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Global edge alliances in graphs

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ABSTRACT

In the paper we introduce and study a new problem of finding a minimum *global edge alliance* in a graph which is related to the global defensive alliance (Haynes et al., 2013; Hedetniemi, 2004) and the global defensive set (Lewoń et al., 2016).

We proved the $\mathcal{NP}$-completeness of the global edge alliance problem for subcubic graphs and we constructed polynomial time algorithms for trees. We found the exact values of the size of the minimum global edge alliance for certain classes: paths, cycles, wheels, complete k-partite graphs and complete k-ary trees. Moreover, we proved some lower bounds for arbitrary graphs.

1. Introduction

In the following we consider only simple nonempty graphs, and we use standard notations of the graph theory. Let G be a graph and let $X \subset V(G)$. By an *open neighborhood* of X in graph G we mean the set $\{v \in V(G): \exists_{u \in X}\{v, u\} \in E(G)\}$, denoted by $N_G(X)$. By a *closed neighborhood* of X in graph G we mean set $X \cup N_G(X)$, denoted by $N_G[X]$. Set X is a *dominating set* of G iff $V(G) = N_G[X]$, and X is a *total dominating set* iff $V(G) = N_G(X)$. By $\gamma_t(G)$ we denote the size of the minimum total dominating set in G.

Let $S \subset V(G)$. We define for any non-empty subset X of S the predicate $SEC_{G,S}(X) = true$ iff $|N_G[X] \cap S| \geq |N_G[X] \setminus S|$. In the following, we use the notation $SEC(X)$ instead of $SEC_{G,S}(X)$ if G and S are clearly given.

By $G[A]$, where $A \subset V(G)$, we denote a subgraph of G induced by set A, and by $G \setminus A$ we mean the graph $G[V \setminus A]$. For the sake of notation simplicity, we write $N_G[v]$ and $N_G[v, u]$ instead of $N_G[\{v\}]$ and $N_G[\{v, u\}]$, respectively, and analogously, $SEC(v)$ and $SEC(v, u)$. Let $\deg_G(v) = |N_G(v)|$ be the degree of a vertex $v \in V(G)$. By $n(G)$, $\Delta(G)$ and $\delta(G)$ we denote the number of vertices of G, the maximum and the minimum degree of a vertex of G, respectively. By an *isolated vertex* (in a graph G) we mean a vertex $v \in V(G)$ with $\deg_G(v) = 0$, and by an *isolated edge* (in a graph G) we mean an edge $\{u, v\}$ such that $\deg_G(u) = \deg_G(v) = 1$. Set $X \subset V(G)$ is an *independent set* iff each vertex of X is isolated in $G[X]$. By a *pendant vertex* we mean a vertex of degree 1. We call each neighbor of a pendant vertex in a tree a *support* vertex. Let $\mathrm{diam}(G) = \max\{d_G(v, u): u, v \in E(G)\}$, where $d_G(v, u)$ is the length of a shortest path in G between v and u.

Let $S \subset V(G)$ for a given graph G. Set S is an *edge alliance* in G iff $G[S]$ has no isolated vertices and for each edge $e = \{v, u\} \in E(G[S])$ we have $SEC(v, u) = true$. An edge alliance S is a *global edge alliance* in G if it also dominates G. By $\gamma_{ea}(G)$ we denote the size of the minimum global edge alliance in graph G.

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Fig. 1. The examples of the global alliance number and the global edge alliance number: (a) $\gamma_a = 2 < \gamma_{ea} = 3$ and (b) $\gamma_{ea} = 3 < \gamma_a = 5$.

1.1. Related problems and our contribution

A set S is a *defensive alliance* (or *alliance*) iff for each vertex $v \in S$ we have $SEC(v) = true$. If S is also a dominating set of G, we say that S is a *global defensive alliance* (or *global alliance*). By $\gamma_a(G)$ we mean the size of the minimum global alliance in G. The concept of alliances and global alliances in graphs is due to [16] and [15]. The problem has certain interesting applications in web communities [12] or fault-tolerant computing [22].

In [15] the authors proved bounds on the minimum global alliance for general graphs (lower bounds: $\frac{\sqrt{4n+1}-1}{2}$ and $\frac{n}{\lceil \frac{\Delta}{2} \rceil + 1}$, upper bound: $n - \lceil \frac{\delta}{2} \rceil$), for bipartite graphs (lower bound: $\lceil \frac{2n}{\Delta+3} \rceil$), and trees (lower bound: $\frac{n+2}{4}$, upper bound: $\frac{3n}{5}$). For the other bounds on trees, see [1,2,8]. The exact values of the minimum global alliance were given in [15] for complete graphs, complete bipartite graphs, cycles, paths and wheels, for k-ary trees ($k \in \{2, 3, 4\}$) in [6], and independently in [14] (for $k \in \{2, 3\}$), and for star graphs in [17]. In [5] the authors proved the $\mathcal{NP}$-completeness of the minimum global alliance problem for general graphs, in [18] the author proved it for bipartite or chordal graphs, and in [20] the authors proved $\mathcal{NP}$-completeness for subcubic bipartite planar graphs. In [21] the authors study the problem of finding two disjoint global alliances in graphs.

Set S is a *defensive set* in G iff for each vertex $v \in S$ we have: $SEC(v) = true$ or there exists a neighbor $u \in S$ of v (i.e. $\{v, u\} \in E(G)$) such that $SEC(v, u) = true$. If S is also a dominating set of G, we say that S is a *global defensive set*. By $\gamma_{ds}(G)$ we denote the size of the minimum global defensive set in G. The concept of defensive sets introduced and studied in [20] arises from the concept of alliances, but is a kind of relaxation of the alliance problem. In [20] the authors proved the $\mathcal{NP}$-completeness of the minimum global defensive set problem for subcubic bipartite planar graphs, they constructed polynomial time algorithm for trees, and proved some bounds on γ_{ds}.

Set S is a *secure set* in G iff $\forall_{X \subset S} SEC(X) = true$. The concept of secure sets was introduced in [3] and studied in certain papers, e.g., [4,9–11].

In the paper we introduce and study the global edge alliance problem. The concept of edge alliance arises from the idea of alliances and it is a restriction of defensive sets. In the alliance problem a vertex being under an attack (say x) can be defended by itself and some of its neighbors, and it is possible iff $SEC(x) = true$. In the defensive set problem [20], if $SEC(x) = false$, we allow one of the neighbors of x, say y, to join 'the war', i.e., an attack can be simultaneously done on two vertices x and y, and in that case each attack on x and y can be defended, whenever $SEC(x, y) = true$. In the edge alliance problem, instead of defending the nodes, we defend the links between them, i.e., the structure being under an attack is an edge which can be defended by its end vertices and some of their neighbors. Note that an edge alliance is a defensive set.

We prove the $\mathcal{NP}$-completeness of the global edge alliance problem for subcubic graphs and we construct polynomial time algorithm for trees. We find the exact values of the size of the minimum global edge alliance for certain classes: paths, cycles, wheels, complete k-partite graphs and complete k-ary trees. We prove the lower bound for arbitrary graphs.

2. Bounds on the minimum global edge alliance

By $\mathcal{G}_{ea}(G)$ we denote $\{S \subset V(G): S$ is a global edge alliance of $G\}$. If $\delta(G) \geq 1$, then $\mathcal{G}_{ea}(G) \neq \emptyset$.

Let G be a graph with $\delta(G) \geq 1$, and let $S \subset V(G)$ be a global edge alliance. Since S is a dominating set of G and $G[S]$ has no isolated vertices, we have that S is a total dominating set of G. Moreover, S is a global defensive set. Thus,

Proposition 2.1. *Let G be a graph with $\delta(G) \geq 1$. Then,*

$$\gamma_{ea}(G) \geq \max\{\gamma_{ds}(G), \gamma_t(G)\}. \quad \square$$

There is no such a relation, in general, between the global alliance number and the global edge alliance number, which is shown in Fig. 1. For subcubic graphs with $\delta \geq 2$ by [15] we have that $\gamma_t = \gamma_a$. Thus,

Proposition 2.2. *Let G be a graph with $\delta(G) \geq 2$ and $\Delta(G) \leq 3$. Then,*

$$\gamma_{ea}(G) \geq \gamma_a(G). \quad \square$$

Following [20], we prove the lower bound on γ_{ea} for arbitrary graphs. By $\nu(G)$ we mean the size of the maximum matching in graph G (i.e., set of edges no two of which have a common end). For a given global edge alliance S in graph G, by $\nu_S(G)$ we mean $\nu(G[S])$. Finally, let us define by $\nu_{ea}(G) = \max\{\nu_S(G): S \in \mathcal{G}_{ea}(G) \wedge \gamma_{ea}(G) = |S|\}$.

Theorem 2.1. *Let G be a graph with $\delta(G) \geq 1$. Then,*

$$\gamma_{ea}(G) \geq \frac{\sqrt{4n(G) + (\nu_{ea}(G) - 1)^2} + \nu_{ea}(G) - 1}{2}.$$

Proof. Let S be any global edge alliance such that $|S| = \gamma_{ea}(G)$ and $\nu_{ea}(G) = \nu_S(G)$, and let $V = V(G)$, $n = n(G)$, $s = |S|$ and $\nu = \nu_{ea}(G)$. Obviously, $\nu > 0$. The thesis is equivalent to $s^2 - (\nu - 1)s - n \geq 0$. Let us assume to the contrary that $s^2 - (\nu - 1)s - n < 0$. Since S is a total dominating set of G we have $|N_G[S] \cap (V \setminus S)| = |V \setminus S| = n - s > s^2 - \nu s$.

On the one hand, let $M = \{\{v_i, u_i\} : i \in \{1, \ldots, \nu\}\}$ be a maximum matching in $G[S]$, and let $U = \bigcup_{i=1}^{\nu} \{v_i, u_i\}$. Since $SEC(v_i, u_i) = true$, we have that $|N_G[v_i, u_i] \cap (V \setminus S)| \leq |N_G[v_i, u_i] \cap S| \leq s$. Hence, by $|N_G[S \setminus U] \cap (V \setminus S)| + \sum_{i=1}^{\nu} |N_G[v_i, u_i] \cap (V \setminus S)| \geq |(N_G[S \setminus U] \cup \bigcup_{i=1}^{\nu} N_G[v_i, u_i]) \cap (V \setminus S)| = |V \setminus S| > s^2 - \nu s$, we get $|N_G[S \setminus U] \cap (V \setminus S)| > s(s - 2\nu)$.

On the other hand, since for every $w \in S$ there is $u \in S$ such that $\{w, u\} \in E(G[S])$ and $SEC(w, u) = true$, we have that $|N_G[w] \cap (V \setminus S)| \leq |N_G[w, u] \cap (V \setminus S)| \leq |N_G[w, u] \cap S| \leq s$. Thus, $|N_G[S \setminus U] \cap (V \setminus S)| \leq \sum_{w \in S \setminus U} |N_G[w] \cap (V \setminus S)| \leq s(s - 2\nu)$, a contradiction. $\square$

Following [20], let us observe that for any $r \geq 1$ and $l \geq r$ there is a graph G with $n(G) = l + r + l(l + r)$ vertices and $\nu_{ea}(G) = r$ such that the lower bound proved in Theorem 2.1 is tight. Let $G = K^*_{r,l}$ be the graph obtained from the complete bipartite graph $K_{r,l}$ by attaching $l + r$ vertices to each vertex of the part with l vertices. It is easy to notice that $\gamma_{ea}(G) = l + r$ and $\gamma_{ea}^2(G) - \gamma_{ea}(G)(\nu_{ea}(G) - 1) = n(G)$.

Proposition 2.3. *Let G be a graph with $\delta(G) \geq 1$. Then,*

$$\nu_{ea}(G) \geq \lceil \frac{diam(G) + 1}{5} \rceil.$$

Proof. Let $d = \text{diam}(G)$, and let v and u be vertices of G such that $d_G(v, u) = d$. Let $A_i = \{w \in V(G) \colon d_G(v, w) = i\}$, for $i \in \{0, \ldots, d\}$. Obviously, $A_0 = \{v\}$ and $u \in A_d$. Let us observe that for each edge $\{x, y\} \in E(G)$, if $x \in A_i$ and $y \in A_j$, then $|i - j| \leq 1$.

Let $S \subset V(G)$ be a global edge alliance of G such that $\gamma_{ea}(G) = |S|$. Since S is a total dominating set we have $S \cap A_1 \neq \emptyset$ and there is $\{v_0, u_0\} \in E(G)$ such that $\{v_0, u_0\} \subset S \cap (A_0 \cup A_1 \cup A_2)$. If $d \leq 4$, the thesis holds.

Let $d = 5k + i$, where $k \geq 1$ and $0 \leq i \leq 4$. Let us observe that for each $l \in \{0, \ldots, d - 2\}$, $S \cap (A_l \cup A_{l+1} \cup A_{l+2}) \neq \emptyset$. Thus, for each $j \in \{1, \ldots, k - 1\}$ there is an edge $\{v_j, u_j\} \in E(G)$ such that $\{v_j, u_j\} \subset S \cap (A_{5j-2} \cup A_{5j-1} \cup A_{5j} \cup A_{5j+1} \cup A_{5j+2})$. If $j = k$, then there is an edge $\{v_k, u_k\} \in E(G)$ such that $\{v_k, u_k\} \subset S \cap (A_{5k-2} \cup A_{5k-1} \cup A_{5k} \cup \cdots \cup A_{5k+i})$.

Since $\{\{v_0, u_0\}, \ldots, \{v_k, u_k\}\}$ is the matching in $G[S]$, we have $\nu_{ea}(G) \geq \nu_S(G) \geq k + 1 = \lceil \frac{\text{diam}(G)+1}{5} \rceil$. $\square$

Corollary 2.4. *Let G be a graph with $\delta(G) \geq 1$. Then,*

$$\gamma_{ea}(G) \geq \frac{\sqrt{4n(G) + (\lceil \frac{diam(G)+1}{5} \rceil - 1)^2} + \lceil \frac{diam(G)+1}{5} \rceil - 1}{2}. \quad \square$$

3. Global edge alliance of certain graph classes

In this section we give the exact formulas for the global edge alliance number of the following classes: paths, cycles, wheels, complete multipartite graphs and complete k-ary trees.

Let G be a graph with $\delta(G) \geq 1$ and $\Delta(G) \leq 2$. Hence, we have that $|N_G[u, v] \setminus \{u, v\}| \leq 2$ for each $\{u, v\} \in E(G)$, and so $\gamma_{ea}(G) = \gamma_t(G)$. Thus,

Proposition 3.1. *Let G be a path of order $n \geq 2$ or a cycle of order $n \geq 3$. Then,*

$$\gamma_{ea}(G) = \begin{cases} \frac{n}{2} AMZ@Pif\ 4 \mid n, \\ \lfloor \frac{n}{2} \rfloor + 1 AMZ@Potherwise. \end{cases} \quad \square$$

Let us recall that by wheel W_k ($k \geq 3$) we mean a graph obtained from cycle C_k by adding the *central vertex* v_c and joining it with all other vertices of the cycle C_k. Hence, we have $|N_{W_k}[v_c]| = |V(W_k)| = k + 1$, $\Delta(W_k) = k$ and for each $v \in V(W_k) \setminus \{v_c\}$ we have $\deg_{W_k}(v) = 3$.

Proposition 3.2. *Let G be a wheel of order n. Then,*

$$\gamma_{ea}(G) = \lceil \frac{n}{2} \rceil.$$

Proof. Let G be a wheel of order n. If $n = 4$, then $\gamma_{ea}(G) = 2$. Let $n \geq 5$ and let $v_c \in V(G)$ be the central vertex of wheel G. Let $S \subset V(G)$ be any global edge alliance of G. If $v_c \in S$, then there is $v \in V(G) \setminus \{v_c\}$ such that $v \in S$. Hence, $N_G[v, v_c] = V(G)$, so $|S| \geq \lceil n/2 \rceil$. Let $v_c \notin S$. For each $\{v, u\} \in E(G[S])$ we have $|N_G[v, u] \setminus \{v, u\}| = 3$, so $|S \cap N_G[u, v]| \geq 3$. Hence, each connected component of graph $G[S]$ has at least three vertices. Since S is a dominating set of G, we have $|S| \geq \lceil 3(n-1)/5 \rceil$. Thus, $|S| \geq \min\{\lceil n/2 \rceil, \lceil 3(n-1)/5 \rceil\} = \lceil n/2 \rceil$. Take any $S \subset V(G)$ such that $v_c \in S$ and $|S \cap V(G)| = \lceil n/2 \rceil$. Thus, S is a global edge alliance and $\gamma_{ea}(G) = \lceil \frac{n}{2} \rceil$. □

Theorem 3.1. *Let G be a complete multipartite graph of order $n \geq 3$. Then,*

$$\gamma_{ea}(G) = \lceil \frac{n}{2} \rceil.$$

Proof. Let G be a complete multipartite graph of order $n \geq 3$. For each edge $\{u, v\} \in E(G)$ we have $N_G[u, v] = V(G)$. Let $S \subset V(G)$ be any global edge alliance of G, and let $\{u, v\} \in E(G[S])$. Then, $SEC(u, v) \Leftrightarrow |V(G) \cap S| \geq |V(G) \setminus S| \Leftrightarrow |S| \geq \lceil n/2 \rceil$. Thus, $\gamma_{ea}(G) \geq \lceil n/2 \rceil$.

Let $V(G) = V_1 \cup \cdots \cup V_k$, where $k \geq 2$, and all V_i are maximal independent sets and pairwise disjoint. Let $r_i = |V_i|$, for each $i \in \{1, \ldots, k\}$. If for each $i \in \{1, \ldots, k\}$ an integer r_i is even, then let $S \subset V(G)$ be any set such that $|S \cap V_i| = r_i/2$. Thus, S is a global edge alliance and $|S| = n/2$.

Without loss of generality, let us assume that $r_1, \ldots, r_l$ are odd integers, and $r_{l+1}, \ldots, r_k$ are even integers, for some $1 \leq l \leq k$. Take any $S \subset V(G)$ such that $|S \cap V_i| = \lceil r_i/2 \rceil$ for each $i \in \{1, \ldots, \lceil l/2 \rceil\}$, $|S \cap V_i| = \lfloor r_i/2 \rfloor$ for each $i \in \{\lceil l/2 \rceil + 1, \ldots, l\}$, and $|S \cap V_i| = r_i/2$ for each $i \in \{l+1, \ldots, k\}$. Hence, $\delta(G[S]) \geq 1$ and $|S| = \lceil n/2 \rceil$. Thus, S is a global edge alliance. □

3.1. Complete k-ary trees

Let us remind that the exact formulas for the minimum global alliance of complete k-ary trees are known only for $k \in \{2, 3, 4\}$ [6]. In this section we give the exact formulas for the minimum global edge alliance of complete k-ary trees for arbitrary $k \geq 2$.

Let $\delta(G) \geq 1$ and $A \subset V(G)$. By $\gamma_{ea}(G, A)$ we mean $\min\{|S \cap A| : S \in \mathcal{G}_{ea}(G)\}$.

Observation 3.3. *Let G be a graph with $\delta(G) \geq 1$, and let $A_i \subset V(G)$, for $i \in \{1, \ldots, p\}$, where $p \geq 1$. If sets $A_1, \ldots, A_p$ are pairwise disjoint, then*

$$\sum_{i=1}^{p} \gamma_{ea}(G, A_i) \leq \gamma_{ea}(G, \bigcup_{i=1}^{p} A_i) \leq \gamma_{ea}(G). \quad \square$$

By a complete k-ary tree of height h, $k \geq 2$, $h \geq 1$ (both integers), denoted by T_k^h, we mean a tree with a vertex set and an edge set, respectively,

$$V(T_k^h) = \{v_1^0, v_1^1, \ldots, v_k^1, v_1^2, \ldots, v_{k^2}^2, \ldots, v_1^{h-1}, \ldots, v_{k^{h-1}}^{h-1}, v_1^h, \ldots, v_{k^h}^h\},$$

$$E(T_k^h) = \bigcup_{l=0}^{h-1} \bigcup_{i=1}^{k^l} \bigcup_{j=(i-1)k+1}^{i \cdot k} \{\{v_i^l, v_j^{l+1}\}\}.$$

Let $T = T_k^h$. Let us observe that $\deg_T(v_1^0) = k$, and for each $i \in \{1, \ldots, h\}$ and $j \in \{1, \ldots, k^i\}$, $d_T(v_1^0, v_j^i) = i$ and, if $i \neq h$, $\deg_T(v_j^i) = k + 1$. Set of all leaves in T is $\{v_1^h, \ldots, v_{k^h}^h\}$, and $d_T(v_1^0, v_j^h) = h$, for each $j \in \{1, \ldots, k^h\}$. By $r(T)$ we mean the only vertex of degree k in T.

For each $l \in \{0, \ldots, h\}$, let us define $L_l = \{v_1^l, \ldots, v_{k^l}^l\}$ (obviously, $L_l = \{v \in V(T) : d_T(v, r(T)) = l\}$) and let $L_{h+i} = \emptyset$, for $i \geq 1$. Let $v \in L_p$, for some $p \in \{0, \ldots, h\}$. By $C(T, v)$ we mean $N_T(v) \cap L_{p+1}$, and for each $A \subset V(G)$, by $C(T, A)$ we mean $\bigcup_{v \in A} C(T, v)$. Obviously, $|C(T, v)| = k$, and $C(T, L_h) = \emptyset$. Let us define $C^0(T, v) = \{v\}$, and for each $l \geq 1$, let $C^l(T, v) = C(T, C^{l-1}(T, v))$. Let $T_v^l = T[\bigcup_{i=0}^{l} C^i(T, v)]$, and $T_v = T_v^{h-p}$.

Lemma 3.4. *Let $T = T_k^h$, $h \geq 2$, $k \geq 2$, and let $l \in \{1, 2, 3\}$. Then,*

$$\gamma_{ea}(T, V(T_{r(T)}^{l-1})) \geq \frac{k^{l-1} - 1}{k - 1}.$$

Proof. Let $r = r(T)$. If $l \leq 2$, then the thesis is obvious. Let $l = 3$ and $S \in \mathcal{G}_{ea}(T)$. If $r \in S$, then there is $u \in L_1 \cap S$. Since $SEC(r, u) = true$ and $|N_T[\{r, u\}]| = 2k+1$, $|N_T[\{r, u\}] \cap S| \geq k+1$. Hence, $\gamma_{ea}(T, V(T_r^2)) \geq k+1$. If $r \notin S$, then for each $u \in L_1$, $\gamma_{ea}(T, V(T_u^1)) \geq 1$. Since $N_T[r] \cap S \neq \emptyset$, there is $u \in L_1 \cap S$. Since $N_T[u] \cap S \neq \emptyset$, $|S \cap V(T_u^1)| \geq 2$. Thus, by Observation 3.3, $\gamma_{ea}(T, V(T_r^2)) \geq \sum_{u \in L_1} \gamma_{ea}(T, V(T_u^1)) \geq k + 1$. □

Lemma 3.5. *Let $T = T_k^h$, $h \geq 2$, $k \geq 2$. Then,*

$$\gamma_{ea}(T, L_{h-2} \cup L_{h-1} \cup L_h) \geq (k+1)k^{h-2}.$$

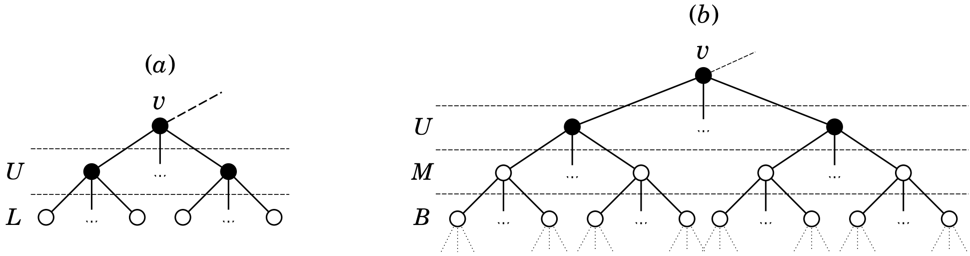

Fig. 2. Subgraphs of T_k^h discussed in Lemmas 3.5 (a) and 3.6 (b).

Proof. Let $v \in L_{h-2}$, and let $L = C^2(T, v)$ and $U = C^1(T, v)$ (see Fig. 2(a)). Let $S \in \mathcal{G}_{ea}(T)$. Since $L \subset L_h$ and $N_T(L) = U$, $U \subset S$. If $v \in S$, then $|S \cap V(T_v)| \geq k + 1$. If $v \notin S$, then for each $u \in U$, $|S \cap V(T_u)| \geq 2$. Thus, $\gamma_{ea}(T, V(T_v)) \geq k + 1$.

Since $|L_{h-2}| = k^{h-2}$, and $\bigcup_{v\in L_{h-2}} V(T_v) = L_{h-2} \cup L_{h-1} \cup L_h$, by Observation 3.3 we get the thesis. $\square$

Lemma 3.6. *Let $T = T_k^h$, $h \geq 5$, $k \geq 2$. For each $l \in \{0, \dots, h-5\}$,*

$$\gamma_{ea}(T, L_l \cup L_{l+1} \cup L_{l+2} \cup L_{l+3}) \geq (k + 1)k^l.$$

Proof. Let $l \in \{0, \dots, h-5\}$ and $v \in L_l$. Let $B = C^3(T, v)$, $M = C^2(T, v)$ and $U = C^1(T, v)$ (see Fig. 2(b)). Let $S \in \mathcal{G}_{ea}(T)$. Since $N_T(M) = B \cup U$, for each $u \in U$, $|S \cap V(T_u^2)| \geq 1$. If $v \in S$, then $|S \cap V(T_v)| \geq k + 1$. If $v \notin S$, then for each $u \in U$, $|S \cap V(T_u^2)| \geq 2$. Thus, $\gamma_{ea}(T, V(T_v)) \geq k + 1$.

Since $|L_l| = k^l$, and $\bigcup_{v\in L_l} V(T_v^3) = L_l \cup L_{l+1} \cup L_{l+2} \cup L_{l+3}$, by Observation 3.3 we get the thesis. $\square$

Lemma 3.7. *Let $T = T_k^h$, $h = 4p + 2$, $p \geq 0$, $k \geq 2$. Then,*

$$\gamma_{ea}(T) = (k + 1)\frac{k^{h+2} - 1}{k^4 - 1}.$$

Proof. Let $A_i = L_{4i} \cup L_{4i+1} \cup L_{4i+2} \cup L_{4i+3}$, for each $i \in \{0, \dots, p-1\}$, and let $A_p = L_{4p} \cup L_{4p+1} \cup L_{4p+2}$. By Observation 3.3 and by Lemmas 3.5 and 3.6, $\gamma_{ea}(T) \geq \sum_{i=0}^{p} \gamma_{ea}(T, A_i) = (k + 1)\sum_{i=0}^{p} k^{4i} = (k + 1)\frac{k^{h+2}-1}{k^4-1}$.

Let $S = \bigcup_{i=0}^{p}(L_{4i} \cup L_{4i+1})$. Since $S \in \mathcal{G}_{ea}(T)$ and $|S| = (k + 1)\frac{k^{h+2}-1}{k^4-1}$, $\gamma_{ea}(T) = |S|$. $\square$

Lemma 3.8. *Let $T = T_k^h$, $k \geq 2$, and let $h = q + 4p + 2 \geq 2$, where $q \in \{1, 2, 3\}$ and $p \geq 0$. Then,*

$$\gamma_{ea}(T) = \frac{k^{q-1} - 1}{k - 1} + (k + 1)\frac{k^{h+2} - k^q}{k^4 - 1}.$$

Proof. Let $h = q + 4p + 2 \geq 2$, where $q \in \{1, 2, 3\}$ and $p \geq 0$. Let $r = r(T)$. Let $A_i = L_{q+4i} \cup L_{q+4i+1} \cup L_{q+4i+2} \cup L_{q+4i+3}$, for each $i \in \{0, \dots, p-1\}$, and let $A_p = L_{q+4p} \cup L_{q+4p+1} \cup L_{q+4p+2}$. Since $V(T) = V(T_r^{q-1}) \cup \bigcup_{i=0}^{p} A_i$, by Observation 3.3 and by Lemmas 3.4 and 3.7 we get $\gamma_{ea}(T) \geq \frac{k^{q-1}-1}{k-1} + (k + 1)\frac{k^{h+2}-k^q}{k^4-1}$.

Let $\hat{S} = \bigcup_{i=0}^{p}(L_{q+4i} \cup L_{q+4i+1})$. Let $S_1 = \emptyset$, $S_2 = \{u\}$, where $u \in N_T(r)$, and $S_3 = V(T_r^1)$. Let us observe that for each $q \in \{1, 2, 3\}$, $S = \hat{S} \cup S_q \in \mathcal{G}_{ea}(T)$, and $|S| = \frac{k^{q-1}-1}{k-1} + (k + 1)\frac{k^{h+2}-k^q}{k^4-1}$. Thus, $\gamma_{ea}(T) = |S|$. $\square$

By Lemmas 3.7 and 3.8 we conclude

Theorem 3.2. *Let $T = T_k^h$, $h \geq 2$, $k \geq 2$. Then,*

1. *if $h \equiv 0 \mod 4$, then $\gamma_{ea}(T) = \frac{k+1}{k^4-1}(k^{h+2} - k^2) + 1$,*
2. *if $h \equiv 1 \mod 4$, then $\gamma_{ea}(T) = \frac{k+1}{k^4-1}(k^{h+2} - k^3) + k + 1$,*
3. *if $h \equiv 2 \mod 4$, then $\gamma_{ea}(T) = \frac{k+1}{k^4-1}(k^{h+2} - 1)$,*
4. *if $h \equiv 3 \mod 4$, then $\gamma_{ea}(T) = \frac{k+1}{k^4-1}(k^{h+2} - k)$.* $\square$

4. $\mathcal{NP}$-completeness for subcubic graphs

In this section we prove the $\mathcal{NP}$-completeness of the global edge alliance problem for subcubic graphs, by the reduction from the $\mathcal{NP}$-complete $\overline{\text{3DM}}$ problem [13].

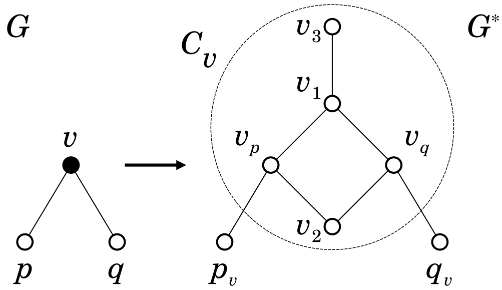

Fig. 3. Graph C_v replacing vertex $v \in V$ of degree 2. Note that $p_v, q_v \notin V(C_v)$, but $p_v \in V(H_p)$, $q_v \in V(H_q)$.

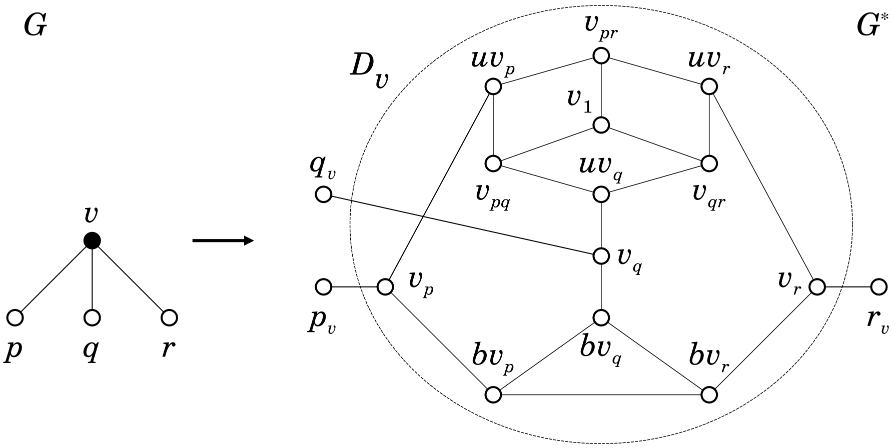

Fig. 4. Graph D_v replacing vertex $v \in V$ of degree 3. Note that $p_v, q_v, r_v \notin V(D_v)$, but $p_v \in V(H_p)$, $q_v \in V(H_q)$, $r_v \in V(H_r)$.

$\overline{\text{3DM}}$	
Instance:	A subcubic bipartite graph $G = (V \cup Q, E)$ without pendant vertices, where V and Q is a bipartition of G, $V = X \cup Y \cup Z$, $\lvert X\rvert = \lvert Y\rvert = \lvert Z\rvert = m$, $\lvert V\rvert = 3m$. For each vertex $q \in Q$, $\deg_G(q) = 3$ and q is adjacent to exactly one vertex from each X, Y and Z, and for each vertex $v \in V$, $\deg_G(v) \in \{2, 3\}$.
Question:	Is there a subset $Q' \subset Q$ of cardinality $\lvert Q'\rvert = m$ dominating all vertices in V, i.e., $N_G(Q') = V$?

Theorem 4.1. *The global edge alliance problem for subcubic graphs is $\mathcal{NP}$-complete.*

Proof. The proof proceeds by the reduction from the problem $\overline{\text{3DM}}$. Let $G = (V \cup Q, E)$ be a subcubic bipartite graph with bipartition V and Q, such that $V = X \cup Y \cup Z$, $|X| = |Y| = |Z| = m$, $|V| = 3m$, and $|Q| = t$. For each vertex $q \in Q$, $\deg_G(q) = 3$ and q is adjacent to exactly one vertex from each of the sets X, Y and Z, and for each vertex $v \in V$, $\deg_G(v) \in \{2, 3\}$. Let $V_i = \{v \in V : \deg_G(v) = i\}$, and $m_i = |V_i|$, for $i \in \{2, 3\}$. We construct a subcubic graph G^* such that there is a subset $Q' \subset Q$ of cardinality $|Q'| = m$ dominating all vertices in V iff there is a global edge alliance S in graph G^* such that $|S| \le 2m_2 + 5m_3 + 9t$.

We shall transform (in polynomial time) graph G into graph G^* in four steps:

(S_1) each $v \in V_2$, where $N_G(v) = \{p, q\} \subset Q$, replace with graph C_v (Fig. 3),
(S_2) each $v \in V_3$, where $N_G(v) = \{p, q, r\} \subset Q$, replace with graph D_v (Fig. 4),
(S_3) each $q \in Q$, where $N_G(q) = \{x, y, z\} \subset V$ and $x \in X, y \in Y, z \in Z$, replace with graph H_q (Fig. 5),
(S_4) each $\{v, q\} \in E(G)$, where $v \in V$, $q \in Q$, replace with edge $\{v_q, q_v\} \in E(G^*)$.

Formally, all graphs $\{C_v\}_{v \in V_2}$ are isomorphic and pairwise vertex disjoint (analogously, $\{D_v\}_{v \in V_3}$ and $\{H_q\}_{q \in Q}$), so

$$V(G^*) = \bigcup_{v \in V_2} V(C_v) \cup \bigcup_{v \in V_3} V(D_v) \cup \bigcup_{q \in Q} V(H_q)$$

$$E(G^*) = \bigcup_{v \in V_2} E(C_v) \cup \bigcup_{v \in V_3} E(D_v) \cup \bigcup_{q \in Q} E(H_q) \cup \bigcup_{v \in V, q \in Q, \{v,q\} \in E(G)} \{\{v_q, q_v\}\}$$

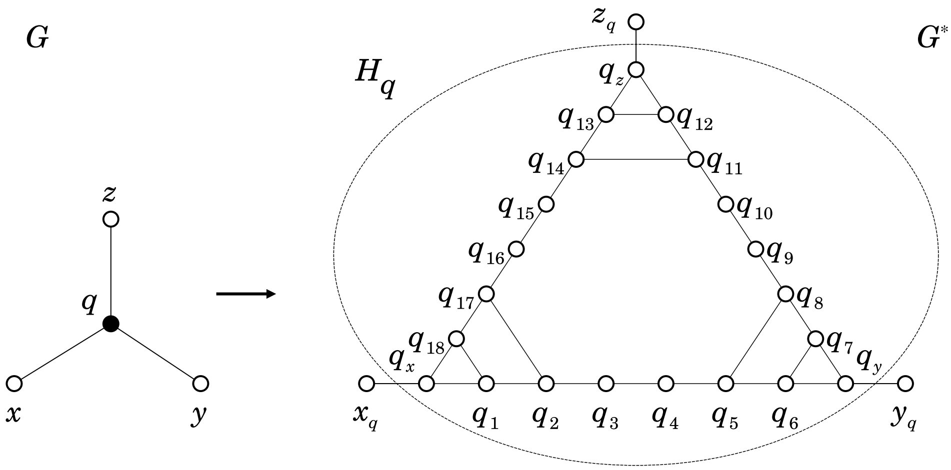

Fig. 5. Graph H_q replacing vertex $q \in Q$. Note that $x_q, y_q, z_q \notin V(H_q)$.

It is easy to observe that $\Delta(G^*) \le 3$.

($\Rightarrow$) Suppose that $Q' \subset Q$ dominates all vertices in V and $|Q'| = m$. Since $|X| = |Y| = |Z| = m$, we have $|N_G(v) \cap Q'| = 1$, for each $v \in V$. Hence, let $\{q(v)\} = N_G(v) \cap Q'$, for each $v \in V$. Moreover, if $v \in V_2$, then let $\{p(v)\} = N_G(v) \cap (Q \setminus Q')$ and if $v \in V_3$, then let $\{p(v), r(v)\} = N_G(v) \cap (Q \setminus Q')$. For each $q \in Q$, if $N_G(q) = \{x, y, z\} \subset V$ and $x \in X, y \in Y, z \in Z$, then let $Q_q^+ = \{q_x, q_y, q_z, q_3, q_4, q_9, q_{10}, q_{15}, q_{16}\} \subset V(H_q)$ and $Q_q^- = \{q_1, q_2, q_3, q_7, q_8, q_9, q_{13}, q_{14}, q_{15}\} \subset V(H_q)$ (see Fig. 5). Let

$$S = \bigcup_{v \in V_2} \{v_1, v_{q(v)}\} \cup \bigcup_{v \in V_3} \{v_{q(v)}, bv_{q(v)}, uv_{p(v)}, v_{p(v)r(v)}, uv_{r(v)}\} \cup \bigcup_{q \in Q'} Q_q^+ \cup \bigcup_{q \in Q \setminus Q'} Q_q^- .$$

For each $v \in V_2$, set $\{v_1, v_{q(v)}\} \subset V(C_v)$ is a dominating set of C_v, and for each $v \in V_3$, set $\{v_{q(v)}, bv_{q(v)}, uv_{p(v)}, v_{p(v)r(v)}, uv_{r(v)}\} \subset V(D_v)$ is a dominating set of D_v. If $q \in Q'$, then Q_q^+ is a dominating set of H_q, and if $q \in Q \setminus Q'$, then Q_q^- is a dominating set of H_q. Thus, S is a dominating set of G^*.

Each edge of $G^*[S]$ belongs to the one graph from $\{C_v\}_{v \in V_2} \cup \{D_v\}_{v \in V_3} \cup \{H_q\}_{q \in Q}$, or is equal to $\{v_q, q_v\}$ for some $q \in Q'$ and $v \in V$.

Let $e \in E(G^*[S])$. If e is one of the edges $\{q_3, q_4\}, \{q_9, q_{10}\}$ or $\{q_{15}, q_{16}\}$ for some $q \in Q'$, then $SEC(e) = true$. Otherwise, e is not an isolated edge in $G^*[S]$. Thus, by $\Delta(G^*) \le 3$, we have that S is a global edge alliance, and $|S| \le 2m_2 + 5m_3 + 9|Q'| + 9|Q \setminus Q'| = 2m_2 + 5m_3 + 9t$.

($\Leftarrow$) Let S be any global edge alliance of G^* such that $|S| \le 2m_2 + 5m_3 + 9t$. In the following we refer to the notation from Figs. 3–5.

Claim 4.1. *For each $v \in V_2$, $|S \cap V(C_v)| \ge 2$. Moreover, if $|S \cap V(C_v)| = 2$, then $|S \cap \{v_p, v_q\}| = 1$, and, $\{v_1, v_p, p_v\} \subset S$ or $\{v_1, v_q, q_v\} \subset S$, where $N_G(v) = \{p, q\}$.*

Proof. Let $v \in V_2$, and let $N_G(v) = \{p, q\}$. Since S is a total dominating set of G^*, $v_1 \in S$. If $v_3 \in S$, then $|N_{G^*}[v_2] \cap S| \ge 1$ and $|S \cap V(C_v)| \ge 3$. Otherwise, $v_p \in S$ or $v_q \in S$. Since S is an edge alliance, $p_v \in S$ and $q_v \in S$. Thus, $\{v_1, v_p, p_v\} \subset S$ or $\{v_1, v_q, q_v\} \subset S$. □

Claim 4.2. *For each $v \in V_3$, $|S \cap V(D_v)| \ge 5$. Moreover, if $|S \cap V(D_v)| = 5$, then $|S \cap \{v_p, v_q, v_r\}| = 1$, and, $\{bv_p, v_p, p_v\} \subset S$ or $\{bv_q, v_q, q_v\} \subset S$ or $\{bv_r, v_r, r_v\} \subset S$, where $N_G(v) = \{p, q, r\}$.*

Proof. Let $v \in V_3$ and $N_G(v) = \{p, q, r\}$. Let us remind that S is a global edge alliance.

Suppose $\{bv_p, bv_q, bv_r\} \cap S = \emptyset$. Hence, $\{v_p, v_q, v_r\} \subset S$. If $v_1 \in S$, then there is $u \in N_{G^*}[v_1] \cap S$ such that $|S \cap N_{G^*}[u]| \ge 3$. Thus, $|S \cap V(D_v)| \ge 6$. Let $v_1 \notin S$, and let us assume without loss of generality that $v_{pr} \in S$. If $|N_{G^*}[v_{pr}] \cap S| \ge 3$, then $|S \cap V(D_v)| \ge 6$. Let $|N_{G^*}[v_{pr}] \cap S| = 2$. If $uv_r \in S$ ($uv_p \in S$, analogously), then $N_{G^*}[v_{pq}] \cap S \ne \emptyset$. Thus, $|S \cap V(D_v)| \ge 6$.

Suppose $S \cap \{bv_p, bv_q, bv_r\} \ne \emptyset$. If $|S \cap \{bv_p, bv_q, bv_r\}| \ge 2$, then analogously to the case $S \cap \{bv_p, bv_q, bv_r\} = \emptyset$ we get $|S \cap V(D_v)| \ge 6$. Thus, without loss of generality, let us assume that $bv_q \in S$ and $S \cap \{bv_p, bv_r\} = \emptyset$. Hence, $v_q \in S$ and we consider two cases $uv_q \in S$ or $q_v \in S$. Let $U = N_{G^*}[v_{pr}]$ and $U_1 = N_{G^*}[U]$.

Let $\{bv_q, v_q, uv_q\} \subset S$. If $v_{pr} \in S$, then $|S \cap U_1| \ge 3$. Since $\{bv_q, v_q, uv_q\} \cap U_1 = \emptyset$, we have $|S \cap V(D_v)| \ge 6$. So, let $v_{pr} \notin S$. Let us observe that $S \cap \{v_{pq}, v_{qr}\} \ne \emptyset$. If $\{v_{pq}, v_{qr}, v_1\} \subset S$, then $|S \cap V(D_v)| \ge 6$. If $v_{qr} \notin S$ ($v_{pq} \notin S$, analogously), then $v_{pq} \in S$ and $|S \cap N_{G^*}[\{v_{pr}, uv_r\}]| \ge 2$. Thus, $|S \cap V(D_v)| \ge 6$. So, let $S \cap \{v_{pq}, v_{qr}, v_1\} = \{v_{pq}, v_{qr}\}$. Since $S \cap U \ne \emptyset$, we have $|S \cap V(D_v)| \ge 6$.

Let $\{bv_q, v_q, q_v\} \subset S$, and let $U_2 = N_{G^*}[U_1] \cap V(D_v)$. Since $S \cap U \neq \emptyset$ and $U_2 \cap \{bv_q, v_q\} = \emptyset$, we have $|S \cap U_2| \geq 3$. Thus, $|S \cap V(D_v)| \geq 5$. □

Claim 4.3. *For each $q \in Q$, $|S \cap V(H_q)| \geq 9$.*

Proof. Let $q \in Q$ and $N_G(q) = \{x, y, z\}$, where $x \in X, y \in Y, z \in Z$. Let us remind that S is a global edge alliance. We prove that $|S \cap \{q_x, q_1, q_2, q_3, q_{16}, q_{17}, q_{18}\}| \geq 3$. Let $q_x \in S$ and $|S \cap \{q_2, q_{17}\}| \leq 1$. If $q_2 \in S$ ($q_{17} \in S$, analogously), then $q_1 \in S$ or $q_3 \in S$. If $S \cap \{q_2, q_{17}\} = \emptyset$, then $S \cap \{q_1, q_3\} \neq \emptyset$ and $S \cap \{q_{16}, q_{18}\} \neq \emptyset$. Let $q_x \notin S$. If $q_1 \in S$ ($q_{18} \in S$, analogously), then there is $u \in N_{G^*}[q_1]$ such that $|S \cap N_{G^*}[u]| \geq 3$. If $S \cap \{q_1, q_{18}\} = \emptyset$, then $\{q_2, q_{17}\} \subset S$, and so $S \cap \{q_3, q_{16}\} \neq \emptyset$. Analogously, we have $|S \cap \{q_y, q_4, q_5, q_6, q_7, q_8, q_9\}| \geq 3$ and $|S \cap \{q_z, q_{10}, q_{11}, q_{12}, q_{13}, q_{14}, q_{15}\}| \geq 3$. Thus, $|S \cap V(H_q)| \geq 9$. □

Claim 4.4. *$|S| = 2m_2 + 5m_3 + 9t$, and*

(i) For each $v \in V_2$, $|S \cap V(C_v)| = 2$. Moreover, $|S \cap \{v_p, v_q\}| = 1$ and $\{v_u, u_v\} \subset S$ for exactly one $u \in N_G(v) = \{p, q\}$.
(ii) For each $v \in V_3$, $|S \cap V(D_v)| = 5$. Moreover, $|S \cap \{v_p, v_q, v_r\}| = 1$ and $\{v_u, u_v\} \subset S$ for exactly one $u \in N_G(v) = \{p, q, r\}$.
(iii) For each $q \in Q$, $|S \cap V(H_q)| = 9$. Moreover, $\{q_x, q_y, q_z, x_q, y_q, z_q\} \subset S$ or $S \cap \{q_x, q_y, q_z, x_q, y_q, z_q\} = \emptyset$, where $N_G(q) = \{x, y, z\}$.

Proof. Since $|S| \leq 2m_2 + 5m_3 + 9t$, by Claims 4.1–4.3 we have that $|S| = 2m_2 + 5m_3 + 9t$, and the properties (i) and (ii) hold.

(iii). Let $q \in Q$ and $N_G(q) = \{x, y, z\}$, where $x \in X, y \in Y, z \in Z$. Since $|S| = 2m_2 + 5m_3 + 9t$, we have by properties (i) and (ii) that $|S \cap V(H_q)| = 9$. From the proof of Claim 4.3 we have

$$|S \cap \{q_x, q_1, q_2, q_3, q_{16}, q_{17}, q_{18}\}| = 3 \tag{1}$$
$$|S \cap \{q_y, q_4, q_5, q_6, q_7, q_8, q_9\}| = 3 \tag{2}$$
$$|S \cap \{q_z, q_{10}, q_{11}, q_{12}, q_{13}, q_{14}, q_{15}\}| = 3 \tag{3}$$

By properties (i) and (ii) we have that

$$\text{for each } v \in N_G(q), \text{ if } v_q \in S, \text{ then } q_v \in S. \tag{4}$$

In the following we prove that

$$\text{for each } v \in N_G(q), \text{ if } q_v \in S, \text{ then } S \cap N_{G^*}(q_v) = \{v_q\}. \tag{5}$$

Without loss of generality, let $v = x$ (see Fig. 5). Let us assume to the contrary that $q_x \in S$ and $S \cap \{q_1, q_{18}\} \neq \emptyset$. Hence by Eq. (1), we have that $|S \cap \{q_2, q_3, q_{16}, q_{17}\}| \leq 1$. Thus, we consider three cases: (a) $|S \cap \{q_2, q_{17}\}| = 1$, or (b) $|S \cap \{q_3, q_{16}\}| = 1$, or (c) $|S \cap \{q_1, q_{18}\}| = 2$.

Case (a). Let $q_{17} \in S$ ($q_2 \in S$, analogously). Hence, $q_{18} \in S$ and $S \cap \{q_2, q_3, q_{16}\} = \emptyset$, and so, $\{q_4, q_5, q_{14}\} \subset S$. If $q_y \in S$, then by Eq. (2) we have $SEC(\{q_4, q_5\}) = false$, a contradiction. Thus, $q_y \notin S$. Hence by Eq. (4), we have that $y_q \notin S$, implying that $q_6 \in S$. Further, by (2), we have that $S \cap \{q_8, q_9\} = \emptyset$, and so $\{q_{10}, q_{11}\} \subset S$. Since $q_{14} \in S$, by (3) we get $S \cap \{q_{12}, q_{13}, q_z\} = \emptyset$, which contradicts (4).

Case (b). Let $q_3 \in S$ ($q_{16} \in S$, analogously). Hence, we have that $q_{18} \in S$, otherwise $S \cap N_{G^*}[q_{17}] = \emptyset$. So, $\{q_4, q_{14}, q_{15}\} \subset S$, and by (3) and (4) we get $q_{13} \in S$. Thus, $\{q_8, q_9\} \subset S$, and by Eq. (2) we get $SEC(\{q_8, q_9\}) = false$, a contradiction.

Case (c). Let $\{q_1, q_{18}\} \subset S$. Hence, $S \cap \{q_2, q_3, q_{16}\} = \emptyset$, and so, $\{q_4, q_5, q_{14}\} \subset S$. The proof goes analogously to case (a). Thus, we proved property (5).

In the following we prove that $\{q_x, q_y, q_z\} \subset S$ or $S \cap \{q_x, q_y, q_z\} = \emptyset$. It suffices to prove that if $q_x \in S$, then $q_y \in S$. Let us assume to the contrary that $q_x \in S$ and $q_y \notin S$. By (1) and (5), we have that $x_q \in S$, $S \cap \{q_1, q_{18}\} = \emptyset$ and $|S \cap \{q_2, q_3, q_{16}, q_{17}\}| = 2$. If $\{q_2, q_{17}\} \subset S$, then by (1) we have $SEC(\{q_2, q_{17}\}) = false$, implying that $|S \cap \{q_2, q_{17}\}| \leq 1$ and $|S \cap \{q_3, q_{16}\}| \geq 1$. If $q_3 \notin S$, then $\{q_{16}, q_{17}\} \subset S$, and so $q_4 \in S$. If $q_3 \in S$, then $q_{17} \notin S$, and so $q_4 \in S$. Analogously, we have $q_{15} \in S$. Thus, $\{q_4, q_{15}\} \subset S$. Since $q_y \notin S$, we have by (4) that $y_q \notin S$. Hence by (2), we have that $1 \leq |S \cap \{q_6, q_7\}| \leq 2$, and so $\{q_4, q_5, q_6\} \subset S$. Further, $S \cap \{q_8, q_9\} = \emptyset$, and so $\{q_{10}, q_{11}\} \subset S$. By (3), we have that $SEC(\{q_{10}, q_{11}\}) = false$, a contradiction.

By (4) and (5), we have that for each $v \in N_G(q)$, $v_q \in S \Leftrightarrow q_v \in S$. Thus, $\{q_x, q_y, q_z, x_q, y_q, z_q\} \subset S$ or $S \cap \{q_x, q_y, q_z, x_q, y_q, z_q\} = \emptyset$. □

For each $q \in Q$, let us define $N_G(q) = \{x(q), y(q), z(q)\}$, where $x(q) \in X, y(q) \in Y, z(q) \in Z$. Let

$$Q' = \{q \in Q : \{q_{x(q)}, q_{y(q)}, q_{z(q)}\} \subset S\}.$$

Let $v \in V$. By Claim 4.4 (i) and (ii) there is $q \in N_G(v)$ such that $\{v_q, q_v\} \subset S$. Hence by Claim 4.4 (iii), we have that $q \in Q'$. Thus, set Q' dominates V, and so $3|Q'| = \sum_{q \in Q'} |N_G(q)| \geq |N_G(Q')| = |V| = 3m$. By Claim 4.4, we have that for every $p, q \in Q'$ and $p \neq q$, $N_G(p) \cap N_G(q) = \emptyset$. Thus, $|Q'| = m$. This establishes Theorem 4.1. □

5. $O(n\Delta^2 \log \Delta)$-time algorithm for trees

In [6] the authors constructed $O(n \log \Delta)$-time algorithm for finding the minimum global alliance in trees. In this section we present $O(n\Delta^2 \log \Delta)$-time algorithm for finding a minimum global edge alliance in trees.

We construct the optimal solution for a given tree T using the bottom-up technique in accordance with a defined orientation of T. First, we orient all edges of T in an *in-tree* manner with a leaf as root, i.e., we choose any leaf r as root and orient all edges of tree T towards the root r. As a result, for each vertex $v \in V(T) \setminus \{r\}$, there is exactly one oriented edge outcoming from a vertex v towards r, let us denote this edge by $e_v = \{v, r_v\}$. By T_v we denote a subtree of T rooted at v and consisting of all (oriented) edges that lead to vertex v. By T_v^* we mean the tree T_v with an attached edge e_v, i.e., $T_v^* = T_v \cup e_v$. Let $p(v) = deg_T(v) - 1$ and let $N_v^b = \{v_1, \ldots, v_{p(v)}\}$ be the set of vertices adjacent to v and different from r_v.

The key idea of the approach is to use the recursive scheme, in which we build a data structure A_v, related to the vertex v, from data structures $A_{v_1}, \ldots, A_{v_{p(v)}}$ related to the children of vertex v (i.e., N_v^b). We will use some auxiliary data structure (B_v) to clarify the process of building A_v from $A_{v_1}, \ldots, A_{v_{p(v)}}$. It is important to ensure that one can apply the data structures associated with all children of vertex v to build A_v. The algorithm goes as follows:

1. Starting from leaves, first build A_v, and go towards root r.
2. Traversing tree T for each vertex $v \neq r$:
   (i) construct an *auxiliary* data structure B_v using $A_{v_1}, \ldots, A_{v_{p(v)}}$,
   (ii) construct A_v from B_v.
3. Use A_s, where s is the only neighbor of root r, to find an optimal solution.

The total time complexity of the algorithm depends on the time complexity of the construction of structures A_v and B_v. In fact, by this schema we calculate the size of the optimal solution. The construction of the optimal solution may be possible by using additional data structures for saving the appropriate information while building structures A_v and B_v, which, however, does not change the time complexity of the algorithm.

In the following, for the sake of notation simplicity, we shall use *gea* instead of *global edge alliance*, and *ea* instead of *edge alliance*. We use the symbol ∞ to denote illegal cases, and assume that $\infty \geq a$, $\infty \pm a = \infty$ and $\min\{\infty, a\} = a$, where a is a number or ∞.

Theorem 5.1. *There exists $O(n\Delta^2 \log \Delta)$ time algorithm finding the minimum global edge alliance for trees with at most n vertices and the maximum degree bounded by Δ.*

Proof. Let $v \in V(T) \setminus \{r\}$, $p = \deg_T(v) - 1$ and $q = \deg_T(r_v) - 1$. We define a tree T_v^l obtained from T_v^* by attaching $l \geq 0$ pendant vertices $L_l = \{u_1, \ldots, u_l\}$ to vertex r_v. Note that $T_v^0 = T_v^*$.

Let us define $A_v = (a_v^{00}, a_v^{01}, a_v^{10}, A_v^{11})$, where a_v^{jh} is an integer or ∞, for $(j, h) \in \{(0, 0), (0, 1), (1, 0)\}$, and A_v^{11} is a matrix of the size $(q + 1) \times 1$, all described as follows:

$$\begin{aligned} a_v^{00} &= \min\{|S \setminus \{r_v\}| : S \text{ is a } gea \text{ in } T_v \wedge v \notin S \wedge r_v \notin S\}, \\ a_v^{01} &= \min\{|S \setminus \{r_v\}| : S \setminus \{r_v\} \text{ is a } gea \text{ in } T_v \setminus \{v\} \text{ and an } ea \text{ in } T_v \wedge \\ &\qquad v \notin S \wedge r_v \in S\}, \\ a_v^{10} &= \min\{|S \setminus \{r_v\}| : S \text{ is a } gea \text{ in } T_v^* \wedge v \in S \wedge r_v \notin S\}. \end{aligned}$$

For each $k \in \{0, \ldots, q\}$, if $k < q/2$, then let

$$A_v^{11}[k] = \min\{|S \setminus \{r_v\}| : S \text{ is a } gea \text{ in } T_v^{q-2k} \wedge \{v, r_v\} \subset S \wedge L_{q-2k} \cap S = \emptyset\},$$

and if $k \geq q/2$, then let

$$A_v^{11}[k] = \min\{|S \setminus (L_{2k-q} \cup \{r_v\})| : S \text{ is a } gea \text{ in } T_v^{2k-q} \wedge \{v, r_v\} \subset S \wedge L_{2k-q} \subset S\}.$$

Let us observe that for a_v^{jh}, we have $j = 1$ iff $v \in S$, and $h = 1$ iff $r_v \in S$. If any $\min(\cdot)$ cannot be legally defined, we preset the value as ∞.

In the next steps we construct A_v in accordance with the given definitions.

Let v be a leaf. Then, by definition we initially put $a_v^{00} = \infty$, $a_v^{01} = 0$, $a_v^{10} = \infty$ and $A_v^{11}[k] = 1$ for $2k + 2 \geq q$, and $A_v^{11}[k] = \infty$ for $2k + 2 < q$.

Let v be a vertex that is not a leaf. Let us define $B_v = (B_v^0, B_v^1)$, where B_v^0 is a matrix of the size $p \times 3$ and B_v^1 is a matrix of the size $(p + 1) \times p \times 4$. Let us remind that $N_v^b = \{v_1, \ldots, v_p\}$.

For each $i \in \{1, \ldots, p\}$ we set $B_v^0[i, 0] = i$, and if $a_{v_i}^{10} > a_{v_i}^{00}$, then let $B_v^0[i, 1] = 1$, otherwise, let $B_v^0[i, 1] = 0$. Finally, $B_v^0[i, 2] = |a_{v_i}^{10} - a_{v_i}^{00}|$. Let us observe that if $a_{v_i}^{10} = \infty$ or $a_{v_i}^{00} = \infty$, then $B_v^0[i, 2] = \infty$. The matrix B_v^0 can be constructed in $O(p)$ time.

For each $k \in \{0, \ldots, p\}$ and $i \in \{1, \ldots, p\}$ we set $B_v^1[k, i, 0] = i$, and if $A_{v_i}^{11}[k] > a_{v_i}^{01}$, then let $B_v^1[k, i, 1] = 1$, otherwise, let $B_v^1[k, i, 1] = 0$. We put $B_v^1[k, i, 2] = \min\{A_{v_i}^{11}[k], a_{v_i}^{01}\}$ and finally, $B_v^1[k, i, 3] = |A_{v_i}^{11}[k] - a_{v_i}^{01}|$. The matrix B_v^1 can be constructed in $O(p^2)$ time.

With all the values B_v, a_v, b_v and c_v already calculated, now we can determine A_v. Let $a_v^0 = \sum_{i=1}^{p}(1 - B_v^0[i,1])$ and $b_v^0 = \sum_{i=1}^{p}\min\{a_{v_i}^{10}, a_{v_i}^{00}\}$. If for some $i \in \{1,\ldots,p\}$ we have $a_{v_i}^{10} = a_{v_i}^{00} = \infty$, then $b_v^0 = \infty$. Let $c_v^0 = \min\{B_v^0[i,2] : i \in \{1,\ldots,p\} \wedge B_v^0[i,1] = 1\}$. The values a_v^0, b_v^0 and c_v^0 can be calculated in $O(p)$ time.

Claim 5.1. *The value a_v^{00} can be determined in $O(p)$ time.*

Proof. We have to ensure that v is dominated by at least one v_i, where $i \in \{1,\ldots,p\}$. If $a_v^0 > 0$, then $a_v^{00} = b_v^0$, otherwise, $a_v^{00} = b_v^0 + c_v^0$. □

Claim 5.2. *The value a_v^{01} can be determined in $O(p)$ time.*

Proof. Since v is dominated by r_v, just take the best solution: $a_v^{01} = b_v^0$. □

Claim 5.3. *The value a_v^{10} can be determined in $O(p^2 \log p)$ time.*

Proof. We have to ensure that for each $v_i \in S$ the edge alliance property holds for an edge $\{v, v_i\}$. For every $k \in \{1,\ldots,p\}$, let us define

$$s_k = \min\{|S \setminus \{r_v\}| : S \text{ is a } gea \text{ in } T_v^* \wedge v \in S \wedge r_v \notin S \wedge |N_v^b \cap S| = k\},$$

or $s_k = \infty$, if there is no such S. Obviously, $a_v^{10} = \min\{s_1,\ldots,s_p\}$.

For $k \in \{1,\ldots,p\}$ we calculate s_k or prove that there is $l > k$, such that $s_l \le s_k$. Let $a = \sum_{i=1}^{p}(1 - B_v^1[k-1,i,1])$, and $b = \sum_{i=1}^{p} B_v^1[k-1,i,2]$. We have to ensure that exactly k edges $\{v, v_i\}$ satisfy the edge alliance condition for $i \in \{1,\ldots,p\}$. The rest of vertices v_i are outside the edge alliance.

If $a > k$, then it is easy to observe that for some $l \ge a$ we have $s_l \le s_k$. Thus, without loss of generality we can assume that $a \le k$.

If $a = k$, then we can put $s_k = b + 1$.

If $a < k$, then we do the following: let $\hat{B}_v$ be a matrix of the size $p \times 4$ obtained from $B_v^1[k-1]$ by sorting rows $B_v^1[k-1,i]$ (for $i \in \{1,\ldots,p\}$) in a non-decreasing order with respect to the value $B_v^1[k-1,i,3]$. Thus, we get $\hat{B}[1,3] \le \hat{B}[2,3] \le \cdots \le \hat{B}[p,3]$. The construction of $\hat{B}_v$ can be done in $O(p \log p)$ time. Let k_0 be the smallest integer such that $k - a = \sum_{i=1}^{k_0} \hat{B}_v[i,1]$, and let $c = \sum_{i=1}^{k_0} \hat{B}_v[i,3] \cdot \hat{B}_v[i,1]$. Hence, we put $s_k = b + c + 1$. Thus, we constructed a_v^{10} in $O(p^2 \log p)$ time. □

Claim 5.4. *The matrix A_v^{11} can be constructed in $O(qp^2 \log p)$ time.*

Proof.

Now, for any $l \in \{0,\ldots,q\}$ we construct $A_v^{11}[l]$ in time $O(p^2 \log p)$.

The main difference between the construction of a_v^{10} and A_v^{11} is that we have to ensure that edge $\{v, r_v\}$ satisfies the edge alliance property (i.e., $SEC(v, r_v) = true$). The proof goes analogously as for a_v^{10}. For every $k \in \{0,\ldots,p\}$ and $l \in \{0,\ldots,q\}$,

- if $l < q/2$, then let $s_{k,l} = \min\{|S \setminus \{r_v \cup L_{q-2l}\}| : S \text{ is a } gea \text{ in } T_v^{q-2l} \wedge v \in S \wedge r_v \in S \wedge |N_v^b \cap S| = k \wedge L_{q-2l} \cap S = \emptyset\}$,
- if $l \ge q/2$, then let $s_{k,l} = \min\{|S \setminus \{r_v \cup L_{2l-q}\}| : S \text{ is a } gea \text{ in } T_v^{2l-q} \wedge v \in S \wedge r_v \in S \wedge |N_v^b \cap S| = k \wedge L_{2l-q} \subset S\}$,
- and in both cases, if there is no such S, then we put $s_{k,l} = \infty$.

Analogously as in the construction of a_v^{10}, we have to ensure that exactly k edges $\{v, v_i\}$ satisfy the edge alliance property for $i \in \{1,\ldots,p\}$. Let us observe that $SEC(v, r_v) = true$ iff $2k + 2l + 2 \ge q + p$. Thus, if $2k + 2l + 2 \ge q + p$, then we calculate the value $s_{k,l}$ analogously as for a_v^{10}, otherwise, we put $s_{k,l} = \infty$. Finally, $A_v^{11}[l] = \min\{s_{1,l},\ldots,s_{p,l}\}$. The construction of matrix A_v^{11} can be done in $O(qp^2 \log p)$ time. □

Claim 5.5. *The equality $\gamma_{ea}(T) = \min\{a_s^{10}, A_s^{11}[0] + 1\}$ holds, where $\{s\} = N_T(r)$*

Proof. The root r needs to be dominated, so for any global edge alliance S it is true that $\{r, s\} \cap S \ne \emptyset$. Thus $a_s^{00} = \infty$. The vertex s is the only neighbor of the root r, so a global edge alliance S such that $r \in S$ and $s \notin S$ cannot exist. Thus $a_s^{01} = \infty$.

Since $q(s) = 0$, therefore the matrix A_s^{11} has only one element, i.e. $A_s^{11}[0]$. Thus $\gamma_{ea}(T) = \min\{a_s^{10}, A_s^{11}[0] + 1\}$. □

To sum up, the algorithm that gives the size of the minimum global edge alliance goes as follows:

(1) For every leaf l in tree build A_v using the following values:

(i) $a_l^{00} = \infty$, $a_l^{01} = 0$, $a_l^{10} = \infty$,
(ii) $A_l^{11}[k] = 1$ for each k such that $2k + 2 \ge q$,
(iii) $A_l^{11}[k] = \infty$ for each k such that $2k + 2 < q$.

(2) Traversing tree T towards root for each vertex $v \ne r$ and v is not a leaf:

(i) construct an *auxiliary* data structure B_v using $A_{v_1}, \ldots, A_{v_{p(v)}}$,
(ii) applying Claims 5.1–5.4 construct A_v from B_v:

(3) By Claim 5.5 the value of an optimal solution is $\min\{a_s^{10}, A_s^{11}[0] + 1\}$, where s is the only neighbor of root r.

By Claims 5.1–5.4 we can deduce that the construction of data structure A_v can be done in $O(qp^2 \log p)$. Therefore, the time complexity of the algorithm is $O(n\Delta^2 \log \Delta)$.

As mentioned before, the construction of an optimal solution may be possible in the same time complexity by using additional data structures for saving the appropriate information while building structures A_v and B_v. □

6. Future work and open questions

Recently, in the paper [19], the authors proved the upper bound on the edge alliance number for trees, i.e., $\gamma_{ea}(T) \leq 2n/3$, and characterized the class of trees reaching this upper bound.

In the papers [7] and [8] the authors proved the upper bound for trees on the minimum total domination number, and the minimum global alliance problem, respectively. Precisely, let $s(T)$ be the number of support vertices in a tree T. If T is a tree of order $n(T) \geq 3$, then $\gamma_t(T) \leq \frac{n(T)+s(T)}{2}$ [7] and $\gamma_a(T) \leq \frac{n(T)+s(T)}{2}$ [8]. In the paper [19] the authors proved that if T is a tree of order $n(T) \geq 2$, then $\gamma_{ea}(T) \leq \frac{n(T)+s(T)}{2}$.

The challenging problem is to give the complexity of the problem of finding the minimum global edge alliance in the class of cubic (bipartite) graphs.

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Further Reading

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