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Note

Greedy T-colorings of graphs

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Abstract

This paper deals with greedy T-colorings of graphs, i.e. T-colorings produced by the greedy (or first-fit) algorithm. We study their parameters, such as the number of colors, the span, the edge span and the values of colors they use. In particular, we show that these T-colorings have three nice properties: (1) their span and edge span are equal; (2) the number of colors they use is independent of T; (3) the set of colors they use is a function of T and the number of colors used, only. As a result of these considerations we receive some necessary and sufficient conditions for a greedy T-coloring to be optimal. The paper ends with some considerations concerning greedy algorithms with color interchange.

Keywords: T-coloring; T-span; Greedy algorithm

1. Introduction

T-coloring of graphs is one of many graph-theoretic problems which arose as mathematical models for a practical problem. Nowadays, there are other potential applications for T-coloring known, but its original one, the channel assignment problem, remains the main application. The channel assignment problem can be formulated as follows. There are several transmitters situated in a region of a plane. Assign channels (operational frequencies) to them in such a way that: (1) there is no interference during transmitting; (2) the span of the used frequency band, i.e. the distance between the smallest and the largest channel used, is minimal. For details of the problem we refer the reader to [1,5].

In this paper we consider a slightly modified definition of a T-coloring, which we call *restricted*. Let $G = (V, E)$ be a simple graph[1] with vertex set V and edge set E, let C be an infinite set of positive integers and T be a T-set, i.e. a finite set of nonnegative integers such that $0 \in T$. A function $c: V \to C$ is a *C-restricted T-coloring* of G if and only if $|c(u) - c(v)| \notin T$ whenever u is a neighbour of v in G, i.e. $\{u, v\} \in E$. The C-restricted T-coloring problem can be formulated as follows: given a simple graph G and a T-set T, find a C-restricted T-coloring of G with minimal span, where the span of a coloring is the difference between the largest and the smallest color used. If $C = \mathbb{N}$, then the above definitions turn into the original definition of a T-coloring and the T-coloring problem. The minimal span

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[1] A graph without loops and multiple edges.

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over all T-colorings of G has its own name and symbol in the literature. Namely, it is called the T-*span* of G and is denoted by $\mathrm{sp}_T(G)$.

The T-coloring problem has been studied for over two decades. A number of facts concerning its computational complexity have been discovered (see [8] for a review of other facts related to the T-coloring problem). For example, there are known several graph classes for which the problem becomes polynomial (bipartite, polygon trees, outerplanar, cacti and thorny graphs [4]) and other classes, for which it is NP-hard in the strong sense (complete [6], r-regular (for any $r \geq 3$) and subcubic graphs [3]). Moreover, Giaro [2] showed that for any $\varepsilon > 0$ there are no polynomial $(1 + \varepsilon)$-ratio approximation algorithms for the problem, unless $P = NP$. Summarizing, the problem is computationally hard and if we need to solve it in practice, we have to apply heuristic algorithms which are fast but do not ensure that their outcome is optimal. The greedy (or first-fit) algorithm is one of these heuristics.

Let us recall how the greedy algorithm T-colors graph G. Let $v_1, v_2, \ldots, v_n$ be any vertex ordering of G. Color v_1 with $\min C$, i.e. set $c(v_1) = \min C$. Having assigned colors $c(v_1), c(v_2), \ldots, c(v_{k-1})$, let $c(v_k)$ be the smallest possible integer that belongs to C and does not violate the definition of a T-coloring, i.e.

$$c(v_k) = \min\{l \in C\colon \forall_{i<k} \{v_i, v_k\} \in E \Rightarrow |l - c(v_i)| \notin T\}.$$

For instance, if $T = \{0, 2, 4, 6\}$, $C = \{n \in \mathbb{Z}\colon n > 2\}$ and G is a 3-vertex complete graph, then the greedy algorithm produces the $\{0, 2, 4, 6\}$-coloring c given by $c(v_1) = 3$, $c(v_2) = 4$ and $c(v_3) = 11$.

The behavior of the greedy algorithm in the case $C = \mathbb{N}$ has been studied by many authors. Liu [7] introduced two collections of sets related to the greedy algorithm. The first one, denoted by $\mathcal{G}$, is the collection of all T-sets T such that for all complete graphs G the greedy algorithm produces T-colorings whose span is minimal, i.e. equals $\mathrm{sp}_T(G)$. The second one, denoted by $\mathcal{E}$, is the collection of all T-sets T satisfying $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_{\chi(G)})$ for all graphs G, where χ is the chromatic number. It may seem that $\mathcal{E}$ is not related to the greedy algorithm but we will show that $\mathcal{E} \cap \mathcal{G}$ is the collection of all T-sets T such that for every graph G there exists a vertex ordering for which the greedy algorithm produces a T-coloring with minimal span.

The paper is devoted to the greedy T-colorings of graphs, i.e. T-colorings produced by the greedy algorithm. We study their properties in the remainder of the paper which consists of three sections. Section 2 presents some preliminaries related to greedy $\{0\}$-colorings and greedy T-colorings of complete graphs. Section 3 contains our main results. In particular, we show that the span and the edge span (to be defined later) of a greedy T-coloring are equal. We present lower and upper bounds for the span of greedy T-colorings and show that $\mathcal{E} \cap \mathcal{G}$ has the property that was mentioned earlier. The last section contains some considerations concerning greedy algorithms with color interchange.

2. Preliminaries

It will be seen later that the behavior of the greedy algorithm is determined by its behavior in two special cases: on complete graphs and in the case $T = \{0\}$. Therefore we begin our considerations with the study of these cases. Notations introduced in Sections 2.1 and 2.2 will be used in the remainder of the paper.

2.1. Complete graphs

Let T be a T-set, C be an infinite subset of $\mathbb{N}$, n be a positive integer and K_n be the only complete n-vertex graph with vertex set $\{1, 2, \ldots, n\}$. We define function $\mathrm{gc}_T(C, \cdot)\colon \mathbb{N} \to C$ by the following recursive equality:

$$\mathrm{gc}_T(C, k) = \min\{l \in C\colon \forall_{i<k} |l - \mathrm{gc}_T(C, i)| \notin T\}. \tag{1}$$

Since C is infinite and bounded from below, $\mathrm{gc}_T(C, \cdot)$ is well-defined. It is easy to see that $\mathrm{gc}_T(C, k)$ $(k \leq n)$ is the color assigned to vertex k of K_n by the greedy algorithm provided that vertex k is the kth colored. Furthermore, the function $\mathrm{gc}_T(C, \cdot)$ is increasing and $\mathrm{gc}_T(C, 1) = \min C$.

Lemma 1. *The equality*

$$\mathrm{gc}_T(C, k) = \min\{l \in C\colon \forall_{i<k} l - \mathrm{gc}_T(C, i) \notin T\} \tag{2}$$

holds for each k.

Proof. Let $g(k) = \min\{l \in C: \forall_{i<k} l - g(i) \notin T\}$ and suppose that $g(k) \neq \mathrm{gc}_T(C, k)$. Without loss of generality we may assume that $g(i) = \mathrm{gc}_T(C, i)$ for every $i < k$. Obviously $k > 1$. There are only two cases to consider.

(1) $g(k) < \mathrm{gc}_T(C, k)$. Then there is $j < k$ such that $|g(k) - \mathrm{gc}_T(C, j)| \in T$. Since $\mathrm{gc}_T(C, j) = g(j)$ and $g(k) - g(j) \notin T$, we have $|g(k) - g(j)| \neq g(k) - g(j)$. Therefore $g(k) < g(j)$, which contradicts the fact that g is an increasing function.
(2) $g(k) > \mathrm{gc}_T(C, k)$. Then there is $j < k$ such that $\mathrm{gc}_T(C, k) - g(j) \in T$. Since $g(j) = \mathrm{gc}_T(C, j)$, we have $\mathrm{gc}_T(C, k) - \mathrm{gc}_T(C, j) \in T$, which is impossible due to the definition of $\mathrm{gc}_T(C, \cdot)$ and the fact that T contains nonnegative integers only. □

Since the greedy algorithm always chooses the smallest possible element of C as a color for a vertex being colored, it should not be a surprise that these colors cannot grow too fast. To make this observation more precise, we need to introduce another notation. Let $r_C: C \to \mathbb{Z}$ be a function defined by $r_C(k) = |\{l \in C:\ l < k\}|$. Then

Lemma 2. *The inequality*

$$r_C(\mathrm{gc}_T(C, k)) \leq |T| \cdot (k - 1) \tag{3}$$

holds for each k.

Proof. Let $A_i = \{l \in C: l - \mathrm{gc}_T(C, i) \in T\}$ for $i < k$. By (2), for every $l \in C$ less than $\mathrm{gc}_T(C, k)$ we have $l \in A_i$ for some $i < k$. Hence

$$r_C(\mathrm{gc}_T(C, k)) = |\{l \in C:\ l < \mathrm{gc}_T(C, k)\}| \leq |\bigcup_{i=1}^{k-1} A_i| \leq \sum_{i=1}^{k-1} |A_i| \leq |T| \cdot (k - 1),$$

which is the desired conclusion. □

2.2. $T = \{0\}$

Let $G = (V, E)$ be any fixed n-vertex graph and $v_1, v_2, \ldots, v_n$ be one of its vertex orderings. We define another function $\mathrm{gr}_T(C, \cdot): V \to C$ by another recursive equality

$$\mathrm{gr}_T(C, v_k) = \min\{l \in C: \forall_{i<k}(\{v_i, v_k\} \in E \Rightarrow |l - \mathrm{gr}_T(C, v_i)| \notin T)\}. \tag{4}$$

It is easy to see that $\mathrm{gr}_T(C, v_k)$ is the color assigned to vertex v_k of G by the greedy algorithm provided that vertex v_k is the kth colored. Obviously, if G is a complete graph and $v_k = k$, then $\mathrm{gr}_T(C, v_k) = \mathrm{gc}_T(C, k)$. One would ask why to introduce two different symbols gc_T and gr_T for a notion that could be denoted by a single one. The reason is that the introduction of two symbols instead of one makes further formulas simpler.

Lemma 3. *Let $f: \mathbb{N} \to \mathbb{N}$ be an increasing function. Then the equality*

$$f(\mathrm{gr}_{\{0\}}(C, v_k)) = \mathrm{gr}_{\{0\}}(f(C), v_k) \tag{5}$$

holds for each $k \leq n$, where $f(C) = \{f(l):\ l \in C\}$.

Proof. Suppose that $f(\mathrm{gr}_{\{0\}}(C, v_k)) \neq \mathrm{gr}_{\{0\}}(f(C), v_k)$. Without loss of generality we may assume that $f(\mathrm{gr}_{\{0\}}(C, v_i)) = \mathrm{gr}_{\{0\}}(f(C), v_i)$ for every $i < k$. Obviously $k > 1$. There are only two cases to consider.

(1) $f(\mathrm{gr}_{\{0\}}(C, v_k)) < \mathrm{gr}_{\{0\}}(f(C), v_k)$. Then there is $i < k$ such that $\{v_i, v_k\} \in E$ and $\mathrm{gr}_{\{0\}}(f(C), v_i) = f(\mathrm{gr}_{\{0\}}(C, v_k))$. Since $\mathrm{gr}_{\{0\}}(f(C), v_i) = f(\mathrm{gr}_{\{0\}}(C, v_i))$ and f is increasing, we have $\mathrm{gr}_{\{0\}}(C, v_i) = \mathrm{gr}_{\{0\}}(C, v_k)$, which is impossible due to the fact that $\mathrm{gr}_{\{0\}}(C, \cdot)$ is a legal coloring of G.
(2) $f(\mathrm{gr}_{\{0\}}(C, v_k)) > \mathrm{gr}_{\{0\}}(f(C), v_k)$. Let $x \in C$ be a positive integer such that $f(x) = \mathrm{gr}_{\{0\}}(f(C), v_k)$. Then $x < \mathrm{gr}_{\{0\}}(C, v_k)$ and, by definition of $\mathrm{gr}_{\{0\}}(C, \cdot)$, there is $i < k$ such that $\{v_i, v_k\} \in E$ and $\mathrm{gr}_{\{0\}}(C, v_i) = x$. Hence $\mathrm{gr}_{\{0\}}(f(C), v_k) = f(x) = f(\mathrm{gr}_{\{0\}}(C, v_i)) = \mathrm{gr}_{\{0\}}(f(C), v_i)$, a contradiction. □

Eq. (5) may not hold if we replace $\{0\}$ by other T-sets. Moreover, it may become false even for complete graphs. For example, if $T = \{0, 1\}$ and $f(k) = 2k$, then $f(\mathrm{gc}_T(\mathbb{N}, k)) = f(2k - 1) = 4k - 2$ and $\mathrm{gc}_T(f(\mathbb{N}), k) = 2k$.

3. Main results

Now we are in a position to formulate and prove our main result. Before we do this, let us note that $\mathrm{gc}_T(C,\cdot)$ has the following properties:

(1) If $i \neq j$ then $|\mathrm{gc}_T(C,i) - \mathrm{gc}_T(C,j)| \notin T$.
(2) If $x < \mathrm{gc}_T(C,k)$ and $x \in C$ then there is $i < k$ such that $|x - \mathrm{gc}_T(C,i)| \in T$.

Let us also note that if $x < \mathrm{gr}_T(C,v_k)$ and $x \in C$ then there is $i < k$ such that $\{v_i, v_k\} \in E$ and $|x - \mathrm{gr}_T(C,v_i)| \in T$.

Theorem 4. *For every $k \leq n$ we have*

$$\mathrm{gr}_T(C,v_k) = \mathrm{gc}_T(C, \mathrm{gr}_{\{0\}}(\mathbb{N}, v_k)). \tag{6}$$

Proof. Suppose that $\mathrm{gr}_T(C,v_k) \neq \mathrm{gc}_T(C,\mathrm{gr}_{\{0\}}(\mathbb{N},v_k))$. Without loss of generality we may assume that $\mathrm{gr}_T(C,v_i) = \mathrm{gc}_T(C,\mathrm{gr}_{\{0\}}(\mathbb{N},v_i))$ for every $i < k$. Since $\mathrm{gr}_T(C,v_1) = \min C$ and $\mathrm{gc}_T(C,1) = \min C$, we have $k > 1$. There are only two cases to consider.

(1) $\mathrm{gr}_T(C,v_k) < \mathrm{gc}_T(C,\mathrm{gr}_{\{0\}}(\mathbb{N},v_k))$. Then there is $i < \mathrm{gr}_{\{0\}}(\mathbb{N},v_k)$ such that $|\mathrm{gr}_T(C,v_k) - \mathrm{gc}_T(C,i)| \in T$. By definition of $\mathrm{gr}_{\{0\}}(\mathbb{N},\cdot)$, there is $j < k$ such that $\{v_j,v_k\} \in E$ and $\mathrm{gc}_{\{0\}}(\mathbb{N},v_j) = i$. Hence $|\mathrm{gr}_T(C,v_k) - \mathrm{gr}_T(C,v_j)| = |\mathrm{gr}_T(C,v_k) - \mathrm{gc}_T(C,\mathrm{gr}_{\{0\}}(\mathbb{N},v_j))| = |\mathrm{gr}_T(C,v_k) - \mathrm{gc}_T(C,i)| \in T$, a contradiction.
(2) $\mathrm{gr}_T(C,v_k) > \mathrm{gc}_T(C,\mathrm{gr}_{\{0\}}(\mathbb{N},v_k))$. Then there is $i < k$ such that $\{v_i,v_k\} \in E$ and $|\mathrm{gc}_T(C,\mathrm{gr}_{\{0\}}(\mathbb{N},v_k)) - \mathrm{gr}_T(C,v_i)| \in T$. We know that $\mathrm{gr}_T(C,v_i) = \mathrm{gc}_T(C,\mathrm{gr}_{\{0\}}(\mathbb{N},v_i))$. Therefore $|\mathrm{gc}_T(C,\mathrm{gr}_{\{0\}}(\mathbb{N},v_k)) - \mathrm{gc}_T(C,\mathrm{gr}_{\{0\}}(\mathbb{N},v_i))| \in T$, which is impossible due to the fact that $\mathrm{gr}_{\{0\}}(\mathbb{N},v_i) \neq \mathrm{gr}_{\{0\}}(\mathbb{N},v_k)$. □

According to Lemma 3 and the fact that $\mathrm{gc}_T(C,\cdot)$ is increasing, Eq. (6) may be written in the following, equivalent form:

$$\mathrm{gr}_T(C,v_k) = \mathrm{gr}_{\{0\}}(\{\mathrm{gc}_T(C,i)\colon i \in \mathbb{N}\}, v_k). \tag{7}$$

This equation explicitly states that the only difference between the behavior of the greedy algorithm in a general case and in case $T = \{0\}$ is caused by the set from which colors are chosen. This has several interesting consequences. Two of them are obvious:

(1) The number of colors that our algorithm uses depends only on G and $v_1, v_2, \ldots, v_n$. In particular, it is independent of T.
(2) The set of colors that our algorithm uses depends only on T and the number of colors used.

To formulate other consequences, we must introduce two other notions. Let us recall that if c is a T-coloring of G then the numbers $\mathrm{sp}(c) = \max c(V) - \min c(V)$ and $\mathrm{esp}(c) = \max\{|c(u) - c(v)|\colon \{u,v\} \in E\}$ are its *span* and *edge span*, respectively.

Corollary 5. *If $a, b \in \{\mathrm{gr}_T(C,v_k)\colon k \leq n\}$ are different numbers then there are adjacent vertices $v_i, v_j \in V$ such that $a = \mathrm{gr}_T(C,v_i)$ and $b = \mathrm{gr}_T(C,v_j)$.*

Proof. Without loss of generality we can assume that $a > b$. Since a and b are elements of $\{\mathrm{gr}_T(C,v_k)\colon k \leq n\}$, there are vertices $v_i, v_l \in V$ such that $a = \mathrm{gr}_T(C,v_i)$ and $b = \mathrm{gr}_T(C,v_l)$. Let $x = \mathrm{gr}_{\{0\}}(\mathbb{N},v_i)$ and $y = \mathrm{gr}_{\{0\}}(\mathbb{N},v_l)$. By (6), $b = \mathrm{gc}_T(C,y)$ and $a = \mathrm{gc}_T(C,x)$, which yield $x > y$. Therefore there is a vertex v_j which is a neighbour of v_i such that $y = \mathrm{gr}_{\{0\}}(\mathbb{N},v_j)$. Using (6) again, we obtain $\mathrm{gr}_T(C,v_j) = \mathrm{gc}_T(C,y) = b$, which is our claim. □

In general, the difference between the span and edge span of a coloring can be arbitrarily large. The above corollary implies that the span and edge span of a greedy T-coloring have to be equal. Moreover, Theorem 4 says that

$$\mathrm{sp}(\mathrm{gr}_T(C,\cdot)) = \mathrm{sp}(\mathrm{gc}_T(C,\cdot)|_{\{1,2,\ldots,s(C)\}}),$$

where $s(C) = |\{\mathrm{gr}_{\{0\}}(C,v_k)\colon k \leq n\}|$. Since $\chi \leq s(C)$ and $\mathrm{sp}_T(K_{s(C)}) \leq \mathrm{sp}(\mathrm{gc}_T(C,\cdot)|_{\{1,2,\ldots,s(C)\}})$, we obtain the following lower bound

$$\mathrm{sp}(\mathrm{gr}_T(C,\cdot)) \geq \mathrm{sp}_T(K_\chi). \tag{8}$$

The above bound is tight—if we set $C = c(\{1, 2, \ldots, \chi\}) \cup \{k : k > \mathrm{sp}_T(K_\chi)\}$ (where c is any T-coloring of K_χ satisfying $\mathrm{sp}(c) = \mathrm{sp}_T(K_\chi)$) and choose $v_1, v_2, \ldots, v_n$ in such a way that $s(C) = \chi$, then $\mathrm{gc}_T(C, k) = c(k)$ for every $k \leq \chi$ and hence $\mathrm{sp}(\mathrm{gr}_T(C, \cdot)) = \mathrm{sp}_T(K_\chi)$.

Corollary 6. *For every $k \leq n$ we have*

$$r_C(\mathrm{gr}_T(C, v_k)) \leq |T| \cdot \deg(v_k), \tag{9}$$

where $\deg(v_k)$ denotes the degree of vertex v_k.

Proof. This is an easy consequence of Theorem 4 and Lemma 2. Indeed, combining equality (6) with inequality (3), we obtain the inequality

$$r_C(\mathrm{gr}_T(C, v_k)) = r_C(\mathrm{gc}_T(C, \mathrm{gr}_{\{0\}}(\mathbb{N}, v_k))) \leq |T| \cdot (\mathrm{gr}_{\{0\}}(\mathbb{N}, v_k) - 1).$$

To complete the proof, it suffices to note that $\mathrm{gr}_{\{0\}}(\mathbb{N}, v_k)$ is at most $\deg(v_k) + 1$. □

Inequality (9) provides an upper bound for the span of a greedy T-coloring. Indeed, since $\min\{\mathrm{gr}_T(C, v_k) : k \leq n\} = \min C$, we have

$$r_C(\mathrm{sp}(\mathrm{gr}_T(C, \cdot)) + \min C) \leq |T| \cdot \Delta, \tag{10}$$

where $\Delta = \max_{k \leq n} \deg(v_k)$. In particular, if $C = \mathbb{N}$ then the above inequality is as follows: $\mathrm{sp}(\mathrm{gr}_T(\mathbb{N}, \cdot)) \leq |T| \cdot \Delta$.

Inequality (9) also allows us to estimate the computational complexity of the greedy algorithm. It says that the algorithm needs to verify at most $|T| \cdot \deg(v_k)$ elements of C during the process of searching for the color of v_k. The cost of determining whether a given integer can be a color of v_k is $O(\log |T| \cdot \deg(v_k))$ provided that T is stored in a sorted table. Therefore the greedy algorithm runs in polynomial time and its computational complexity is $O(n^2 |T| \log |T|)$.

Theorem 4 can be also used to determine what conditions should be fulfilled to make $\mathrm{gc}_T(C, \cdot)$ optimal, i.e. $\mathrm{sp}(\mathrm{gr}_T(C, \cdot)) = \mathrm{sp}_T G$. To this end, let us recall that it is known [8] that $\mathrm{sp}_T(G) \leq \mathrm{sp}_T(K_\chi)$. Combining this with (8) we obtain the three necessary conditions:

(1) $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_\chi)$;
(2) $s(C) = \chi$;
(3) $\mathrm{sp}(\mathrm{gc}_T(C, \cdot)|_{\{1,2,\ldots,s(C)\}}) = \mathrm{sp}_T(K_\chi)$.

On the other hand, (1) and (3) guarantee that for every vertex ordering of G using exactly χ colors, $\mathrm{gc}_T(C, \cdot)$ will be optimal. Thus, the above conditions are also sufficient.

Furthermore, this allows us to prove that $\mathcal{E} \cap \mathcal{G}$ is the collection of all T-sets T such that for every graph G there is a vertex ordering for which the greedy algorithm produces a T-coloring with minimal possible span. The fact that every such T-set must belong to $\mathcal{E} \cap \mathcal{G}$ follows immediately from the above three conditions and the definition of $\mathcal{E}$ and $\mathcal{G}$. To prove the opposite, suppose that $T \in \mathcal{E}$ and $T \in \mathcal{G}$. Let c be a T-coloring of K_χ such that $\mathrm{sp}(c) = \mathrm{sp}_T(K_\chi)$, $C = c(\{1, 2, \ldots, \chi\}) \cup \{k : k > \mathrm{sp}_T(K_\chi)\}$, and $v_1, v_2, \ldots, v_n$ be such an ordering of vertices of G that $s(C) = \chi$. Then $\mathrm{sp}(\mathrm{gc}_T(C, \cdot)) = \mathrm{sp}(\mathrm{gc}_T(C, \cdot)|_{\{1,2,\ldots,\chi\}}) = \mathrm{sp}_T(K_\chi) = \mathrm{sp}_T(G)$, which is the desired conclusion.

4. Color interchange

Color interchange is one of the methods that was invented to improve the quality of colorings produced by various coloring algorithms. It is used to decrease the number of colors used and it is applied in a moment when a coloring algorithm tries to use a new color. If the method succeeds, one of the previously used colors is freed and it can be used to color the vertex being colored. Details of the method are not important here. All we need to know is that it takes two colors a, b and subgraph H induced by vertices colored with these colors, and interchanges colors in some connected components of H (vertices colored with a receive color b and vice versa).

First of all, let us note that color interchange can be applied in the greedy algorithm. Colors used by it are always a subset of $\{\mathrm{gc}_T(C, i) : i \in \mathbb{N}\}$, so interchange does not lead to the violation of the conditions that T-coloring must satisfy. Thus it is reasonable to study the greedy algorithm with color interchange. Using methods similar to that of the previous section, one can show that the algorithm has the following properties:

(1) the set of colors it uses is a subset of $\{\mathrm{gc}_T(C, i)\colon\ i \in \mathbb{N}\}$ (more precisely, it has a form $\{\mathrm{gc}_T(C, i)\colon\ i \leq k\}$ for some k);
(2) the span of a coloring produced by it must be greater than or equal to $\mathrm{sp}_T(K_\chi)$;
(3) if $C = \mathbb{N}$ then the span of a coloring produced by it must be less than or equal to $|T| \cdot \Delta$.

The first property can be easily proved by induction on the number of vertices; (2) and (3) are immediate consequences of (1) and inequality (10). We leave the details of the proofs to the reader.

5. Summary

The most important of the results of this paper can be formulated informally as follows: T-colorings produced by the greedy algorithm (with or without color interchange) are rarely optimal. For sets T not belonging to $\mathcal{E} \cap \mathcal{G}$ there are graphs G for which they cannot be optimal; for other T-sets the problem of finding the proper set C and sequence $v_1, v_2, \ldots, v_n$ is NP-hard.

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