Hat problem on a graph

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Abstract

The topic of our paper is the hat problem. In that problem, each of n people is randomly fitted with a blue or red hat. Then everybody can try to guess simultaneously his own hat color looking at the hat colors of the other people. The team wins if at least one person guesses his hat color correctly and no one guesses his hat color wrong, otherwise the team loses. The aim is to maximize the probability of win. In this version every person can see everybody excluding him. In this paper we consider such problem on a graph, where vertices are people and a person can see these people, to which he is connected by an edge. We prove some general theorems about the hat problem on a graph and solve the problem on trees. We also consider the hat problem on a graph with given degrees of vertices. We give an upper bound that is based only on the degrees of vertices on the chance of success of any strategy for the graph G. We show that this upper bound together with integrality constraints is tight on some toy examples.

Keywords: hat problem, graph, path, tree, vertex degree.
$\mathcal{AMS}$ **Subject Classification:** 05C05, 05C38, 05C57, 91A43.

1 Introduction

In the hat problem, a team of n people enters a room and a blue or red hat is randomly placed on the head of each person. Each person can see the hats of all of the other people but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each person must simultaneously guess the color of his own hat or pass. The team wins if at least one person guesses his hat color correctly and no one guesses his hat color wrong, otherwise the team loses. The aim is to maximize the probability of win.

The hat problem with seven people called ”seven prisoners puzzle” was formulated by T. Ebert in his Ph.D. Thesis [10]. The hat problem was also the subject of articles in The New York Times [20], Die Zeit [5], and abcNews [19]. The hat problem with n people and two colors of hat was investigated in [6]. It was solved for $2^k - 1$ people in [12]. The hat problem and Hamming codes were also the subject of an article in Polish math−physics−informatic magazine [9].

There are also known some variants and generalizations of hat problem. The authors of [18] investigate the generalized hat problem with $q \geq 2$ colors, they also consider variants in which there are arbitrary input distributions, randomized playing strategies, and symmetric strategies. In the papers [1], [8], and [17] there is considered another variant of hat problem in which passing is not allowed, thus everybody has to try to guess his hat color. The aim is to maximize the number of correct guesses. In [14] the authors investigate several variants of hat problem in which the aim is to design a strategy such that the number of correct guesses is greater than or equal to the given positive integer. In the paper [15] there is considered the hat problem, and also a variant in which the probabilities of getting hats of each colors do not have to be equal. The authors of [2] investigate a problem similar to the hat problem. There are n people which have random bits on foreheads, and they have to vote on the parity of the n bits.

The hat problem and its variants have many applications and connections to other areas of science, for example: information technology [4], linear programming [14, 16], genetic programming [7], economy [1, 17], biology [15], approximating Boolean functions [2], and autoreducibility of random sequences [3,10–13]. Therefore, it is hoped that the hat problem on a graph considered in this paper, as a natural generalization, is worth exploring, and may also have many applications.

We consider the hat problem on a graph, where vertices are people and a person can see these people, to which he is connected by an edge. We prove some general theorems about the hat problem on a graph and solve the problem on trees. We also consider the hat problem on a graph with given degrees of vertices. We give an upper bound that is based only on the degrees of vertices on the chance of success of any strategy for the graph G. We show that this upper bound together with integrality constraints is tight on some toy examples.

The paper is organized as follows. In Section 2 we give the notation and terminology used. In Section 3 first we make some general observations about the hat problem on a graph. In Theorem 4 we solve that problem on paths, and in Theorem 5 we solve the hat problem on trees. Then we consider the hat problem on a graph with given degrees of vertices.

2 Preliminaries

For a graph G, by $V(G)$ and $E(G)$ we denote the set of vertices and the set of edges of this graph, respectively. If H is a subgraph of G, then we write $H \subseteq G$. Let $v \in V(G)$. By $N_G(v)$ we denote the open neighbourhood of v, that is $N_G(v) = \{x \in V(G)\colon vx \in E(G)\}$. By $N_G[v]$ we denote the closed neighbourhood of v, that is $N_G[v] = N_G(v) \cup \{v\}$. By $d_G(v)$ we denote the degree of the vertex v, that is the number of its neighbours, thus $d_G(v) = |N_G(v)|$. By P_n we denote the path with n vertices. By C_n we denote the cycle with n vertices. By K_n we denote the complete graph with n vertices. Let $f\colon X \to Y$ be a function. If $Z \subseteq X$, then by $f_{|Z}$ we denote the restriction of f to Z. If $y \in Y$, then by $f \equiv y$ we denote that for every $x \in X$ we have $f(x) = y$.

Without loss of generality we may assume an ordering of the vertices of a graph G, that is $V(G) = \{v_1, v_2, \ldots, v_n\}$.

Let $\{b, r\}$ be the set of colors (b means blue and r means red). If $v_i \in V(G)$, then $c(v_i)$ is the color of v_i, so $c\colon V(G) \to \{b, r\}$ is a function. By a case for the graph G we mean a sequence $(c(v_1), c(v_2), \ldots, c(v_n))$. The set of all cases for the graph G we denote by $C(G)$, of course $|C(G)| = 2^{|V(G)|}$.

If $v_i \in V(G)$, then by s_i we denote a function $s_i\colon V(G) \to \{b, r, *\}$, where $s_i(v_j) \in \{b, r\}$ is the color of v_j if v_i sees v_j, and mark $*$ otherwise, that is, $s_i(v_j) = c(v_j)$ if $v_j \in N_G(v_i)$, while $s_i(v_j) = *$ if $v_j \in V(G) \setminus N_G(v_i)$. By a situation of the vertex v_i in the graph G we mean the sequence $(s_i(v_1), s_i(v_2), \ldots, s_i(v_n))$. The set of all possible situations of v_i in the graph G we denote by $St_i(G)$. Of course, $|St_i(G)| = 2^{|N_G(v_i)|}$.

Let $v_i \in V(G)$. We say that a case $(c_1, c_2, \ldots, c_n)$ for the graph G corresponds to a situation $(t_1, t_2, \ldots, t_n)$ of the vertex v_i in the graph G if it is created from this situation only by changing every mark $*$ to b or r. So, a case corresponds to a situation of v_i if every vertex adjacent to v_i, in that case has the same color as in that situation. To every situation of the vertex v_i in the graph G correspond $2^{|V(G)|-|N_G(v_i)|}$ cases, because every situation of v_i has $|V(G)|-|N_G(v_i)|$ marks $*$.

Let G and H be graphs such that $V(H)=\{v_1,v_2,\dots,v_m\}$, $V(G)=\{v_1,v_2,\dots,v_m,\dots,v_n\}$, and $E(H)\subseteq E(G)$. We say that a case $(a_1,a_2,\dots,a_m,\dots,a_n)$ for the graph G corresponds to a case $(b_1,b_2,\dots,b_m)$ for the graph H if $(a_1,a_2,\dots,a_m) = (b_1,b_2,\dots,b_m)$, that is, every vertex from the graph H in both cases has the same color. Of course, to every case for the graph H correspond 2^{n-m} cases for the graph G.

Let G and H be graphs such that $V(H)=\{v_1,v_2,\dots,v_m\}$, $V(G)=\{v_1,v_2,\dots,v_m,\dots,v_n\}$, and $E(H)\subseteq E(G)$. Let $i\in\{1,2,\dots,m\}$. We say that a situation $(t_1,t_2,\dots,t_m,\dots,t_n)$ of the vertex v_i in the graph G corresponds to a situation $(u_1,u_2,\dots,u_m)$ of the vertex v_i in the graph H if $(t_1,t_2,\dots,t_m)=(u_1,u_2,\dots,u_m)$, that is, every vertex adjacent to v_i in the graph H, in both of these situations has the same color.

By a statement of a vertex we mean its declaration about the color it guesses it is. By the result of a case we mean a win or a loss. According to the definition of the hat problem, the result of a case is a win if at least one vertex states its color correctly and no vertex states its color wrong. The result of a case is a loss if no vertex states its color or some vertex states its color wrong.

By a guessing instruction for the vertex $v_i\in V(G)$ (denoted by g_i) we mean a function $g_i\colon St_i(G)\to\{b,r,p\}$ which, for a given situation, gives b or r meaning the color v_i guesses it is, or the letter p if v_i passes. Thus a guessing instruction is a rule which determines the behavior of the vertex v_i in every situation. By a strategy for the graph G we mean a sequence $(g_1,g_2,\dots,g_n)$. By $\mathcal{F}(G)$ we denote the family of all strategies for the graph G.

Let $v_i\in V(G)$ and $S\in\mathcal{F}(G)$. We say that v_i never states its color in the strategy S if v_i passes in every situation, that is $g_i\equiv p$. We say that v_i always states its color in the strategy S if v_i states its color in every situation, that is, for every $T\in St_i(G)$ we have $g_i(T)\in\{b,r\}$ ($g_i(T)\neq p$, equivalently).

If $S\in\mathcal{F}(G)$, then by $Cw(S)$ and $Cl(S)$ we denote the sets of cases for the graph G in which the team wins or loses, respectively. Of course, $|Cw(S)|+|Cl(S)| = |C(G)|$. Consequently, by the chance of success of the strategy S we mean the number $p(S)=\frac{|Cw(S)|}{|C(G)|}$. By the hat number of the graph G we mean the number $h(G)=\max\{p(S)\colon S\in\mathcal{F}(G)\}$. Certainly $p(S)\leq h(G)$. We say that the strategy S is optimal for the graph G if $p(S)=h(G)$. By $\mathcal{F}^0(G)$ we denote the family of all optimal strategies for the graph G.

Let $t, m_1, m_2, \ldots, m_t \in \{1, 2, \ldots, n\}$ be such that $m_j \neq m_k$ and $c_{m_j} \in \{b, r\}$, for every $j, k \in \{1, 2, \ldots, t\}$.

By $C(G, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, \ldots, v_{m_t}^{c_{m_t}})$ we denote the set of cases for the graph G such that the color of v_{m_j} is c_{m_j}.

Let $S \in \mathcal{F}(G)$. By $Cw(S, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, \ldots, v_{m_t}^{c_{m_t}})$ ($Cl(S, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, \ldots, v_{m_t}^{c_{m_t}})$, respectively) we denote the set of cases for G which belong to the set $C(G, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, \ldots, v_{m_t}^{c_{m_t}})$, and in which the team wins (loses, respectively).

Let $v_i \in V(G)$. If for every $j \in \{1, 2, \ldots, t\}$ we have $v_{m_j} \in N_G(v_i)$, then by $St_i(G, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, \ldots, v_{m_t}^{c_{m_t}})$ we denote the set of possible situations of v_i in the graph G such that the color of v_{m_j} is c_{m_j}.

3 Results

First let us observe that indeed we can confine to deterministic strategies (that is strategies such that the decision of each person is determined uniquely by the hat colors of other people). We can do this since for any randomized strategy there exists a not worse deterministic one. It is true, because every randomized strategy is a convex combination of some deterministic strategies. The probability of winning is a linear function on the convex polyhedron corresponding to the set of all randomized strategies which can be achieved combining those deterministic strategies. It is well known that this function achieves its maximum on a vertex of the polyhedron which corresponds to a deterministic strategy.

Let G and H be graphs. Assume that $H \subseteq G$. Since every vertex from the set $V(G) \setminus V(H)$ can always pass, and every vertex $v_i \in V(H)$ can ignore the colors of vertices from the set $N_G(v_i) \setminus N_H(v_i)$, it is easy to see that the hat number of the graph G is greater than or equal to the hat number of the graph H. It is that if $H \subseteq G$, then $h(H) \leq h(G)$.

Since K_1 is a subgraph of every graph, we get $h(G) \geq \frac{1}{2}$.

Let S be an optimal strategy for the graph G. By definition we have $p(S) = h(G)$. Since $h(G) \geq \frac{1}{2}$, we get $p(S) \geq \frac{1}{2}$.

Now we prove a fact characterizing the number of cases in which the loss of the team is caused by a statement of a vertex.

Fact 1 *Let G be a graph and let v_i be a vertex of G. Let $S \in \mathcal{F}(G)$. If v_i states its color in a situation, then the team loses in at least half of all cases corresponding to this situation.*

Proof. Assume that v_i states its color in a situation T. Without loss of generality we assume that in this situation v_i states it is blue, that is $g_i(T) = b$. In half of all cases corresponding to T we have $c(v_i) = r$, it means that v_i is red. Thus, the team loses in every one of these cases, because v_i states its color wrong, as $g_i(T) = b \neq r = c(v_i)$. ∎

Corollary 2 *Let G be a graph and let v be a vertex of G. If $S \in \mathcal{F}^0(G)$ is a strategy such that v always states its color, then $h(G) = \frac{1}{2}$.*

Proof. Assumption indicates that in every case v states its color, so by Fact 1 we have $|Cl(S)| \geq \frac{|C(G)|}{2}$. Consequently,

$$p(S) = \frac{|Cw(S)|}{|C(G)|} = \frac{|C(G)| - |Cl(S)|}{|C(G)|} \leq \frac{|C(G)| - \frac{|C(G)|}{2}}{|C(G)|} = \frac{1}{2}.$$

Since $p(S) \leq \frac{1}{2}$ and $S \in \mathcal{F}^0(G)$, we have $h(G) \leq \frac{1}{2}$ (by definition). On the other hand we have $h(G) \geq \frac{1}{2}$. ∎

In the following theorem we give a sufficient condition for deleting a vertex of a graph without changing its hat number.

Theorem 3 *Let G be a graph and let v be a vertex of G. If $S \in \mathcal{F}^0(G)$ is a strategy such that v never states its color, then $h(G) = h(G - v)$.*

Proof. Let $S' \in \mathcal{F}(G - v)$ be the strategy as follows: Every vertex not adjacent to v in G behaves in the same way as in S, that is, if $v_i \notin N_G(v)$, then $g'_i = g_i$, where g'_i and g_i are the guessing instructions for the vertex v_i in the strategies S' and S, respectively. First assume that $|Cw(S, v^b)| \geq |Cw(S, v^r)|$. Let every vertex adjacent to v in G behave in the same way as in S when v is blue, that is, if $v_i \in N_G(v)$, then $g'_i = g_{i|St_i(G,v^b)}$. The result of any case C' in the strategy S' is the same as the result of the case C in the strategy S, where C is the corresponding case in which v is blue, because in both strategies S' and S the vertex v never states its color and every vertex in the strategy S' behaves in the same way as in S when v is blue. This implies that $|Cw(S')| = |Cw(S, v^b)|$. Now we get

$$p(S') = \frac{|Cw(S')|}{2^{|V(G-v)|}} = \frac{|Cw(S, v^b)|}{2^{|V(G)|-1}} = \frac{2|Cw(S, v^b)|}{2^{|V(G)|}}$$

$$\geq \frac{|Cw(S,v^b)| + |Cw(S,v^r)|}{2^{|V(G)|}} = p(S).$$

If $|Cw(S,v^b)| < |Cw(S,v^r)|$, then similarly we get a strategy S' such that $p(S') > p(S)$. Since $S \in \mathcal{F}^0(G)$ and $S' \in \mathcal{F}(G-v)$, we have $h(G) = p(S) \leq p(S') \leq h(G-v)$. On the other hand we have $h(G) \geq h(G-v)$. ■

Let S be a strategy for the graph G. Let C be a case in which some vertex states its color. Since the rules of the hat problem are such that one correct statement suffices to win, and one wrong statement causes the loss, it is easy to see that a statement of any other vertex cannot improve the result of the case C.

Now we solve the hat problem on paths.

Theorem 4 *For every path P_n we have $h(P_n) = \frac{1}{2}$.*

Proof. Let $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. We distinguish six possibilities: $n = 1$, $n = 2$, $n = 3$, $n = 4$, $n = 5$, and $n \geq 6$.

First, we assume that $n = 1$. Since $P_1 = K_1$, we have $h(P_1) = h(K_1) = \frac{1}{2}$.

Now assume that $n = 2$. Let S be an optimal strategy for P_2. If some vertex, say v_i, never states its color, then by Theorem 3 we have $h(P_2) = h(P_2 - v_i)$. Since $P_2 - v_i = P_1$, we have $h(P_2) = h(P_1) = \frac{1}{2}$. Now assume that v_1 and v_2 state their colors. If one of them always states its color, then by Corollary 2 we have $h(P_2) = \frac{1}{2}$. If, neither v_1 nor v_2 always states its color, then without loss of generality we assume that v_1 states its color when v_2 is blue, and in this situation it states it is blue. We consider the following four possibilities: $g_2(b,*) = b$ (Table 1); $g_2(b,*) = r$ (Table 2); $g_2(r,*) = b$ (Table 3); $g_2(r,*) = r$ (Table 4). In the next tables b means blue, r means red, + means correct statement (success), $-$ means wrong statement (loss), and blank square means passing.

In Tables 1, 2, and 3 we have $|Cw(S)| = 1$, $|C(P_2)| = 4$, so $p(S) = \frac{1}{4} < \frac{1}{2}$, a contradiction.

Table 1

No	The color of		The statement of		Result
	v_1	v_2	v_1	v_2	
1	b	b	+	+	+
2	b	r		$-$	$-$
3	r	b	$-$		$-$
4	r	r			$-$

Table 2

No	The color of		The statement of		Result
	v_1	v_2	v_1	v_2	
1	b	b	+	−	−
2	b	r		+	+
3	r	b	−		−
4	r	r			−

Table 3

No	The color of		The statement of		Result
	v_1	v_2	v_1	v_2	
1	b	b	+		+
2	b	r			−
3	r	b	−	+	−
4	r	r		−	−

Table 4

No	The color of		The statement of		Result
	v_1	v_2	v_1	v_2	
1	b	b	+		+
2	b	r			−
3	r	b	−	−	−
4	r	r		+	+

In Table 4 we have $|Cw(S)| = 2$, $|C(P_2)| = 4$, so $p(S) = \frac{2}{4} = \frac{1}{2}$. Since $S \in \mathcal{F}^0(P_2)$, we have $h(P_2) = \frac{1}{2}$.

Now assume that $n = 3$. Let S be an optimal strategy for P_3. If v_1 or v_3 never states its color, then without loss of generality we assume that it is v_1. By Theorem 3 we have $h(P_3) = h(P_3 - v_1)$. Since $P_3 - v_1 = P_2$, we have $h(P_3) = h(P_2) = \frac{1}{2}$. Now assume that v_1 and v_3 state their colors. If v_1 or v_3 always states its color, then by Corollary 2 we have $h(P_3) = \frac{1}{2}$. If neither v_1 nor v_3 always states its color, then without loss of generality we assume that v_1 states its color when v_2 is blue, and in this situation it states it is blue. We have the following two possibilities: (1) v_3 states its color when v_2 is blue; (2) v_3 does not state its color when v_2 is blue.

(1) Let the strategy S' differ from S only in that v_3 does not state its color when v_2 is blue. Since in every case in which v_2 is blue v_1 states its color,

the statement of v_3 cannot improve the result of any of these cases. Therefore, $p(S) \leq p(S')$. Since $S \in \mathcal{F}^0(P_3)$, the strategy S' is also optimal for P_3. If v_3 never states its color in the strategy S', then we have the possibility already considered. The other possibility when v_3 states its color we consider in the next paragraph.

(2) Certainly, v_3 states its color when v_2 is red. Since v_1 (v_3, respectively) states its color when v_2 is blue (red, respectively), by Fact 1 we have

$$|Cl(S, v_2^b)| \geq \frac{|C(P_3, v_2^b)|}{2} \qquad \left(|Cl(S, v_2^r)| \geq \frac{|C(P_3, v_2^r)|}{2}, \text{ respectively}\right).$$

This implies that

$$|Cl(S)| = |Cl(S, v_2^b)| + |Cl(S, v_2^r)| \geq \frac{|C(P_3, v_2^b)|}{2} + \frac{|C(P_3, v_2^r)|}{2} = \frac{|C(P_3)|}{2}.$$

Consequently,

$$p(S) = \frac{|Cw(S)|}{|C(P_3)|} = \frac{|C(P_3)| - |Cl(S)|}{|C(P_3)|} \leq \frac{|C(P_3)| - \frac{|C(P_3)|}{2}}{|C(P_3)|} = \frac{1}{2}.$$

Since $S \in \mathcal{F}^0(P_3)$, we have $h(P_3) \leq \frac{1}{2}$. Since $h(P_3) \geq \frac{1}{2}$, we get $h(P_3) = \frac{1}{2}$.

Now assume that $n = 4$. Let S be an optimal strategy for P_4. If some vertex, say v_i, never states its color, then by Theorem 3 we have $h(P_4) = h(P_4 - v_i)$. If $i \in \{1, 4\}$, then $P_4 - v_i = P_3$, so $h(P_4) = h(P_3) = \frac{1}{2}$. If $i \in \{2, 3\}$, then $P_4 - v_i = P_1 \cup P_2$. Since $P_1 \cup P_2 \subseteq P_3$, we have $h(P_1 \cup P_2) \leq h(P_3) = \frac{1}{2}$. Therefore, $h(P_4) = h(P_1 \cup P_2) \leq \frac{1}{2}$. Since $h(P_4) \geq \frac{1}{2}$, we get $h(P_4) = \frac{1}{2}$. Now assume that every vertex states its color. If some vertex always states its color, then by Corollary 2 we have $h(P_4) = \frac{1}{2}$. If no vertex always states its color, then without loss of generality we assume that v_1 states its color when v_2 is blue, and in this situation it states it is blue. Similarly, since $N_{P_4}[v_1] \cap N_{P_4}[v_4] = \emptyset$, we may assume that v_4 states its color when v_3 is blue, and in this situation it states it is blue. We consider the following two possibilities: (1) v_2 states its color when v_3 is blue, or v_3 states its color when v_2 is blue; (2) v_2 does not state its color when v_3 is blue, and v_3 does not state its color when v_2 is blue.

(1) Let the strategy S' differ from S only in that v_2 does not state its color when v_3 is blue, and v_3 does not state its color when v_2 is blue. Since in every case in which v_3 (v_2, respectively) is blue v_4 (v_1, respectively) states its color, the statement of v_2 (v_3, respectively) cannot improve the result of any of these cases. Therefore, $p(S) \leq p(S')$. Since $S \in \mathcal{F}^0(P_4)$, the strategy S' is also optimal for P_4. If v_2 or v_3 never states its color in the strategy S', then we have the possibility

already considered. The other possibility when v_2 and v_3 state their colors we consider in the next paragraph.

(2) If $c(v_1) = r$ and $c(v_2) = b$, or $c(v_3) = b$ and $c(v_4) = r$, then in each of the 7 cases, the team loses. Certainly, v_2 can state its color only when v_3 is red. Thus there are the following four possibilities: (2.1) $g_2(b, *, r, *) = b$; (2.2) $g_2(b, *, r, *) = r$; (2.3) $g_2(r, *, r, *) = b$; (2.4) $g_2(r, *, r, *) = r$.

(2.1) Since $|Cl(S, v_1^b, v_2^r, v_3^r)| = |C(P_4, v_1^b, v_2^r, v_3^r)| = 2$ and $C(P_4, v_1^b, v_2^r, v_3^r) \cap (C(P_4, v_1^r, v_2^b) \cup C(P_4, v_3^b, v_4^r)) = \emptyset$, the team loses in at least $7+2 = 9$ cases, and wins in at most 7 cases. It means that $p(S) \leq \frac{7}{16} < \frac{1}{2}$, a contradiction.

Possibilities (2.2) and (2.3) are similar to (2.1).

(2.4) Certainly, v_3 can state its color only when v_2 is red. Thus we have the following four possibilities: (2.4.1) $g_3(*, r, *, b) = b$; (2.4.2) $g_3(*, r, *, b) = r$; (2.4.3) $g_3(*, r, *, r) = b$; (2.4.4) $g_3(*, r, *, r) = r$.

In possibilities (2.4.1), (2.4.2), and (2.4.3), without considering the consequences of statements of v_2, we get a similar contradiction as in (2.1), (2.2), and (2.3).

(2.4.4) In this possibility, analyzed in Table 5, we have $|Cw(S)| = 8$, $|C(P_4)| = 16$, so $p(S) = \frac{8}{16} = \frac{1}{2}$. Since $S \in \mathcal{F}^0(P_4)$, we have $h(P_4) = \frac{1}{2}$.

Now assume that $n = 5$. Let S be an optimal strategy for P_5. If for some $i \in \{1, 3, 5\}$ the vertex v_i never states its color, then by Theorem 3 we have $h(P_5) = h(P_5 - v_i)$. If $i \in \{1, 5\}$, then $P_5 - v_i = P_4$, so $h(P_5) = h(P_4) = \frac{1}{2}$. If $i = 3$, then $P_5 - v_3 = P_2 \cup P_2$. Since $P_2 \cup P_2 \subseteq P_4$, we have $h(P_2 \cup P_2) \leq h(P_4) = \frac{1}{2}$, so $h(P_5) = h(P_2 \cup P_2) \leq \frac{1}{2}$. Since $h(P_5) \geq \frac{1}{2}$, we get $h(P_5) = \frac{1}{2}$. Now assume that every vertex from the set $\{v_1, v_3, v_5\}$ states its color. If some of these vertices always states its color, then by Corollary 2 we have $h(P_5) = \frac{1}{2}$. If no vertex from the set $\{v_1, v_3, v_5\}$ always states its color, then without loss of generality we assume that v_1 states its color when v_2 is blue, and in this situation it states it is blue. Similarly, since $N_{P_5}[v_1] \cap N_{P_5}[v_5] = \emptyset$, we may assume that v_5 states its color when v_4 is blue, and in this situation it states it is blue. We consider the following two possibilities: (1) v_3 states its color when v_2 or v_4 is blue; (2) v_3 does not state its color when v_2 or v_4 is blue.

(1) Let the strategy S' differ from S only in that v_3 does not state its color when v_2 or v_4 is blue. Since in every case in which v_2 (v_4, respectively) is blue, v_1 (v_5, respectively) states its color, the statement of v_3 cannot improve the result of any of these cases. Therefore, $p(S) \leq p(S')$. Since $S \in \mathcal{F}^0(P_5)$, the strategy S' is also optimal for P_5. If v_3 never states its color in the strategy S', then we

Table 5

No	The color of				The statement of				Result
	v_1	v_2	v_3	v_4	v_1	v_2	v_3	v_4	
1	b	b	b	b	+			+	+
2	b	b	b	r	+			−	−
3	b	b	r	b	+				+
4	b	b	r	r	+				+
5	b	r	b	b				+	+
6	b	r	b	r			−	−	−
7	b	r	r	b					−
8	b	r	r	r			+		+
9	r	b	b	b	−			+	−
10	r	b	b	r	−			−	−
11	r	b	r	b	−	−			−
12	r	b	r	r	−	−			−
13	r	r	b	b				+	+
14	r	r	b	r			−	−	−
15	r	r	r	b		+			+
16	r	r	r	r		+	+		+

have the possibility already considered. The other possibility when v_3 states its color we consider in the next paragraph.

(2) If $c(v_1) = r$ and $c(v_2) = b$, or $c(v_4) = b$ and $c(v_5) = r$, then in each of the $2^3 + 2^3 - 2 = 14$ cases the team loses. Certainly, v_3 states its color only when v_2 and v_4 are red. Without loss of generality we assume that in this situation v_3 states it is blue. If $c(v_2) = c(v_3) = c(v_4) = r$, then in each of the 4 cases, the team loses. Since $(C(P_5, v_1^r, v_2^b) \cup C(P_5, v_4^b, v_5^r)) \cap C(P_5, v_2^r, v_3^r, v_4^r) = \emptyset$, the team loses in at least $14 + 4 = 18$ cases, and wins in at most 14 cases. This implies that $p(S) \leq \frac{14}{32} < \frac{1}{2}$, a contradiction.

The result for $n \geq 6$ we prove by the induction on the number of vertices of a path. Let us assume that n is an integer such that $n \geq 6$, and $h(P_{n-1}) = \frac{1}{2}$. We will prove that $h(P_n) = \frac{1}{2}$. Let S be an optimal strategy for P_n. If for some $i \in \{1, 3, n\}$ the vertex v_i never states its color, then by Theorem 3 we have $h(P_n) = h(P_n - v_i)$. If $i \in \{1, n\}$, then $P_n - v_i = P_{n-1}$, so $h(P_n) = h(P_{n-1}) = \frac{1}{2}$. If $i = 3$, then $P_n - v_3 = P_2 \cup P_{n-3}$. Since $P_2 \cup P_{n-3} \subseteq P_{n-1}$, we have $h(P_2 \cup P_{n-3}) \leq h(P_{n-1}) = \frac{1}{2}$, so $h(P_n) = h(P_2 \cup P_{n-3}) \leq \frac{1}{2}$. Since

$h(P_n) \geq \frac{1}{2}$, we get $h(P_n) = \frac{1}{2}$. Now assume that every vertex from the set $\{v_1, v_3, v_n\}$ states its color. If some from these vertices always states its color, then by Corollary 2 we have $h(P_n) = \frac{1}{2}$. If no vertex from the set $\{v_1, v_3, v_n\}$ always states its color, then without loss of generality we assume that v_1 states its color when v_2 is blue, and in this situation it states it is blue. Similarly, since $N_{P_n}[v_1] \cap N_{P_n}[v_n] = \emptyset$, we may assume that v_n states its color when v_{n-1} is blue, and in this situation it states it is blue. We consider the following two possibilities: (1) v_3 states its color when v_2 is blue; (2) v_3 does not state its color when v_2 is blue.

(1) Let the strategy S' differ from S only in that v_3 does not state its color when v_2 is blue. Since in every case in which v_2 is blue, v_1 states its color, the statement of v_3 cannot improve the result of any of these cases. Therefore, $p(S) \leq p(S')$. Since $S \in \mathcal{F}^0(P_n)$, the strategy S' is also optimal for P_n. If v_3 never states its color in the strategy S', then we have the possibility already considered. The other possibility when v_3 states its color we consider in the next paragraph.

(2) If $c(v_1) = r$ and $c(v_2) = b$, or $c(v_{n-1}) = b$ and $c(v_n) = r$, then in each of the $(\frac{1}{4} + \frac{1}{4} - \frac{1}{4^2})|C(P_n)| = \frac{7}{16}|C(P_n)|$ cases the team loses. Certainly, v_3 can state its color only when v_2 is red. Without loss of generality we assume that v_3 states its color when v_2 is red and v_4 is blue, and in this situation it states it is blue. If $c(v_2) = c(v_3) = r$ and $c(v_4) = b$, then the team loses. All the cases in which $c(v_{n-1}) = b$ and $c(v_n) = r$ have been counted, so it remains to count the such ones that $c(v_2) = c(v_3) = r$, $c(v_4) = b$, and $(c(v_{n-1}) = r$ or $c(v_n) = b)$. There are $\frac{1}{2^3} \cdot \frac{3}{4} \cdot |C(P_n)| = \frac{3}{32}|C(P_n)|$ such cases. This implies that the team loses in at least $(\frac{7}{16} + \frac{3}{32})|C(P_n)| = \frac{17}{32}|C(P_n)|$ cases, and wins in at most $\frac{15}{32}|C(P_n)|$ cases. It means that $p(S) \leq \frac{15}{32} < \frac{1}{2}$, a contradiction. ■

Now we solve the hat problem on trees.

Theorem 5 *For every tree T we have $h(T) = \frac{1}{2}$.*

Proof. The result we prove by induction on the number of vertices of a tree. If T has one vertex, that is $T = K_1$, it is obvious that the theorem is true. Let T be any tree with $n \geq 2$ vertices, and let us assume that $h(T') = \frac{1}{2}$ for every tree T' with $n - 1$ vertices. Every tree has at least two leafs (that is vertices of a tree having exactly one neighbour). If T has exactly two leafs, then T is a path, and by Theorem 4 we have $h(T) = \frac{1}{2}$. If T has at least three leafs, then let v_1, v_2, and v_3 be any three of them. Let S be an optimal strategy for T. Since v_1, v_2, and v_3 are leafs, there are exactly two possible situations of each of them. If for some $i \in \{1, 2, 3\}$ the vertex v_i never states its color, then by Theorem 3 we have $h(T) = h(T - v_i)$. Since $T - v_i$ is a tree with $n - 1$ vertices, by the inductive assumption we have $h(T - v_i) = \frac{1}{2}$, and therefore $h(T) = \frac{1}{2}$. Now assume that every vertex from the set $\{v_1, v_2, v_3\}$ states its color. If one of them always states its color, then by Corollary 2 we have $h(T) = \frac{1}{2}$. Now assume that every vertex from the set $\{v_1, v_2, v_3\}$ states its color in exactly one situation. We consider the following two possibilities: (1) at least two leafs from the set $\{v_1, v_2, v_3\}$ have the same neighbour, that is, $N_T(v_i) = N_T(v_j)$ for certain $i, j \in \{1, 2, 3\}$, $i \neq j$; (2) every leaf from the set $\{v_1, v_2, v_3\}$ has another neighbour, that is, $N_T(v_1) \neq N_T(v_2) \neq N_T(v_3)$ and $N_T(v_1) \neq N_T(v_3)$.

(1) Let us denote $\{x\} = N_T(v_i) = N_T(v_j)$. We consider the following two possibilities: (1.1) v_i and v_j state their colors in the same situation; (1.2) v_i and v_j state their colors in different situations.

(1.1) Without loss of generality we assume that v_i and v_j state their colors when x is blue. Let the strategy S' differ from S only in that v_j does not state its color when x is blue, that is, v_j never states its color. Since in every case in which x is blue v_i states its color, the statement of v_j cannot improve the result of any of these cases. Therefore, $p(S) \leq p(S')$. Since $S \in \mathcal{F}^0(T)$, the strategy S' is also optimal for T. Since v_j never states its color in the strategy S', we have the possibility already considered.

(1.2) Without loss of generality we assume that v_i states its color when x is blue and v_j states its color when x is red. By Fact 1 we have

$$|Cl(S, x^b)| \geq \frac{|C(T, x^b)|}{2} \text{ and } |Cl(S, x^r)| \geq \frac{|C(T, x^r)|}{2}.$$

This implies that

$$|Cl(S)| = |Cl(S, x^b)| + |Cl(S, x^r)| \geq \frac{|C(T, x^b)|}{2} + \frac{|C(T, x^r)|}{2} = \frac{|C(T)|}{2}.$$

Consequently,

$$p(S) = \frac{|Cw(S)|}{|C(T)|} = \frac{|C(T)| - |Cl(S)|}{|C(T)|} \leq \frac{|C(T)| - \frac{|C(T)|}{2}}{|C(T)|} = \frac{1}{2}.$$

Since $S \in \mathcal{F}^0(T)$, we have $h(T) \leq \frac{1}{2}$. Since $h(T) \geq \frac{1}{2}$, we get $h(T) = \frac{1}{2}$.

(2) If $i \in \{1, 2, 3\}$, then let us denote $N_T(v_i) = \{v'_i\}$. Without loss of generality we assume that v_1 states its color when v'_1 is blue, and in this situation it states

it is blue. Similarly, since $v_1' \neq v_2' \neq v_3'$ and $v_1' \neq v_3'$, we may assume that v_2 states its color when v_2' is blue and in this situation it states it is blue, and v_3 states its color when v_3' is blue and in this situation it states it is blue. No vertex from the set $\{v_1, v_2, v_3\}$ states its color if and only if $c(v_1') = c(v_2') = c(v_3') = r$. If ($c(v_1) = r$ and $c(v_1') = b$) or ($c(v_2) = r$ and $c(v_2') = b$), or ($c(v_3) = r$ and $c(v_3') = b$), then in each of the $(1-(1-\frac{1}{4})^3)|C(T)| = \frac{37}{64}|C(T)|$ cases the team loses. This implies that the team wins in at most $\frac{27}{64}|C(T)|$ cases. Consequently,

$$p(S) = \frac{|Cw(S)|}{|C(T)|} \leq \frac{\frac{27}{64}|C(T)|}{|C(T)|} = \frac{27}{64} < \frac{1}{2},$$

a contradiction. ∎

Now we consider the hat problem on a graph such that the only information we know about are the degrees of vertices. In the following theorem we give an upper bound on the chance of success of any strategy for the hat problem on a graph with given degrees of vertices.

Theorem 6 *Let G be a graph and let S be any strategy for this graph. Then*

$$|Cw(S)| \leq \sum_{v \in V(G)} \left\lfloor 2^{d_G(v)+1} - \frac{|Cw(S)|}{2^{|V(G)|-d_G(v)-1}} \right\rfloor \cdot 2^{|V(G)|-d_G(v)-1}.$$

Proof. Let v_i be a vertex of G. Every statement of the color in any situation done by v_i is wrong in exactly $2^{|V(G)|-d_G(v_i)-1}$ cases, because to every situation of v_i correspond $2^{|V(G)|-d_G(v_i)}$ cases, and in the half of them v_i has another color than it states it has. The vertex v_i cannot state its color in at least

$$\left\lfloor 2^{d_G(v_i)+1} - \frac{|Cw(S)|}{2^{|V(G)|-d_G(v_i)-1}} \right\rfloor + 1$$

situations, otherwise its statements are wrong in at least

$$2^{|V(G)|-d_G(v_i)-1}\left(\left\lfloor 2^{d_G(v_i)+1} - \frac{|Cw(S)|}{2^{|V(G)|-d_G(v_i)-1}} \right\rfloor + 1\right)$$

$$> 2^{|V(G)|-d_G(v_i)-1}\left(2^{d_G(v_i)+1} - \frac{|Cw(S)|}{2^{|V(G)|-d_G(v_i)-1}}\right) = 2^{|V(G)|} - |Cw(S)|$$

cases. This implies that the team loses in more than $2^{|V(G)|} - |Cw(S)|$ cases, and wins in less than

$$|C(G)| - (2^{|V(G)|} - |Cw(S)|) = 2^{|V(G)|} - 2^{|V(G)|} + |Cw(S)| = |Cw(S)|$$

cases, but $|Cw(S)|$ is the number of cases in which the team wins, a contradiction. Since the vertex v_i does not state its color in at least

$$\left\lfloor 2^{d_G(v_i)+1} - \frac{|Cw(S)|}{2^{|V(G)|-d_G(v_i)-1}} \right\rfloor + 1$$

situations, it states its color in at most

$$\left\lfloor 2^{d_G(v_i)+1} - \frac{|Cw(S)|}{2^{|V(G)|-d_G(v_i)-1}} \right\rfloor$$

situations. Every statement of the color in any situation done by v_i is correct in exactly $2^{|V(G)|-d_G(v_i)-1}$ cases, because to every situation of v_i correspond $2^{|V(G)|-d_G(v_i)-1}$ cases, and in the half of them v_i has the color it states it has. Therefore, the statements of v_i are correct in at most

$$\left\lfloor 2^{d_G(v_i)+1} - \frac{|Cw(S)|}{2^{|V(G)|-d_G(v_i)-1}} \right\rfloor \cdot 2^{|V(G)|-d_G(v_i)-1}$$

cases. This implies that the team wins in at most

$$\sum_{v\in V(G)} \left\lfloor 2^{d_G(v)+1} - \frac{|Cw(S)|}{2^{|V(G)|-d_G(v)-1}} \right\rfloor \cdot 2^{|V(G)|-d_G(v)-1}$$

cases. ■

In the following three facts we show that the upper bound from the previous theorem together with integrality constraints is tight on complete graphs with two, three, and four vertices, respectively.

Fact 7 $h(K_2) = \frac{1}{2}$.

Proof. Let S be any strategy for K_2. By Theorem 6 we have

$$|Cw(S)| \le \sum_{v\in V(K_2)} \left\lfloor 2^{d_{K_2}(v)+1} - \frac{|Cw(S)|}{2^{|V(K_2)|-d_{K_2}(v)-1}} \right\rfloor \cdot 2^{|V(K_2)|-d_{K_2}(v)-1}.$$

Since $|V(K_2)| = 2$ and every vertex in K_2 has exactly one neighbour, we get

$$|Cw(S)| \le 2 \cdot \lfloor 2^2 - |Cw(S)| \rfloor \Leftrightarrow |Cw(S)| \le 8 - 2|Cw(S)| \Leftrightarrow |Cw(S)| \le 2\frac{2}{3}.$$

This implies that $|Cw(S)| \le 2$, as $n \in N$. Consequently,

$$p(S) = \frac{|Cw(S)|}{|C(K_2)|} \le \frac{2}{2^2} = \frac{1}{2}.$$

Since S is any strategy for K_2, we have $h(K_2) \le \frac{1}{2}$. Since $h(K_2) \ge \frac{1}{2}$, we get $h(K_2) = \frac{1}{2}$. ■

Fact 8 $h(K_3) = \frac{3}{4}$.

Proof. Let S be any strategy for K_3. By Theorem 6 we have

$$|Cw(S)| \leq \sum_{v \in V(K_3)} \left\lfloor 2^{d_{K_3}(v)+1} - \frac{|Cw(S)|}{2^{|V(K_3)|-d_{K_3}(v)-1}} \right\rfloor \cdot 2^{|V(K_3)|-d_{K_3}(v)-1}.$$

Since $|V(K_3)| = 3$ and every vertex in K_3 has exactly two neighbours, we get

$$|Cw(S)| \leq 3 \cdot \lfloor 2^3 - |Cw(S)| \rfloor \Leftrightarrow |Cw(S)| \leq 24 - 3|Cw(S)| \Leftrightarrow |Cw(S)| \leq 6.$$

Consequently,

$$p(S) = \frac{|Cw(S)|}{|C(K_3)|} \leq \frac{6}{2^3} = \frac{3}{4}.$$

Since S is any strategy for K_3, we have $h(K_3) \leq \frac{3}{4}$. Let $S_1 \in \mathcal{F}(K_3)$ be the strategy such that every vertex considers colors of its two neighbours, and if they are the same, it states it has the opposite color. If they are different, it passes. It is easy to verify that $|Cw(S_1)| = 6$. Since $|C(K_3)| = 2^3 = 8$, we have $p(S_1) = \frac{|Cw(S)|}{|C(K_3)|} = \frac{6}{8} = \frac{3}{4}$. Since $p(S_1) \leq h(K_3)$, we have $h(K_3) \geq \frac{3}{4}$. Since $h(K_3) \geq \frac{3}{4}$ and $h(K_3) \leq \frac{3}{4}$, we get $h(K_3) = \frac{3}{4}$. ∎

Fact 9 $h(K_4) = \frac{3}{4}$.

Proof. Let S be any strategy for K_4. By Theorem 6 we have

$$|Cw(S)| \leq \sum_{v \in V(K_4)} \left\lfloor 2^{d_{K_4}(v)+1} - \frac{|Cw(S)|}{2^{|V(K_4)|-d_{K_4}(v)-1}} \right\rfloor \cdot 2^{|V(K_4)|-d_{K_4}(v)-1}.$$

Since $|V(K_4)| = 4$ and every vertex in K_4 has three neighbours, we get

$$|Cw(S)| \leq 4 \cdot \lfloor 2^4 - |Cw(S)| \rfloor \Leftrightarrow |Cw(S)| \leq 64 - 4|Cw(S)| \Leftrightarrow |Cw(S)| \leq 12\frac{4}{5}.$$

This implies that $|Cw(S)| \leq 12$, as $|Cw(S)| \in N$. Consequently,

$$p(S) = \frac{|Cw(S)|}{|C(K_4)|} \leq \frac{12}{2^4} = \frac{3}{4}.$$

Since S is any strategy for K_4, we have $h(K_4) \leq \frac{3}{4}$. Since $K_3 \subseteq K_4$ and $h(K_3) = \frac{3}{4}$, we get $h(K_3) \leq h(K_4)$. Since $h(K_3) = \frac{3}{4}$, we have $h(K_4) \geq \frac{3}{4}$. This implies that $h(K_4) = \frac{3}{4}$. ∎

In the next fact we solve the hat problem on the graph $K_3 \cup K_2$.

Fact 10 $h(K_3 \cup K_2) = \frac{3}{4}$.

Proof. Let $E(K_3 \cup K_2) = \{v_1v_2, v_2v_3, v_3v_1, v_4v_5\}$. Let S be any strategy for the graph $K_3 \cup K_2$. By Theorem 6 we have

$$|Cw(S)| \leq \sum_{v \in V(K_3 \cup K_2)} \left\lfloor 2^{d_{K_3 \cup K_2}(v)+1} - \frac{|Cw(S)|}{2^{|V(K_3 \cup K_2)| - d_{K_3 \cup K_2}(v) - 1}} \right\rfloor \cdot 2^{|V(K_3 \cup K_2)| - d_{K_3 \cup K_2}(v) - 1}.$$

Since $d_{K_3 \cup K_2}(v_1) = d_{K_3 \cup K_2}(v_2) = d_{K_3 \cup K_2}(v_3) = 2$ and $d_{K_3 \cup K_2}(v_4) = d_{K_3 \cup K_2}(v_5) = 1$, we get

$$\begin{aligned} |Cw(S)| &\leq 3 \cdot 2^2 \cdot \left\lfloor 2^3 - \frac{|Cw(S)|}{2^2} \right\rfloor + 2 \cdot 2^3 \cdot \left\lfloor 2^2 - \frac{|Cw(S)|}{2^3} \right\rfloor \\ &= 12 \left\lfloor 8 - \frac{|Cw(S)|}{4} \right\rfloor + 16 \left\lfloor 4 - \frac{|Cw(S)|}{8} \right\rfloor. \end{aligned}$$

This implies that

$$\begin{aligned} |Cw(S)| &\leq 12 \left(8 - \frac{|Cw(S)|}{4} \right) + 16 \left(4 - \frac{|Cw(S)|}{8} \right) \\ &= 96 - 3|Cw(S)| + 64 - 2|Cw(S)| = 160 - 5|Cw(S)|. \end{aligned}$$

Now we easily get $|Cw(S)| \leq \frac{160}{6} = 26\frac{2}{3}$. Since $|Cw(S)|$ is an integer, we have $|Cw(S)| \leq 26$. Assume that $|Cw(S)| = 26$. We have $26 \leq 12\lfloor 8 - \frac{26}{4} \rfloor + 16\lfloor 4 - \frac{26}{8} \rfloor$ $= 12 \cdot 1 + 16 \cdot 0 = 12$, a contradiction. Now assume that $|Cw(S)| = 25$. We have $25 \leq 12\lfloor 8 - \frac{25}{4} \rfloor + 16\lfloor 4 - \frac{25}{8} \rfloor = 12 \cdot 1 + 16 \cdot 0 = 12$, a contradiction. This implies that $|Cw(S)| \leq 24$, and consequently, $p(S) = \frac{|Cw(S)|}{|C(K_3 \cup K_2)|} \leq \frac{24}{32}$. Since S is any strategy for $K_3 \cup K_2$, we have $h(K_3 \cup K_2) \leq \frac{3}{4}$. Since $K_3 \subseteq K_3 \cup K_2$ and $h(K_3) = \frac{3}{4}$, we get $h(K_3 \cup K_2) \geq h(K_3) = \frac{3}{4}$. This implies that $h(K_3 \cup K_2) = \frac{3}{4}$. ■

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