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Heteroclinic solutions for a class of the second order Hamiltonian systems ☆

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Abstract

We shall be concerned with the existence of heteroclinic orbits for the second order Hamiltonian system $\ddot{q} + V_q(t,q) = 0$, where $q \in \mathbb{R}^n$ and $V \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, $V \leqslant 0$. We will assume that V and a certain subset $\mathcal{M} \subset \mathbb{R}^n$ satisfy the following conditions. $\mathcal{M}$ is a set of isolated points and $\#\mathcal{M} \geqslant 2$. For every sufficiently small $\varepsilon > 0$ there exists $\delta > 0$ such that for all $(t,z) \in \mathbb{R} \times \mathbb{R}^n$, if $d(z, \mathcal{M}) \geqslant \varepsilon$ then $-V(t,z) \geqslant \delta$. The integrals $\int_{-\infty}^{\infty} -V(t,z)\,dt$, $z \in \mathcal{M}$, are equi-bounded and $-V(t,z) \to \infty$, as $|t| \to \infty$, uniformly on compact subsets of $\mathbb{R}^n \setminus \mathcal{M}$. Our result states that each point in $\mathcal{M}$ is joined to another point in $\mathcal{M}$ by a solution of our system.

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Keywords: Heteroclinic solution; Nonautonomous Hamiltonian system; Action functional; Minimizing sequence

1. Introduction

In this work, we shall study the existence of heteroclinic orbits for the second order Hamiltonian system:

$$\ddot{q} + V_q(t,q) = 0, \tag{1}$$

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where $t \in \mathbb{R}$, $q \in \mathbb{R}^n$. We will suppose that $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{M} \subset \mathbb{R}^n$ satisfy the following assumptions:

(A_1) $V \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, $V \leqslant 0$,
(A_2) $\#\mathcal{M} \geqslant 2$ and $\gamma := \frac{1}{3} \inf\{|x - y|:\ x, y \in \mathcal{M},\ x \neq y\} > 0$,
(A_3) for every $0 < \varepsilon \leqslant \gamma$ there is $\delta > 0$ such that for all $(t, z) \in \mathbb{R} \times \mathbb{R}^n$, if $d(z, \mathcal{M}) \geqslant \varepsilon$ then $-V(t, z) \geqslant \delta$,
(A_4) $-V(t, z) \to \infty$, if $|t| \to \infty$, uniformly on every compact subset of $\mathbb{R}^n \setminus \mathcal{M}$,
(A_5) for every $x \in \mathcal{M}$,

$$\int_{-\infty}^{\infty} -V(t, x)\, dt < \gamma\sqrt{2\alpha},$$

where $\alpha := \inf\{-V(t, z):\ t \in \mathbb{R},\ d(z, \mathcal{M}) \geqslant \gamma\}$.

Here and subsequently, $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the Euclidean metric and $|\cdot| : \mathbb{R}^n \to \mathbb{R}$ is the corresponding norm.

We will say that a solution $q : \mathbb{R} \to \mathbb{R}^n$ of (1) is *a heteroclinic solution* (*heteroclinic orbit*) if there exist $x, y \in \mathbb{R}^n$, $x \neq y$, such that q joins x to y, i.e.

$$q(-\infty) := \lim_{t \to -\infty} q(t) = x$$

and

$$q(\infty) := \lim_{t \to \infty} q(t) = y.$$

Theorem 1.1. *Under assumptions* (A_1)–(A_5), *for every $x \in \mathcal{M}$ there exists at least one heteroclinic solution of* (1) *such that $q(-\infty) = x$ and $q(\infty) \in \mathcal{M} \setminus \{x\}$, i.e. q emanates from x and terminates at a certain $y \in \mathcal{M} \setminus \{x\}$.*

In the last years several authors studied connecting (i.e. homoclinic and heteroclinic) orbits for Hamiltonian systems by the use of variational methods and critical points theory. In the survey [9], P. Rabinowitz, who has given fundamental contributions to this field, presents the main results obtained in the last twenty years, describes some methods and lists some open problems. Among the previous studies of heteroclinic orbits are those of [2–4,6,8,10,12,13]. Homoclinic solutions are considered for example in [1,5,7,11,13].

We are motivated by [6] written by P. Rabinowitz. He considered the autonomous second order Hamiltonian system:

$$\ddot{q} + V'(q) = 0, \tag{2}$$

where $q = (q_1, \dots, q_n) \in \mathbb{R}^n$ and a function $V : \mathbb{R}^n \to \mathbb{R}$ satisfies:

(R_1) $V \in C^1(\mathbb{R}^n, \mathbb{R})$,
(R_2) V is periodic in q_i with a period T_i, $1 \leqslant i \leqslant n$.

Conditions (R_1)–(R_2) imply that V possesses a global maximum on $\mathbb{R}^n$. Without loss of generality we may assume that the global maximum of V is 0. Let $\mathcal{M} = \{y \in \mathbb{R}^n\colon\ V(y) = 0\}$. The condition on $\mathcal{M}$ is the following:

(R_3) $\mathcal{M}$ consists only of isolated points.

Theorem 1.2. *(See P.H. Rabinowitz [6].) Under assumptions* (R_1)–(R_3), *for every $x \in \mathcal{M}$, there exist at least two heteroclinic orbits of* (2) *joining x to $\mathcal{M} \setminus \{x\}$. At least one of these orbits emanates from x and at least one terminates at x.*

Theorem 1.1 is an analogue of Rabinowitz's theorem in a nonautonomous case. There is a class of natural examples to apply this theorem. Consider for instance a potential $V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$V(t, q) = \left(1 + t^2\right)(-1 + \cos q)$$

and $\mathcal{M} = \{2k\pi\colon\ k \in \mathbb{Z}\}$. It is easy to check that the map V and the set $\mathcal{M}$ satisfy assumptions (A_1)–(A_5). Clearly, $\hat{V}(q) = -1 + \cos q$ and $\mathcal{M}$ satisfy assumptions (R_1)–(R_3). In general, if $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is of the form

$$V(t, q) = f(t)\hat{V}(q),$$

where $\hat{V}$ satisfies (R_1)–(R_3), $f > 0$ and $f(t) \to \infty$, as $|t| \to \infty$, then (A_1)–(A_5) are fulfilled.

Let us observe that if q is a heteroclinic solution of (2) such that q emanates from x and terminates at y, then $q(-t)$ is a solution emanating from y and terminating at x. This fact was used by Rabinowitz to show the multiplicity result. Unfortunately, the same argument does not work for the time dependent Hamiltonian system (1).

The main idea of the proof is the same as in Rabinowitz [6].

Let $E = \{q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)\colon\ \int_{-\infty}^{\infty} |\dot{q}(t)|^2\,dt < \infty\}$. The space E under the norm

$$\|q\|^2 = \int_{-\infty}^{\infty} \left|\dot{q}(t)\right|^2 dt + \left|q(0)\right|^2 \tag{3}$$

is a Hilbert space. If $q \in E$ then for all $t, t_0 \in \mathbb{R}$, $q(t) = \int_{t_0}^{t} \dot{q}(s)\,ds + q(t_0)$. In consequence, q is continuous.

Fix $x \in \mathcal{M}$. For $y \in \mathcal{M} \setminus \{x\}$ and $\varepsilon > 0$, we will denote by $\Gamma_\varepsilon(y)$ the set of all functions $q \in E$ that satisfy the following conditions:

(i) $q(-\infty) = x$,
(ii) $q(\infty) = y$,
(iii) $q(t) \notin B_\varepsilon(\mathcal{M} \setminus \{x, y\})$.

Here and subsequently, $B_\varepsilon(A)$ denotes an open ε-neighbourhood of a set $A \subset \mathbb{R}^n$, and $B_\varepsilon(z)$ stands for an open ball of radius ε, centered at a point $z \in \mathbb{R}^n$.

Let us remark that if $0 < \varepsilon \leqslant \gamma$ then $\Gamma_\varepsilon(y)$ is nonempty for all $y \in \mathcal{M} \setminus \{x\}$. If $q(t) = x$ for $t \leqslant 0$, q is piecewise linear for $t \in [0, 1]$, $q(t) \notin B_\varepsilon(\mathcal{M} \setminus \{x, y\})$, and $q(t) = y$ for $t \geqslant 1$, then $q \in \Gamma_\varepsilon(y)$. Let $I : E \to \mathbb{R} \cup \{\infty\}$ be given by

$$I(q) = \int_{-\infty}^{\infty} \left[\frac{1}{2}\left|\dot{q}(t)\right|^2 - V\big(t, q(t)\big)\right] dt. \tag{4}$$

(A_1) implies that $I \geqslant 0$. Set

$$c_\varepsilon(y) = \inf_{q \in \Gamma_\varepsilon(y)} I(q). \tag{5}$$

From (A_5) it follows that if $0 < \varepsilon \leqslant \gamma$ then $I(q) < \infty$ for $q \in \Gamma_\varepsilon(y)$ piecewise linear, and hence $c_\varepsilon(y) < \infty$.

We will show that for ε small enough there exists $y \in \mathcal{M} \setminus \{x\}$ such that $c_\varepsilon(y) = I(q_{\varepsilon,y})$ for a certain $q_{\varepsilon,y} \in \Gamma_\varepsilon(y)$ which is a desired heteroclinic solution.

In the proof of Theorem 1.2, Rabinowitz several times, in an essential way, applied the fact that (2) is autonomous. Part of the difficulty in treating (1) is caused by the fact that it is nonautonomous.

2. Proof of Theorem 1.1

Our proof is divided into a sequence of lemmas. Set

$$\alpha_\varepsilon := \inf\big\{-V(t, z)\colon\ t \in \mathbb{R},\ z \notin B_\varepsilon(\mathcal{M})\big\}. \tag{6}$$

By (A_3) it follows that $\alpha_\varepsilon > 0$. In particular, if $\varepsilon = \gamma$ then $\alpha_\varepsilon = \alpha$ (see (A_5)).

Lemma 2.1. *Let $0 < \varepsilon \leqslant \gamma$. Suppose that $w \in E$ and $w(t) \notin B_\varepsilon(\mathcal{M})$ for each $t \in \bigcup_{i=1}^k [r_i, s_i]$, where $[r_i, s_i] \cap [r_j, s_j] = \emptyset$ for $i \neq j$. Then*

$$I(w) \geqslant \sqrt{2\alpha_\varepsilon} \sum_{i=1}^{k} \big|w(s_i) - w(r_i)\big|. \tag{7}$$

Lemma 2.1 was proved by Rabinowitz in [6]. It was also applied by him and Tanaka in [10]. Since this lemma is important for our further considerations, we enclose its proof below.

Proof of Lemma 2.1. Let $l = \sum_{i=1}^k |w(s_i) - w(r_i)|$ and $\tau = \sum_{i=1}^k (s_i - r_i)$. Then

$$\begin{aligned} l &= \sum_{i=1}^{k} \left|\int_{r_i}^{s_i} \dot{w}(t)\, dt\right| \leqslant \sum_{i=1}^{k} \int_{r_i}^{s_i} \big|\dot{w}(t)\big|\, dt = \int_{\bigcup_{i=1}^k [r_i, s_i]} \big|\dot{w}(t)\big|\, dt \\ &\leqslant \sqrt{\tau} \Bigg(\int_{\bigcup_{i=1}^k [r_i, s_i]} \big|\dot{w}(t)\big|^2\, dt \Bigg)^{\frac{1}{2}}. \end{aligned}$$

In consequence,

$$I(w) \geqslant \frac{1}{2} \int\limits_{\bigcup_{i=1}^{k}[r_i,s_i]} \left|\dot{w}(t)\right|^2 dt + \int\limits_{\bigcup_{i=1}^{k}[r_i,s_i]} \bigl(-V\bigl(t,w(t)\bigr)\bigr)\, dt$$
$$\geqslant \frac{l^2}{2\tau} + \alpha_\varepsilon \tau.$$

Since a function $(0,\infty) \ni t \to \frac{l^2}{2t} + \alpha_\varepsilon t$ achieves at the point $(\frac{l^2}{2\alpha_\varepsilon})^{1/2}$ its minimum, we have

$$I(w) \geqslant \sqrt{2\alpha_\varepsilon}\, l,$$

which completes the proof. □

Here is a consequence of Lemma 2.1.

Corollary 2.2. *If $w \in E$ and $I(w) < \infty$ then $w \in L^\infty(\mathbb{R},\mathbb{R}^n)$.*

Proof. Assume that $w \notin L^\infty(\mathbb{R},\mathbb{R}^n)$. Then for every $n \in \mathbb{N}$ there exists $t_n \in \mathbb{R}$ such that $|w(t_n)| > n$. (There is no loss of generality in assuming that $t_n \to \infty$, as $n \to \infty$.) Let us consider two cases.

Case 1. Suppose that $\#\mathcal{M} < \infty$.

There is $R > 0$ such that $B_\gamma(\mathcal{M}) \subset B_R(0)$. Assume that $\{t \in \mathbb{R}\colon\ w(t) \in \partial B_R(0)\} \neq \emptyset$. If $N \in \mathbb{N}$ is large enough then for each $n \geqslant N$ there is $r_n < t_n$ such that if $t \in [r_n, t_n]$ then $w(t) \notin B_R(0)$. It is sufficient to take $r_n = \max\{t < t_n\colon\ w(t) \in \partial B_R(0)\}$. From Lemma 2.1,

$$I(w) \geqslant \sqrt{2\alpha}\left|w(t_n) - w(r_n)\right| \geqslant \sqrt{2\alpha}(n - R).$$

Letting $n \to \infty$, we have $I(w) = \infty$, a contradiction.

If $\{t \in \mathbb{R}\colon\ w(t) \in \partial B_R(0)\} = \emptyset$, then by (7) we get

$$I(w) \geqslant \sqrt{2\alpha}\left|w(t_n) - w(t_1)\right| \geqslant \sqrt{2\alpha}\bigl(\left|w(t_n)\right| - \left|w(t_1)\right|\bigr)$$
$$\geqslant \sqrt{2\alpha}\bigl(n - \left|w(t_1)\right|\bigr)$$

for $n \in \mathbb{N}$, contrary to $I(w) < \infty$.

Case 2. Suppose that $\#\mathcal{M} = \infty$.

Since w is continuous, for every $k \in \mathbb{N}$ there exist $r_i < s_i$, $i = 1, 2, \ldots, k$, such that $w(t) \notin B_\gamma(\mathcal{M})$ for every $t \in [r_i, s_i]$, $|w(s_i) - w(r_i)| \geqslant \gamma$ and $[r_i, s_i] \cap [r_j, s_j] = \emptyset$ for $i \neq j$. By Lemma 2.1,

$$I(w) \geqslant \gamma k \sqrt{2\alpha}.$$

From this, $I(w) = \infty$, a contradiction. □

Lemma 2.3. *If $w \in E$ then for every $r, s \in \mathbb{R}$*

$$2I(w)|s-r| \geqslant \left|w(s)-w(r)\right|^2. \tag{8}$$

The proof is similar to the proof of Lemma 2.1. We leave it to the reader.

Lemma 2.4. *If $w \in E$ and $I(w) < \infty$ then there are $y_1, y_2 \in \mathcal{M}$ such that w joins y_1 to y_2.*

Lemma 2.4 is analogous to Proposition 3.11 of [6]. In spite of different assumptions on V, the claims are identical.

Proof of Lemma 2.4. Let $A(w)$ denote the set of limit points of $w(t)$, as $t \to -\infty$. From Corollary 2.2 we conclude that $A(w) \neq \emptyset$. Assume that there are $\varepsilon > 0$ and $\varrho \in \mathbb{R}$ such that if $t < \varrho$ then $w(t) \notin B_\varepsilon(\mathcal{M})$. By ($A_3$), we obtain

$$I(w) \geqslant \int_{-\infty}^{\varrho} -V\big(t, w(t)\big)\, dt = \infty,$$

a contradiction. Thus $A(w) \cap \mathcal{M} \neq \emptyset$. It is sufficient to notice that $A(w)$ consists of a point, say $y_1 \in \mathbb{R}^n$. If not, there is $\varepsilon > 0$ such that $w(t)$ intersects $\partial B_{\frac{\varepsilon}{2}}(y_1)$ and $\partial B_\varepsilon(y_1)$ infinitely many times. Applying Lemma 2.1, we obtain $I(w) \geqslant \frac{\varepsilon}{2} n \sqrt{2\alpha_{\frac{\varepsilon}{2}}}$ for each $n \in \mathbb{N}$, and hence $I(w) = \infty$, a contradiction.

In the same manner we can see that there is $y_2 \in \mathcal{M}$ such that $w(\infty) = y_2$. □

From now on we assume that $0 < \varepsilon \leqslant \gamma$.

Lemma 2.5. *Fix $y \in \mathcal{M} \setminus \{x\}$. If $\{q_m\}_{m=1}^{\infty}$ is a minimizing sequence for* (5) *such that $q_m \to q$ in $L^{\infty}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$, $q \in E$ and $I(q) < \infty$, then $q \in \Gamma_\varepsilon(y)$.*

Proof. By Lemma 2.4, there are $y_1, y_2 \in \mathcal{M}$ such that $q(-\infty) = y_1$ and $q(\infty) = y_2$. Since $q_m \to q$ in $L^{\infty}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ and $q_m \in \Gamma_\varepsilon(y)$, we have $q(t) \notin B_\varepsilon(\mathcal{M} \setminus \{x, y\})$ for all $t \in \mathbb{R}$ and $q(\pm\infty) \in \{x, y\}$. Moreover, for every $t \in \mathbb{R}$ there is $m(t) \in \mathbb{N}$ such that if $m \geqslant m(t)$ then $|q_m(t) - q(t)| < \frac{\varepsilon}{2}$.

Suppose, contrary to our claim, that $q(\infty) = x$. Then there exists $T \in \mathbb{R}$ such that for all $t \geqslant T$, $|q(t) - x| < \frac{\varepsilon}{2}$. In consequence, for every $t \geqslant T$ and for every $m \geqslant m(t)$ we have $|q_m(t) - x| \leqslant |q_m(t) - q(t)| + |q(t) - x| < \varepsilon$.

Let $r_m = \max\{s \in \mathbb{R}\colon\ q_m(s) \in \partial B_\varepsilon(x)\}$ and $s_m = \min\{s > r_m\colon\ q_m(s) \in \partial B_{\varepsilon+\gamma}(x)\}$. Since $\{q_m\}_{m=1}^{\infty}$ is a minimizing sequence for (5), there is $M > 0$ such that $I(q_m) \leqslant M$ for all $m \in \mathbb{N}$. From Lemma 2.3,

$$s_m - r_m \geqslant \frac{|q_m(s_m) - q_m(r_m)|^2}{2M} \geqslant \frac{\gamma^2}{2M} =: a > 0.$$

Fix $t \geqslant T$ and consider $q_{m(t)}$. Then

$$I(q_{m(t)}) \geqslant \int_{r_{m(t)}}^{r_{m(t)}+a} -V\big(s, q_{m(t)}(s)\big)\,ds = -V\big(\tau_{m(t)}, q_{m(t)}(\tau_{m(t)})\big)a \tag{9}$$

for a certain $\tau_{m(t)} \in (r_{m(t)}, r_{m(t)} + a)$, and $\varepsilon < |q_{m(t)}(\tau_{m(t)}) - x| < \varepsilon + \gamma$. From the above it follows that $t < \tau_{m(t)}$. Applying (A$_4$) and letting $t \to \infty$ in (9), we receive $I(q_{m(t)}) \to \infty$, a contradiction. Consequently, $q(\infty) = y$.

Similarly, we can prove that $q(-\infty) = x$. □

Lemma 2.6. *For every $y \in \mathcal{M} \setminus \{x\}$ there exists $q_{\varepsilon,y} \in \Gamma_\varepsilon(y)$ such that*

$$I(q_{\varepsilon,y}) = c_\varepsilon(y).$$

Proof. Let $\{q_m\}_{m=1}^\infty$ be a minimizing sequence for (5). Then there is $M > 0$ such that $I(q_m) \leqslant M$ for all $m \in \mathbb{N}$, and hence $\int_{-\infty}^{\infty} |\dot{q}_m(t)|^2\,dt \leqslant 2M$ for all $m \in \mathbb{N}$.

Assume that $\{q_m(0)\}_{m=1}^\infty$ is an unbounded sequence in $\mathbb{R}^n$. Then for each $k \in \mathbb{N}$ there is $m_k \in \mathbb{N}$ such that $|q_{m_k}(0)| \geqslant k$. Thus $\lim_{k\to\infty} |q_{m_k}(0)| = \infty$, and, in consequence, there is $k_0 \in \mathbb{N}$ such that if $k > k_0$ then $q_{m_k}(0) \notin B_\varepsilon(\mathcal{M})$. Set $t_k = \min\{t > 0\colon\ q_{m_k}(t) \in \partial B_\varepsilon(y)\}$. From Lemma 2.1 it follows that

$$I(q_{m(k)}) \geqslant \sqrt{2\alpha_\varepsilon}\,\big|q_{m(k)}(0) - q_{m(k)}(t_k)\big|$$

for $k > k_0$. Letting $k \to \infty$, $I(q_{m(k)}) \to \infty$, a contradiction. Therefore $\{q_m(0)\}_{m=1}^\infty$ is a bounded sequence in $\mathbb{R}^n$, which gives that $\{q_m\}_{m=1}^\infty$ is bounded in E. Since E is a Hilbert space, going to a subsequence if necessary, there is $q_{\varepsilon,y} \in E$ such that $q_m \rightharpoonup q_{\varepsilon,y}$ in E, and hence $q_m \to q_{\varepsilon,y}$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$.

We show now that $I(q_{\varepsilon,y}) < \infty$. Fix $-\infty < r < s < \infty$. Let us define $I_{r,s} : E \to \mathbb{R}$ as follows

$$I_{r,s}(w) = \int_r^s \left[\frac{1}{2}\big|\dot{w}(t)\big|^2 - V\big(t, w(t)\big)\right] dt. \tag{10}$$

$I_{r,s}$ is easily checked to be weakly lower semicontinuous. For every $m \in \mathbb{N}$

$$I_{r,s}(q_m) \leqslant I(q_m) \leqslant M.$$

Letting to the limit inferior, we receive

$$I_{r,s}(q_{\varepsilon,y}) \leqslant \liminf_{m\to\infty} I_{r,s}(q_m) \leqslant c_\varepsilon(y) = \lim_{m\to\infty} I(q_m) \leqslant M.$$

Since $q_{\varepsilon,y} \in E$ and r, s are arbitrary, $V(\cdot, q_{\varepsilon,y}(\cdot)) \in L^1(\mathbb{R}, \mathbb{R}^n)$ and $I(q_{\varepsilon,y}) \leqslant c_\varepsilon(y)$. Moreover, by Lemma 2.5, it follows that $q_{\varepsilon,y} \in \Gamma_\varepsilon(y)$, and consequently, $I(q_{\varepsilon,y}) = c_\varepsilon(y)$. □

Lemma 2.6 is an analogue of Proposition 3.12 of [6]. The idea to introduce the family of functionals $I_{r,s}$ is adapted from Rabinowitz. However, the second part of our proof involves Lemma 2.5 since the methods from [6] are not applicable.

For $y \in \mathcal{M} \setminus \{x\}$ and $0 < \varepsilon \leqslant \gamma$, let $\mathcal{F}(\varepsilon, y) = \{\sigma \in \mathbb{R}\colon q_{\varepsilon,y}(\sigma) \in \partial B_\varepsilon(\mathcal{M} \setminus \{x, y\})\}$.

Lemma 2.7. *For every $y \in \mathcal{M} \setminus \{x\}$, $q_{\varepsilon,y}$ is a classical solution of the system* (1) *on* $\mathbb{R} \setminus \mathcal{F}(\varepsilon, y)$.

The proof of Lemma 2.7 is based on the concept of Rabinowitz (see Proposition 3.18 of [6]).

Proof. Fix $\sigma \in \mathbb{R} \setminus \mathcal{F}(\varepsilon, y)$. Let $\Theta \subset \mathbb{R} \setminus \mathcal{F}(\varepsilon, y)$ be the maximal open interval such that $\sigma \in \Theta$. Assume that $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^n)$ and $\operatorname{supp}\varphi \subset \Theta$. If $\delta \in \mathbb{R}$ is sufficiently small then $q_{\varepsilon,y} + \delta\varphi \in \Gamma_\varepsilon(y)$, and so $I(q_{\varepsilon,y}) \leqslant I(q_{\varepsilon,y} + \delta\varphi)$. Consequently,

$$\begin{aligned} I'(q_{\varepsilon,y})\varphi &:= \lim_{\delta\to 0} \frac{I(q_{\varepsilon,y} + \delta\varphi) - I(q_{\varepsilon,y})}{\delta} \\ &= \int_{-\infty}^{\infty} \bigl[\bigl(\dot q_{\varepsilon,y}(t), \dot\varphi(t)\bigr) - \bigl(V_q\bigl(t, q_{\varepsilon,y}(t)\bigr), \varphi(t)\bigr)\bigr]\, dt = 0. \end{aligned}$$

Let $r, s \in \Theta$ and $r < s$. From the above it follows that for every $\varphi \in W_0^{1,2}([r, s], \mathbb{R}^n)$

$$\int_{-\infty}^{\infty} \bigl[\bigl(\dot q_{\varepsilon,y}(t), \dot\varphi(t)\bigr) - \bigl(V_q\bigl(t, q_{\varepsilon,y}(t)\bigr), \varphi(t)\bigr)\bigr]\, dt = 0, \tag{11}$$

and hence $q_{\varepsilon,y}|_{[r,s]}$ is a weak solution of the problem

$$\begin{cases} \ddot w(t) + V_q(t, q_{\varepsilon,y}(t)) = 0, & r < t < s, \\ w(r) = q_{\varepsilon,y}(r), \quad w(s) = q_{\varepsilon,y}(s). \end{cases} \tag{12}$$

This linear system has a unique C^2-solution $u : [r, s] \to \mathbb{R}^n$. By (12), we obtain

$$\int_r^s \bigl[\bigl(\dot u(t), \dot\varphi(t)\bigr) - \bigl(V_q\bigl(t, q_{\varepsilon,y}(t)\bigr), \varphi(t)\bigr)\bigr]\, dt = 0 \tag{13}$$

for all $\varphi \in W_0^{1,2}([r, s], \mathbb{R}^n)$. Combining (11) with (13) we receive

$$\int_r^s \bigl(\dot u(t) - \dot q_{\varepsilon,y}(t), \dot\varphi(t)\bigr)\, dt = 0$$

for all $\varphi \in W_0^{1,2}([r, s], \mathbb{R}^n)$. Since $u - q_{\varepsilon,y} \in W_0^{1,2}([r, s], \mathbb{R}^n)$,

$$\int_r^s \bigl|\dot u(t) - \dot q_{\varepsilon,y}(t)\bigr|^2\, dt = 0,$$

and hence $u \equiv q_{\varepsilon,y}$ on $[r, s]$. Therefore $q_{\varepsilon,y} \in C^2([r, s], \mathbb{R}^n)$. Summarizing, since r and s are arbitrary points in Θ, $q_{\varepsilon,y} \in C^2(\mathbb{R} \setminus \mathcal{F}(\varepsilon, y), \mathbb{R}^n)$ and it satisfies (1) on $\mathbb{R} \setminus \mathcal{F}(\varepsilon, y)$. □

Lemma 2.8. *Let M be a positive constant and y be a point in $\mathcal{M} \setminus \{x\}$. Then for each $q \in E$ joining x to y and satisfying $I(q) \leqslant M$ and for every $t \in \mathbb{R}$ such that $q(t) \notin \overline{B_\gamma(\mathcal{M})}$ the following inequality holds*

$$\left|q(t) - x\right| \leqslant \frac{3M}{\sqrt{2\alpha}} - \gamma. \tag{14}$$

Proof. Take $q \in E$ and $t \in \mathbb{R}$ satisfying the assumptions of our lemma. Set

$$t_0 = \max\left\{s \in \mathbb{R}\colon\ q(s) \in \partial B_\gamma(x) \wedge q\big((-\infty, s]\big) \cap \overline{B_\gamma\big(\mathcal{M} \setminus \{x\}\big)} = \emptyset\right\}.$$

Assume that $t_0 < t$. Since q is continuous, the set $q([t_0, t])$ is compact in $\mathbb{R}^n$, and hence there are finitely many points in $\mathbb{R}$:

$$t_0 < s_1 \leqslant t_1 < s_2 \leqslant t_2 < \cdots < s_{k-1} \leqslant t_{k-1} < s_k = t$$

such that for each $i = 1, 2, \ldots, k-1$:

- $q(s_i), q(t_i) \in \partial B_\gamma(\xi)$ for some $\xi \in \mathcal{M}$,
- if $q(t_{i-1}) \in \partial B_\gamma(\xi)$ and $q(s_i) \in \partial B_\gamma(\eta)$ then $\xi \neq \eta$,
- $q([t_{i-1}, s_i]) \cap B_\gamma(\mathcal{M}) = \emptyset$

and $q([t_{k-1}, s_k]) \cap B_\gamma(\mathcal{M}) = \emptyset$. We have

$$\left|q(t) - x\right| \leqslant \sum_{i=1}^{k} \left|q(s_i) - q(t_{i-1})\right| + (2k-1)\gamma. \tag{15}$$

From Lemma 2.1 it follows that

$$I(q) \geqslant \sqrt{2\alpha} \sum_{i=1}^{k} \left|q(s_i) - q(t_{i-1})\right| \geqslant k\gamma\sqrt{2\alpha}.$$

Hence

$$\sum_{i=1}^{k} \left|q(s_i) - q(t_{i-1})\right| \leqslant \frac{M}{\sqrt{2\alpha}} \tag{16}$$

and

$$k \leqslant \frac{M}{\gamma\sqrt{2\alpha}}. \tag{17}$$

Combining (16) and (17) with (15), we receive (14).

If $t < t_0$ then (14) is an immediate consequence of the inequality $M \geqslant \gamma\sqrt{2\alpha}$, which is clear from Lemma 2.1. $\square$

Here is an immediate consequence of Lemma 2.8.

Corollary 2.9. *Let M be a positive constant and y be a point in $\mathcal{M}\setminus\{x\}$. Then for each $q\in E$ joining x to y and satisfying $I(q)\leqslant M$ and for every $t\in\mathbb{R}$ the following inequality holds*

$$\left|q(t)-x\right|\leqslant\frac{3M}{\sqrt{2\alpha}}+\gamma. \tag{18}$$

For every $0<\varepsilon\leqslant\gamma$, set

$$c_\varepsilon=\inf\left\{c_\varepsilon(y):\ y\in\mathcal{M}\setminus\{x\}\right\}. \tag{19}$$

By Lemma 2.1, for each $y\in\mathcal{M}\setminus\{x\}$, $c_\varepsilon(y)>0$. Moreover, if $|y|\to\infty$ then $c_\varepsilon(y)\to\infty$. Hence there are $0<R_\varepsilon<\infty$ and $y_\varepsilon\in B_{R_\varepsilon}(x)$ such that $c_\varepsilon=c_\varepsilon(y_\varepsilon)$.

Let us consider a sequence $\{c_{\varepsilon_j}\}_{j=1}^\infty$ such that $\varepsilon_j\searrow 0$, as $j\to\infty$. Assume that $\varepsilon_1\leqslant\gamma$. It is easily seen that $c_{\varepsilon_j}\geqslant c_{\varepsilon_{j+1}}$ for every $j\in\mathbb{N}$. Choose $y_{\varepsilon_j}\in\mathcal{M}\setminus\{x\}$ and $q_{\varepsilon_j,y_{\varepsilon_j}}\in\varGamma_{\varepsilon_j}(y_{\varepsilon_j})$ such that $c_{\varepsilon_j}=c_{\varepsilon_j}(y_{\varepsilon_j})=I(q_{\varepsilon_j,y_{\varepsilon_j}})$. It follows from (18) that the sequence $\{y_{\varepsilon_j}\}_{j=1}^\infty$ is bounded. Consequently, $\{y_{\varepsilon_j}\}_{j=1}^\infty$ possesses a constant subsequence. Without loss of generality, we can assume that $\{y_{\varepsilon_j}\}_{j=1}^\infty$ is constant, and so there is $y\in\mathcal{M}\setminus\{x\}$ such that $y_{\varepsilon_j}=y$ for all $j\in\mathbb{N}$, and so $q_{\varepsilon_j,y_{\varepsilon_j}}=q_{\varepsilon_j,y}$. From the above, we have

$$c_{\varepsilon_j}=I(q_{\varepsilon_j,y}). \tag{20}$$

Lemma 2.10. *Let $q_j:=q_{\varepsilon_j,y}$. For $j\in\mathbb{N}$ large enough, q_j is a heteroclinic solution of* (1) *joining x to y.*

Proof. From what has already been proved, we see that it is sufficient to show that for $j\in\mathbb{N}$ large enough, $q_j(t)\notin\partial B_{\varepsilon_j}(\mathcal{M}\setminus\{x,y\})$ for all $t\in\mathbb{R}$.

On the contrary, suppose that there are a sequence $j_m\to\infty$, as $m\to\infty$, $\{t_{j_m}\}_{m=1}^\infty\subset\mathbb{R}$ and $\{\eta_{j_m}\}_{m=1}^\infty\subset\mathcal{M}\setminus\{x,y\}$ such that $q_{j_m}(t_{j_m})\in\partial B_{\varepsilon_{j_m}}(\eta_{j_m})$ and for all $t<t_{j_m}$, $q_{j_m}(t)\notin\partial B_{\varepsilon_{j_m}}(\eta_{j_m})$. By Lemma 2.1, we conclude that the set of values of $\{\eta_{j_m}\}_{m=1}^\infty$ is finite. Therefore, going if necessary to a subsequence, we can assume that $\eta_{j_m}=\eta$. Two cases are possible.

Case 1. There exists a subsequence of $\{j_m\}_{m=1}^\infty$, for simplicity of notation it is also denoted by $\{j_m\}_{m=1}^\infty$, such that $q_{j_m}(t)\notin\overline{B_{\varepsilon_{j_m}}(y)}$ for all $t<t_{j_m}$.

Let us consider

$$Q_{j_m}(t)=\begin{cases} q_{j_m}(t) & \text{if } t\leqslant t_{j_m},\\ q_{j_m}(t_{j_m})+(t-t_{j_m})\varepsilon_{j_m}^{-1}(\eta-q_{j_m}(t_{j_m})) & \text{if } t_{j_m}<t\leqslant t_{j_m}+\varepsilon_{j_m},\\ \eta & \text{if } t>t_{j_m}+\varepsilon_{j_m}.\end{cases}$$

One can see that $Q_{j_m}\in\varGamma_{\varepsilon_{j_m}}(\eta)$. An easy computation shows that

$$I(q_{j_m})-I(Q_{j_m})=\int_{t_{j_m}}^{\infty}\left[\frac{1}{2}\left|\dot q_{j_m}(t)\right|^2-V\big(t,q_{j_m}(t)\big)\right]dt$$
$$-\frac{1}{2}\varepsilon_{j_m}-\int_{t_{j_m}}^{t_{j_m}+\varepsilon_{j_m}}-V\big(t,Q_{j_m}(t)\big)\,dt-\int_{t_{j_m}+\varepsilon_{j_m}}^{\infty}-V(t,\eta)\,dt.$$

By Lemma 2.1, we conclude

$$I(q_{j_m}) - I(Q_{j_m}) \geqslant \gamma\sqrt{2\alpha} - \frac{1}{2}\varepsilon_{j_m} - \int_{t_{j_m}}^{t_{j_m}+\varepsilon_{j_m}} -V\bigl(t, Q_{j_m}(t)\bigr)\,dt - \int_{-\infty}^{\infty} -V(t,\eta)\,dt. \quad (21)$$

We will show that $\{t_{j_m}\}_{m=1}^{\infty}$ is bounded. To obtain a contradiction, suppose that $\{t_{j_m}\}_{m=1}^{\infty}$ is unbounded. Assume that $\{t_{j_m}\}_{m=1}^{\infty}$ is unbounded from above. Let $r_{j_m} = \max\{t \in \mathbb{R}\colon\ q_{j_m}(t) \in \partial B_\gamma(\eta)\}$ and $s_{j_m} = \min\{t > r_{j_m}\colon\ q_{j_m}(t) \in \partial B_\gamma(y)\}$. From Lemma 2.3

$$s_{j_m} - r_{j_m} \geqslant \frac{\gamma^2}{2c_{\varepsilon_{j_m}}} \geqslant \frac{\gamma^2}{2c_{\varepsilon_1}} > 0.$$

We have

$$c_{\varepsilon_{j_m}} = I(q_{j_m}) \geqslant \int_{r_{j_m}}^{s_{j_m}} -V\bigl(t, q_{j_m}(t)\bigr)\,dt.$$

Applying the mean value theorem, there is $\hat{t}_{j_m} \in (r_{j_m}, s_{j_m})$ such that

$$\int_{r_{j_m}}^{s_{j_m}} -V\bigl(t, q_{j_m}(t)\bigr)\,dt = (s_{j_m} - r_{j_m})\bigl(-V\bigl(\hat{t}_{j_m}, q_{j_m}(\hat{t}_{j_m})\bigr)\bigr),$$

and hence

$$c_{\varepsilon_{j_m}} \geqslant \frac{\gamma^2}{2c_{\varepsilon_1}}\bigl(-V\bigl(\hat{t}_{j_m}, q_{j_m}(\hat{t}_{j_m})\bigr)\bigr).$$

By Corollary 2.9, we conclude that $\{q_{j_m}(\hat{t}_{j_m})\}_{m=1}^{\infty}$ is a bounded sequence. Since $\hat{t}_{j_m} \to \infty$, as $m \to \infty$, from the above inequality and (A_4) it follows that $c_{\varepsilon_{j_m}} \to \infty$, as $m \to \infty$, a contradiction. In consequence, $\{t_{j_m}\}_{m=1}^{\infty}$ is bounded from above.

In the same manner we can see that $\{t_{j_m}\}_{m=1}^{\infty}$ is bounded from below. It is sufficient to consider $\hat{s}_{j_m} = \max\{t \in \mathbb{R}\colon\ q_{j_m}(t) \in \partial B_\gamma(x)\}$ and $\hat{r}_{j_m} = \min\{t > \hat{s}_{j_m}\colon\ q_{j_m}(t) \in \partial B_\gamma(\eta)\}$.

For m sufficiently large,

$$\begin{aligned} 0 &< \frac{1}{2}\varepsilon_{j_m} + \int_{t_{j_m}}^{t_{j_m}+\varepsilon_{j_m}} -V\bigl(t, Q_{j_m}(t)\bigr)\,dt \\ &< \frac{1}{2}\left(\gamma\sqrt{2\alpha} - \int_{-\infty}^{\infty} -V(t,\eta)\,dt\right). \end{aligned}$$

Hence

$$I(q_{j_m}) - I(Q_{j_m}) > \frac{1}{2}\left(\gamma\sqrt{2\alpha} - \int_{-\infty}^{\infty} -V(t,\eta)\,dt\right)$$

and

$$I(q_{j_m}) > I(Q_{j_m}),$$

by (21) and (A_5). A contradiction with (20).

Case 2. For every $m \in \mathbb{N}$ there is $\tau_{j_m} < t_{j_m}$ such that $q_{j_m}(\tau_{j_m}) \in \partial B_{\varepsilon_{j_m}}(y)$.

Let us consider now

$$\hat{Q}_{j_m}(t) = \begin{cases} q_{j_m}(t) & \text{if } t \leqslant \tau_{j_m}, \\ q_{j_m}(\tau_{j_m}) + (t-\tau_{j_m})\varepsilon_{j_m}^{-1}(y - q_{j_m}(\tau_{j_m})) & \text{if } \tau_{j_m} < t \leqslant \tau_{j_m} + \varepsilon_{j_m}, \\ y & \text{if } t > \tau_{j_m} + \varepsilon_{j_m}. \end{cases}$$

In this case,

$$I(q_{j_m}) - I(\hat{Q}_{j_m}) \geqslant \gamma\sqrt{2\alpha} - \frac{1}{2}\varepsilon_{j_m} - \int_{\tau_{j_m}}^{\tau_{j_m}+\varepsilon_{j_m}} -V\big(t, \hat{Q}_{j_m}(t)\big)\,dt - \int_{-\infty}^{\infty} -V(t,y)\,dt. \tag{22}$$

Using the analogical arguments as in the first case, we have $I(q_{j_m}) > I(\hat{Q}_{j_m})$, a contradiction with (20). $\square$

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