma(q)}\int_0^{t_0}(t_0-s)^{q-1}M(s)p(s)ds \\ &= -\frac{1}{\Gamma(q)}\left[\int_0^{t_2}(t_0-s)^{q-1}M(s)p(s)ds + \int_{t_2}^{t_0}(t_0-s)^{q-1}M(s)p(s)ds\right] \\ &< -\frac{p(t_0)}{\Gamma(q)}\int_0^{t_0}(t_0-s)^{q-1}M(s)ds \\ &= -\frac{p(t_0)}{\Gamma(q)}t_0^q\int_0^1(1-\sigma)^{q-1}M(\sigma t_0)d\sigma \\ &\leq \frac{p(t_0)}{\Gamma(q)}t_0^q\int_0^1(1-\sigma)^{q-1}\bar{M}(\sigma T)d\sigma \\ &= \frac{p(t_0)}{\Gamma(q)}\frac{t_0^q}{T^q}\int_0^T(T-s)^{q-1}\bar{M}(s)ds \\ &\leq \frac{p(t_0)}{\Gamma(q)}\int_0^T(T-s)^{q-1}\bar{M}(s)ds. \end{aligned}$$

Hence,

$$p(t_0)\left[1 - \frac{1}{\Gamma(q)}\int_0^T(T-s)^{q-1}\bar{M}(s)ds\right] < 0.$$

Using condition (4), it shows that $p(t_0) < 0$. It is a contradiction, so the assertion holds. □

Remark 2. If $M(t) = M,\ t \in J$, then the assertion of Lemma 2 holds and condition (4) is superfluous, see for example papers [7,9].

Lemma 2 is an essential improvement both of Lemma 2.1 [10], Lemma 2.3 [9] and Lemma 2.3 [7].

Remark 3. Because $M \in C(J, \mathbb{R})$, so in case $M(t) \leq 0,\ t \in J$ there exists a nonnegative constant M_0 such that $-M(t) \leq M_0$, $t \in J$. Then, condition (4) takes the form $M_0T^q < \Gamma(q+1)$.

We say that u is called a lower solution of (1) if

$$D^q u(t) \leq \mathcal{F}u(t), \quad t \in J_0, \qquad \tilde{u}(0) \leq 0,$$

and it is an upper solution of (1) if the above inequalities are reversed.

Let us introduce the following assumptions:

H_5: $q \in (0, 1),\ f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}),\ k \in C(J \times J, \mathbb{R})$,
H_6: there exists a function $M \in C(J, \mathbb{R})$ such that:

$$f(t, u_1, u_2) - f(t, v_1, v_2) \leq M(t)[v_1 - u_1]$$

if $y_0(t) \leq u_1 \leq v_1 \leq z_0(t),\ u_2 \leq v_2$.

Theorem 2. *Let assumption H_5 hold. Let $y_0, z_0 \in C_{1-q}(J, \mathbb{R})$ be lower and upper solutions of problem (1), respectively and $y_0(t) \le z_0(t),\ t \in J$. In addition, we assume that assumption s H_6, H_3, H_4 are satisfied.*
Then problem (1) has, in the sector $[y_0, z_0]$, solutions, where

$$[y_0, z_0] = \{z \in C_{1-q}(J, \mathbb{R}) :\ y_0(t) \le z(t) \le z_0(t),\ t \in J_0,\ \tilde{y}_0(0) \le \tilde{z}(0) \le \tilde{z}_0(0)\}.$$

Proof. Let $\eta, \xi \in [y_0, z_0]$. Put $\varphi(t) = \min[\eta(t), \xi(t)],\ \Phi(t) = \max[\eta(t), \xi(t)]$. Consider the boundary value problems

$$\begin{cases} D^q v(t) = \mathcal{F}\varphi(t) - M(t)[v(t) - \varphi(t)], & t \in J_0, \\ \tilde{v}(0) = r, \end{cases} \tag{7}$$

$$\begin{cases} D^q w(t) = \mathcal{F}\Phi(t) - M(t)[w(t) - \Phi(t)], & t \in J_0, \\ \tilde{w}(0) = r. \end{cases} \tag{8}$$

By Lemma 1, problems (7), (8) have a unique solution. Therefore, we can define the operator

$$B : \bar{\Omega} \to C_{1-q}(J, \mathbb{R}) \times C_{1-q}(J, \mathbb{R}), \qquad [y_0, z_0] \subset C_{1-q}(J, \mathbb{R}), \quad B(\eta, \xi) = (v, w),$$

where v, w are solutions of (7) and (8), respectively with $\bar{\Omega} = [y_0, z_0] \times [y_0, z_0]$.
Now, we want to show that

$$y_0(t) \le v(t) \le w(t) \le z_0(t), \quad t \in J.$$

Put $p = y_0 - v$. Then

$$\begin{aligned} D^q p(t) &\le \mathcal{F}y_0(t) - \mathcal{F}\varphi(t) + M(t)[v(t) - \varphi(t)], \\ &\le M(t)[\varphi(t) - y_0(t)] + M(t)[v(t) - \varphi(t)] \\ &= -M(t)p(t), \end{aligned}$$

and $\tilde{p}(0) \le 0$.
This and Lemma 2 show that $y_0(t) \le v(t),\ t \in J$. Similarly we can show that $w(t) \le z_0(t),\ t \in J$. To show that $v(t) \le w(t),\ t \in J$, we put $p = v - w$. Then

$$\begin{aligned} D^q p(t) &= \mathcal{F}\varphi(t) - \mathcal{F}\Phi(t) - M(t)[v(t) - \varphi(t) - w(t) + \Phi(t)] \\ &\le M(t)[\Phi(t) - \varphi(t)] - M(t)[v(t) - \varphi(t) - w(t) + \Phi(t)] \\ &= -M(t)p(t) \end{aligned}$$

and $\tilde{p}(0) = 0$. Hence $B : \bar{\Omega} \to \bar{\Omega}$.
In order to apply Schauder's fixed point theorem we need to show that the operator B is continuous and compact. Put $\sigma(t) = \mathcal{F}\varphi(t) + M(t)\varphi(t)$. Then problem (7) takes the form

$$\begin{cases} D^q v(t) = -M(t)v(t) + \sigma(t) \equiv \mathcal{G}v(t), & t \in J_0, \\ \tilde{v}(0) = r. \end{cases}$$

Then the solution of problem (7) is a fixed point of operator $\mathcal{N}$, where operator $\mathcal{N}$ is defined by

$$\mathcal{N}x(t) = rt^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \mathcal{G}v(s)ds.$$

Operator $\mathcal{N}$ is continuous in view of continuity of $\mathcal{G}$.
In fact $\mathcal{N}$ is a compact map. For given $\epsilon > 0$, we take

$$\delta = \min\left[T, \left(\frac{\epsilon\Gamma(2q)}{4D\Gamma(q)}\right)^{\frac{1}{q}}\right].$$

Then for each $v \in C_{1-q}(J, \mathbb{R}),\ t_1, t_2 \in J,\ t_1 < t_2$ and $t_2 - t_1 < \delta$, we have $|t_1^{1-q}\mathcal{N}v(t_1) - t_2^{1-q}\mathcal{N}v(t_2)| < \epsilon$.
In fact, there exists a positive constant D such that $\max_{s \in J} s^{1-q}|\mathcal{G}v(s)| \le D$ and

$$\begin{aligned} |t_1^{1-q}\mathcal{N}v(t_1) - t_2^{1-q}\mathcal{N}v(t_2)| &\le \frac{1}{\Gamma(q)} \left| t_1^{1-q} \int_0^{t_1} (t_1 - s)^{q-1}\mathcal{G}v(s)ds - t_2^{1-q} \int_0^{t_2} (t_2 - s)^{q-1}\mathcal{G}v(s)ds \right| \\ &\le \frac{1}{\Gamma(q)} \left| \int_0^{t_1} [t_1^{1-q}(t_1 - s)^{q-1} - t_2^{1-q}(t_2 - s)^{q-1}]\mathcal{G}v(s)ds \right| \\ &\quad + \frac{1}{\Gamma(q)} \left| \int_{t_1}^{t_2} t_2^{1-q}(t_2 - s)^{q-1}\mathcal{G}v(s)ds \right| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{D}{\Gamma(q)} \int_0^{t_1} [t_1^{1-q}(t_1-s)^{q-1} - t_2^{1-q}(t_2-s)^{q-1}] s^{q-1} ds \\
&\quad + \frac{D}{\Gamma(q)} \int_{t_1}^{t_2} t_2^{1-q}(t_2-s)^{q-1} s^{q-1} ds \\
&= \frac{D}{\Gamma(q)} \left(\int_0^{t_1} t_1^{1-q}(t_1-s)^{q-1} s^{q-1} ds - \int_0^{t_2} t_2^{1-q}(t_2-s)^{q-1} s^{q-1} ds \right. \\
&\quad \left. + \, 2 \int_{t_1}^{t_2} t_2^{1-q}(t_2-s)^{q-1} s^{q-1} ds \right) \\
&\leq \frac{D\Gamma(q)}{\Gamma(2q)} \left[|t_1^q - t_2^q| + 2(t_2-t_1)^q \right]
\end{aligned}
$$

because

$$
\begin{aligned}
\int_{t_1}^{t_2} t_2^{1-q}(t_2-s)^{q-1} s^{q-1} ds &= \int_0^{t_2-t_1} t_2^{1-q}(t_2-t_1-u)^{q-1}(u+t_1)^{q-1} du \\
&= \int_0^1 t_2^{1-q}(t_2-t_1)^q (1-\sigma)^{q-1} [\sigma t_2 + t_1(1-\sigma)]^{q-1} d\sigma \\
&\leq \int_0^1 t_2^{1-q}(t_2-t_1)^q (1-\sigma)^{q-1} (\sigma t_2)^{q-1} d\sigma \\
&= (t_2-t_1)^q \frac{\Gamma^2(q)}{\Gamma(2q)}.
\end{aligned}
$$

Now we consider two cases.

Case 1. Let $\delta \leq t_1 < t_2 < T$. Use a mean value theorem to get

$$t_2^q - t_1^q \leq q\delta^{q-1}(t_2-t_1) \leq q\delta^q.$$

Case 2. Let $0 \leq t_1 < \delta, \ t_2 < 2\delta$. Then

$$t_2^q - t_1^q \leq t_2^q \leq (2\delta)^q.$$

Consequently, we have

$$|t_1^{1-q}\mathcal{N}v(t_1) - t_2^{1-q}\mathcal{N}v(t_2)| < \frac{4D\Gamma(q)}{\Gamma(2q)}\delta^q \leq \epsilon.$$

We see that the operator $B : \bar{\Omega} \to \bar{\Omega}$ is equicontinuous on J. The Arzeli–Ascoli theorem guarantees that B is compact. Hence, by Schauder's fixed point theorem, the operator B has a fixed point, i.e. there exist $(v, w) \in \bar{\Omega}$ such that $B(v, w) = (v, w)$ and $v \leq w$.

Now, by (7) and (8), we see that v, w satisfy the following relations

$$
\begin{cases}
D^q v(t) = \mathcal{F}v(t) - M(t)[v(t) - v(t)], & t \in J_0, \\
\tilde{v}(0) = r,
\end{cases}
$$

$$
\begin{cases}
D^q w(t) = \mathcal{F}w(t) - M(t)[w(t) - w(t)], & t \in J_0, \\
\tilde{w}(0) = r.
\end{cases}
$$

It shows that $v, w \in C_{1-q}(J)$ are solutions of problem (1). It ends the proof. □

Example 1. Let $A, B \in C([0, 1], (0, \infty))$ and $B(t) \leq A(t), \ t \in [0, 1]$. Consider the problem:

$$
\begin{cases}
D^q x(t) = \mathcal{F}x(t), & t \in J_0 = (0, 1], \\
\tilde{x}(0) = 0
\end{cases}
\tag{9}
$$

with

$$\mathcal{F}x(t) = \frac{t^{-q}}{\Gamma(1-q)} + A(t)[t - x(t)]^3 + \frac{1}{2}B(t)\int_0^t (\sin ts)^4 x(s) ds.$$

Let $y_0(t) = 0,\ z_0(t) = 1 + t,\ t \in J = [0, 1]$. It is not difficult to show that y_0 is a lower solution of problem (9). Moreover

$$\begin{aligned}\mathcal{F} z_0(t) &= \frac{t^{-q}}{\Gamma(1-q)} - A(t) + \frac{1}{2}B(t)\int_0^t (\sin ts)^4(1+s)ds \\ &\le \frac{t^{-q}}{\Gamma(1-q)} < \frac{t^{-q}}{\Gamma(1-q)} + \frac{t^{1-q}}{\Gamma(2-q)} = D^q z_0(t).\end{aligned}$$

It proves that z_0 is an upper solution of problem (9). Moreover $M(t) = 3A(t)$. In view of Theorem 2, problem (9) has solutions in $[y_0, z_0]$ if $\frac{1}{2} < q < 1$. In case when $0 < q \le \frac{1}{2}$, we have to extra assume that

$$\frac{\Gamma(q)}{\Gamma(2q)} \max_{t\in[0,1]} A(t) < 1;$$

for example if $q = \frac{1}{2}$, so $\max_{t\in[0,1]} A(t) < \frac{1}{\sqrt{\pi}}$.

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