Discrete Mathematics 339 (2016) 1878–1883

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Independence in uniform linear triangle-free hypergraphs

Piotr Borowiecki [a], Michael Gentner [b], Christian Löwenstein [b], Dieter Rautenbach [b,*]

[a] *Department of Algorithms and System Modeling, Faculty of Electronics, Telecommunications and Informatics, Gdańsk University of Technology, 80-233 Gdańsk, Poland*

[b] *Institute of Optimization and Operations Research, Ulm University, D-89069 Ulm, Germany*

ARTICLE INFO

Article history:
Received 2 October 2014
Accepted 5 January 2016
Available online 1 February 2016

Keywords:
Independence
Hypergraph
Linear
Uniform
Double linear
Triangle-free

ABSTRACT

The independence number $\alpha(H)$ of a hypergraph H is the maximum cardinality of a set of vertices of H that does not contain an edge of H. Generalizing Shearer's classical lower bound on the independence number of triangle-free graphs Shearer (1991), and considerably improving recent results of Li and Zang (2006) and Chishti et al. (2014), we show that

$$\alpha(H) \geq \sum_{u\in V(H)} f_r(d_H(u))$$

for an r-uniform linear triangle-free hypergraph H with $r \geq 2$, where

$$f_r(0) = 1, \quad \text{and}$$

$$f_r(d) = \frac{1 + \big((r-1)d^2 - d\big)f_r(d-1)}{1 + (r-1)d^2} \quad \text{for } d \geq 1.$$

1. Introduction

We consider finite *hypergraphs* H, which are ordered pairs $(V(H), E(H))$ of two sets, where $V(H)$ is the finite set of *vertices of* H and $E(H)$ is the *set of edges of* H, which are subsets of $V(H)$. The *order* $n(H)$ *of* H is the cardinality of $V(H)$. The *degree* $d_H(u)$ *of a vertex* u *of* H is the number of edges of H that contain u. The *average degree* $d(H)$ *of* H is the arithmetic mean of the degrees of its vertices. Two distinct vertices of H are *adjacent* or *neighbors* if some edge of H contains both. The *neighborhood* $N_H(u)$ *of a vertex* u *of* H is the set of vertices of H that are adjacent to u. For a set X of vertices of H, the hypergraph $H - X$ arises from H by removing from $V(H)$ all vertices in X and removing from $E(H)$ all edges that intersect X. If every two distinct edges of H share at most one vertex, then H is *linear*. If H is linear and for every two distinct non-adjacent vertices u and v of H, every edge of H that contains u contains at most one neighbor of v, then H is *double linear*. If there are not three distinct vertices u_1, u_2, and u_3 of H and three distinct edges e_1, e_2, and e_3 of H such that $\{u_1, u_2, u_3\} \setminus \{u_i\} \subseteq e_i$ for $i \in \{1, 2, 3\}$, then H is *triangle-free*. A set I of vertices of H is a *(weak) independent set of* H if no edge of H is contained in I. The *(weak) independence number* $\alpha(H)$ *of* H is the maximum cardinality of an independent set of H. If all edges of H have cardinality r, then H is r*-uniform*. If H is 2-uniform, then H is referred to as a *graph*.

* Corresponding author.
E-mail addresses: pborowie@eti.pg.gda.pl (P. Borowiecki), michael.gentner@uni-ulm.de (M. Gentner), christian.loewenstein@uni-ulm.de (C. Löwenstein), dieter.rautenbach@uni-ulm.de (D. Rautenbach).

http://dx.doi.org/10.1016/j.disc.2016.01.006
0012-365X/

The independence number of (hyper)graphs is a well studied computationally hard parameter. Caro [4] and Wei [14] proved a classical lower bound on the independence number of graphs, which was extended to hypergraphs by Caro and Tuza [5]. Specifically, for an r-uniform hypergraph H, Caro and Tuza [5] proved

$$\alpha(H) \geq \sum_{u \in V(H)} f_{CT(r)}(d_H(u)),$$

where $f_{CT(r)}(d) = \binom{d+\frac{1}{r-1}}{d}^{-1}$. Thiele [13] generalized Caro and Tuza's bound to general hypergraphs; see [3] for a very simple probabilistic proof of Thiele's bound. Originally motivated by Ramsey theory, Ajtai et al. [2] showed that $\alpha(G) = \Omega\left(\frac{\ln d(G)}{d(G)} n(G)\right)$ for every triangle-free graph G. Confirming a conjecture from [2] concerning the implicit constant, Shearer [11] improved this bound to $\alpha(H) \geq f_{S_1}(d(G))n(G)$, where $f_{S_1}(d) = \frac{d \ln d - d + 1}{(d-1)^2}$. In [11] the function f_{S_1} arises as a solution of the differential equation

$$(d+1)f(d) = 1 + (d - d^2)f'(d) \quad \text{and} \quad f(0) = 1.$$

In [12] Shearer showed that

$$\alpha(G) \geq \sum_{u \in V(G)} f_{S_2}(d_G(u))$$

for every triangle-free graph G, where f_{S_2} solves the difference equation

$$(d+1)f(d) = 1 + (d - d^2)\Big(f(d) - f(d-1)\Big) \quad \text{and} \quad f(0) = 1.$$

Since $f_{S_1}(d) \leq f_{S_2}(d)$ for every non-negative integer d, and f_{S_1} is convex, Shearer's bound from [12] is stronger than his bound from [11].

Li and Zang [10] adapted Shearer's approach to hypergraphs and obtained the following.

Theorem 1 (*Li and Zang [10]*). *Let r and m be positive integers with $r \geq 2$.*

If H is an r-uniform double linear hypergraph such that the maximum degree of every subhypergraph of H induced by the neighborhood of a vertex of H is less than m, then

$$\alpha(H) \geq \sum_{u \in V(H)} f_{LZ(r,m)}(d_H(u)),$$

where

$$f_{LZ(r,m)}(x) = \frac{m}{B} \int_0^1 \frac{(1-t)^{\frac{a}{m}}}{t^b(m-(x-m)t)} dt,$$

$a = \frac{1}{(r-1)^2}$, $b = \frac{r-2}{r-1}$, *and* $B = \int_0^1 (1-t)^{(\frac{a}{m}-1)} t^{-b} dt$.

Note that for $r \geq 2$, an r-uniform linear hypergraph H is triangle-free if and only if it is double linear and the maximum degree of every subhypergraph of H induced by the neighborhood of a vertex of H is less than 1. Therefore, since $f_{S_1} = f_{LZ(2,1)}$ and f_{S_1} is convex, Theorem 1 implies Shearer's bound from [11]. Nevertheless, since $f_{S_1}(d) < f_{S_2}(d)$ for every integer d with $d \geq 2$, Shearer's bound from [12] does not quite follow from Theorem 1.

In [6] Chishti et al. presented another version of Shearer's bound from [11] for hypergraphs.

Theorem 2 (*Chishti et al. [6]*). *Let r be an integer with $r \geq 2$.*

If H is an r-uniform linear triangle-free hypergraph, then

$$\alpha(H) \geq f_{CZPI(r)}(d(H))n(H),$$

where

$$f_{CZPI(r)}(x) = \frac{1}{r-1} \int_0^1 \frac{1-t}{t^b(1-((r-1)x-1)t)} dt$$

and $b = \frac{r-2}{r-1}$.

Since $f_{S_1} = f_{CZPI(2)}$, for $r = 2$, the last result coincides with Shearer's bound from [11].

A drawback of the bounds in Theorems 1 and 2 is that they are very often weaker than Caro and Tuza's bound [5], which holds for a more general class of hypergraphs. See Fig. 1 for an illustration.

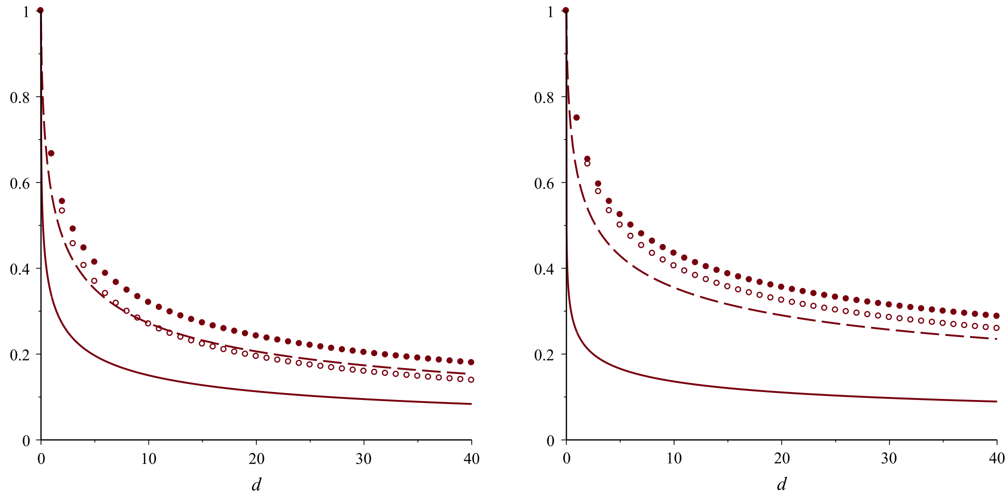

Fig. 1. The values of $f_{LZ(r,1)}(d)$ (line), $f_{CZPI(r)}(d)$ (dashed line), $f_{CT(r)}(d)$ (empty circles), and $f_r(d)$ (solid circles) for $0 \leq d \leq 40$ and $r = 3$ (left) and $r = 4$ (right).

In the present paper we extend Shearer's approach from [12] and establish a lower bound on the independence number of a uniform linear triangle-free hypergraph that considerably improves Theorems 1 and 2 and is systematically better than Caro and Tuza's bound.

For further related results we refer to Ajtai et al. [1], Duke et al. [7], Dutta et al. [8] and Kostochka et al. [9]. Note that our main result provides explicit values when applied to a specific hypergraph but that we do not completely understand its asymptotics. In contrast to that, results as in [1,7,8] are essentially asymptotic statements but are of limited value when applied to a specific hypergraph.

2. Results

For an integer r with $r \geq 2$, let $f_r : \mathbb{N}_0 \to \mathbb{R}_0$ be such that

$$f_r(0) = 1 \quad \text{and}$$

$$f_r(d) = \frac{1 + \Big((r-1)d^2 - d\Big) f_r(d-1)}{1 + (r-1)d^2}$$

for every positive integer d.

Lemma 3. *If r and d are integers with $r \geq 2$ and $d \geq 0$, then $f_r(d) - f_r(d+1) \geq f_r(d+1) - f_r(d+2)$.*

Proof. Substituting within the inequality $f_r(d) - 2f_r(d+1) + f_r(d+2) \geq 0$ first $f_r(d+2)$ with

$$\frac{1 + \Big((r-1)(d+2)^2 - (d+2)\Big) f_r(d+1)}{1 + (r-1)(d+2)^2}$$

and then $f_r(d+1)$ with

$$\frac{1 + \Big((r-1)(d+1)^2 - (d+1)\Big) f_r(d)}{1 + (r-1)(d+1)^2},$$

and solving it for $f_r(d)$, it is straightforward but tedious to verify that it is equivalent to $f_r(d) \geq L(r, d)$ where

$$L(r, d) = \frac{(2r-1)d + 3r}{r(d^2 + 5d + 5)}.$$

Therefore, in order to complete the proof, it suffices to show $f_r(d) \geq L(r, d)$. For $d = 0$, we have $f_r(0) = 1 > \frac{3}{5} = L(r, 0)$. Now, let $f(d) \geq L(r, d)$ for some non-negative integer d. Since $(r-1)(d+1)^2 - (d+1) \geq 0$, we obtain by a straightforward yet tedious calculation

$$f(d+1) - L(r, d+1) = \frac{1 + \big((r-1)(d+1)^2 - (d+1)\big) f(d)}{1 + (r-1)(d+1)^2} - L(r, d+1)$$

$$\geq \frac{\left(1+\left((r-1)(d+1)^2-(d+1)\right)L(r,d)\right)}{1+(r-1)(d+1)^2} - L(r,d+1)$$
$$= \frac{2(1+(r-1)(d+2)^2)}{r(d^2+7d+11)(d^2+5d+5)},$$

which is positive for $r \geq 2$. Therefore, $f(d+1) \geq L(r,d+1)$, which completes the proof by an inductive argument. □

The following is our main result.

Theorem 4. *Let r be an integer with $r \geq 2$.*
If H is an r-uniform linear triangle-free hypergraph, then

$$\alpha(H) \geq \sum_{u \in V(H)} f_r(d_H(u)).$$

Before we proceed to the proof, we compare our bound to the bounds of Caro and Tuza [5], Li and Zang [10], and Chishti et al. [6]. Fig. 1 illustrates some specific values. An inspection of Li and Zang's proof in [10] reveals that they actually prove a lower bound on the so-called *strong independence number*, which is defined as the maximum cardinality of a set of vertices that does not contain two adjacent vertices. Therefore, especially for large values of r, Theorem 1 is much weaker than Theorem 2. In fact, it is quite natural that it is worse by a factor of about $r-1$.

As we show now, our bound is systematically better than Caro and Tuza's bound [5].

Lemma 5. *If r and d are integers with $r \geq 3$ and $d \geq 2$, then $f_r(d) > f_{CT(r)}(d)$.*

Proof. Note that $f_r(0) = f_{CT(r)}(0) = 1$, $f_r(1) = f_{CT(r)}(1) = \frac{r-1}{r}$, and $f_{CT(r)}(d) = \frac{d}{d+\frac{1}{r-1}} f_{CT(r)}(d-1)$ for $d \in \mathbb{N}$, which immediately implies that $f_{CT(r)}(d) < \frac{r-1}{r}$ for $d \geq 2$. Now, if $f_r(d-1) \geq f_{CT(r)}(d-1)$ for some $d \geq 2$, then

$$\begin{aligned}
f_r(d) - f_{CT(r)}(d) &= \frac{1+\left((r-1)d^2-d\right)f_r(d-1)}{1+(r-1)d^2} - f_{CT(r)}(d) \\
&\geq \frac{1+\left((r-1)d^2-d\right)f_{CT(r)}(d-1)}{1+(r-1)d^2} - f_{CT(r)}(d) \\
&= \frac{1+\left((r-1)d^2-d\right)\frac{1+(r-1)d}{(r-1)d} f_{CT(r)}(d)}{1+(r-1)d^2} - f_{CT(r)}(d) \\
&= \frac{1-\frac{r}{r-1} f_{CT(r)}(d)}{1+(r-1)d^2} \\
&> 0,
\end{aligned}$$

that is, $f_r(d) > f_{CT(r)}(d)$, which completes the proof by an inductive argument. □

For $r = 2$, Lemma 5 would state that Shearer's bound [12] is better than Caro [4] and Wei's bound [14], which is known.
We proceed to the proof of Theorem 4.

Proof of Theorem 4. We prove the statement by induction on $n(H)$. If H has no edge, then $\alpha(H) = n(H)$, which implies the desired result for $n(H) \leq r-1$. Now let $n(H) \geq r$. If H has a vertex x with $d_H(x) = 0$, then $f_r(d_H(x)) = 1$ and, by induction,

$$\alpha(H) \geq 1 + \alpha(H-x) \geq f_r(d_H(x)) + \sum_{u \in V(H)\setminus\{x\}} f_r(d_{H-x}(u)) = \sum_{u \in V(H)} f_r(d_H(u)).$$

Hence we may assume that H has no vertex of degree 0.

Since H is r-uniform and linear, for every two edges e_1 and e_2 with $e_1 \cap e_2 = \{u\}$ for some vertex u of H, the sets $e_1 \setminus \{u\}$ and $e_2 \setminus \{u\}$ are disjoint and of order $r-1$. Therefore, for every vertex u of H, there is a set $\mathcal{R}(u)$ of $r-1$ sets of neighbors of u such that every neighbor of u belongs to exactly one of the sets in $\mathcal{R}(u)$, and $|e \cap R| = 1$ for every edge e of H with $u \in e$ and every $R \in \mathcal{R}(u)$.

If x is a vertex of H and $R \in \mathcal{R}(x)$ is such that

$$1 + \sum_{u \in V(H)\setminus(\{x\}\cup R)} f_r(d_{H-(\{x\}\cup R)}(u)) \geq \sum_{u \in V(H)} f_r(d_H(u)),$$

then the statement follows by induction, because $\alpha(H) \geq 1 + \alpha(H - (\{x\} \cup R))$. Therefore, in order to complete the proof, it suffices to show that the following term is non-negative:

$$P = \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \left(1 + \sum_{u \in V(H) \setminus (\{x\} \cup R)} f_r(d_{H-(\{x\} \cup R)}(u)) - \sum_{u \in V(H)} f_r(d_H(u))\right).$$

Since H is linear and triangle-free, we have $d_{H-(\{x\}\cup R)}(z) = d_H(z) - |N_H(z) \cap R|$ for every vertex z in $V(H) \setminus (\{x\} \cup R)$. Trivially, $d_{H-(\{x\}\cup R)}(z) = d_H(z)$ for $z \notin N_H(R)$, and hence P equals $P_1 + P_2$, where

$$P_1 = \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \left(1 - f_r(d_H(x)) - \sum_{y \in R} f_r(d_H(y))\right) \quad \text{and}$$

$$P_2 = \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{z \in N_H(R) \setminus \{x\}} \Big(f_r(d_H(z) - |N_H(z) \cap R|) - f_r(d_H(z))\Big).$$

Since for every vertex u of H, there are exactly $(r-1)d_H(u)$ many vertices v of H such that u belongs to exactly one of the sets in $\mathcal{R}(v)$, we have

$$P_1 = \sum_{x \in V(H)} \Big((r-1) - (r-1)(d_H(x)+1)f_r(d_H(x))\Big).$$

Since $f_r(d-1) - f_r(d)$ is decreasing by Lemma 3, we have $f_r(d-n) - f_r(d) \geq n(f_r(d-1) - f_r(d))$ for all positive integers d and n with $n < d$. Therefore,

$$\begin{aligned}
P_2 &\geq \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{z \in N_H(R) \setminus \{x\}} |N_H(z) \cap R| \Big(f_r(d_H(z)-1) - f_r(d_H(z))\Big) \\
&= \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{z \in N_H(R) \setminus \{x\}} \sum_{y \in R} |N_H(z) \cap \{y\}| \Big(f_r(d_H(z)-1) - f_r(d_H(z))\Big) \\
&= \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{y \in R} \sum_{z \in N_H(R) \setminus \{x\}} |N_H(z) \cap \{y\}| \Big(f_r(d_H(z)-1) - f_r(d_H(z))\Big) \\
&= \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{y \in R} \sum_{z \in N_H(y) \setminus \{x\}} \Big(f_r(d_H(z)-1) - f_r(d_H(z))\Big).
\end{aligned}$$

Let T be the set of all 4-tuples (x, R, y, z) with $x \in V(H)$, $R \in \mathcal{R}(x)$, $y \in R$, and $z \in N_H(y) \setminus \{x\}$. Note that $y \in N_H(z)$ for every (x, R, y, z) in T. Since H is linear, for a given vertex z of H and a given neighbor y of z, there are $(r-1)d_H(y) - 1$ many vertices x of H with $y \in R$ for some R in $\mathcal{R}(x)$ and $z \in N_H(y) \setminus \{x\}$. Furthermore, by the properties of $\mathcal{R}(x)$, given x and y, the set R in $\mathcal{R}(x)$ with $y \in R$ is unique. Therefore,

$$\begin{aligned}
P_2 &\geq \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{y \in R} \sum_{z \in N_H(y) \setminus \{x\}} \Big(f_r(d_H(z)-1) - f_r(d_H(z))\Big) \\
&= \sum_{z \in V(H)} \sum_{y \in N_H(z)} \Big((r-1)d_H(y) - 1\Big)\Big(f_r(d_H(z)-1) - f_r(d_H(z))\Big).
\end{aligned}$$

Let $\mathcal{E}$ be the edge set of the graph that arises from H by replacing every edge of H by a clique, that is, $\mathcal{E}$ is the set of all sets containing exactly two adjacent vertices of H.

We obtain

$$\begin{aligned}
P_2 &\geq \sum_{z \in V(H)} \sum_{y \in N_H(z)} \Big((r-1)d_H(y) - 1\Big)\Big(f_r(d_H(z)-1) - f_r(d_H(z))\Big) \\
&= \sum_{\{y,z\} \in \mathcal{E}} \Big(h_1(y)h_2(z) + h_1(z)h_2(y)\Big), \quad \text{where}
\end{aligned}$$

$$h_1(x) = (r-1)d_H(x) - 1 \quad \text{and}$$

$$h_2(x) = f_r(d_H(x)-1) - f_r(d_H(x)).$$

If $d_H(y) \geq d_H(z)$, then $h_1(y) \geq h_1(z)$ and, by Lemma 3, $h_2(z) \geq h_2(y)$, which implies

$$\Big(h_1(y) - h_1(z)\Big)\Big(h_2(z) - h_2(y)\Big) \geq 0.$$

Therefore, $h_1(y)h_2(z) + h_1(z)h_2(y) \geq h_1(y)h_2(y) + h_1(z)h_2(z)$.

Since, for every vertex y of H, there are exactly $(r-1)d_H(y)$ many vertices z of H with $\{y,z\} \in \mathcal{E}$, we obtain

$$
\begin{aligned}
P_2 &\geq \sum_{\{y,z\}\in\mathcal{E}} \Big(h_1(y)h_2(z) + h_1(z)h_2(y)\Big) \\
&\geq \sum_{\{y,z\}\in\mathcal{E}} \Big(h_1(y)h_2(y) + h_1(z)h_2(z)\Big) \\
&= \sum_{x\in V(H)} (r-1)d_H(x)h_1(x)h_2(x) \\
&= \sum_{x\in V(H)} (r-1)d_H(x)\Big((r-1)d_H(x)-1\Big)\Big(f_r(d_H(x)-1) - f_r(d_H(x))\Big).
\end{aligned}
$$

Combining these estimates, we see that

$$
\begin{aligned}
P &= P_1 + P_2 \\
&\geq \sum_{x\in V(H)} \Big((r-1) - (r-1)(d_H(x)+1)f_r(d_H(x)) \\
&\quad + (r-1)d_H(x)\Big((r-1)d_H(x)-1\Big)\Big(f_r(d_H(x)-1) - f_r(d_H(x))\Big)\Big),
\end{aligned}
$$

which is 0 by the definition of f_r. This completes the proof. □

It seems a challenging task to extend the presented results to non-uniform and/or non-linear triangle-free hypergraphs.

Acknowledgment

The first author has been partially supported by National Science Centre under contract DEC-2011/02/A/ST6/00201.

References

[1] M. Ajtai, J. Komlos, J. Pintz, J. Spencer, E. Szemeredi, Extremal uncrowded hypergraphs, J. Comb. Theory, Ser. A 32 (1982) 321–335.
[2] M. Ajtai, J. Komlós, E. Szemerédi, A note on Ramsey numbers, J. Comb. Theory, Ser. A 29 (1980) 354–360.
[3] P. Borowiecki, F. Göring, J. Harant, D. Rautenbach, The potential of greed for independence, J. Graph Theory 71 (2012) 245–259.
[4] Y. Caro, New Results on the Independence Number, Technical Report, Tel-Aviv University, 1979.
[5] Y. Caro, Zs. Tuza, Improved lower bounds on k-independence, J. Graph Theory 15 (1991) 99–107.
[6] T.A. Chishti, G. Zhou, S. Pirzada, A. Iványi, On vertex independence number of uniform hypergraphs, Acta Univ. Sapientiae, Informatica 6 (2014) 132–158.
[7] R. Duke, H. Lefmann, V. Rödl, On uncrowded hypergraphs, random struct, Algorithms 6 (1995) 209–212.
[8] K. Dutta, D. Mubayi, C.R. Subramanian, New lower bounds for the independence number of sparse graphs and hypergraphs, SIAM J. Discrete Math. 26 (2012) 1134–1147.
[9] A. Kostochka, D. Mubayi, J. Verstraëte, On independent sets in hypergraphs, Random Structures Algorithms 44 (2014) 224–239.
[10] Y. Li, W. Zang, Differential methods for finding independent sets in hypergraphs, SIAM J. Discrete Math. 20 (2006) 96–104.
[11] J.B. Shearer, A note on the independence number of triangle-free graphs, Discrete Math. 46 (1983) 83–87.
[12] J.B. Shearer, A note on the independence number of triangle-free graphs. II, J. Combin. Theory Ser. B 53 (1991) 300–307.
[13] T. Thiele, A lower bound on the independence number of arbitrary hypergraphs, J. Graph Theory 30 (1999) 213–221.
[14] V.K. Wei, A Lower Bound on the Stability Number of a Simple Graph, Technical memorandum, TM 81 - 11217 - 9, Bell Laboratories, 1981.