AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **53** (2012), Pages 19–30

Influence of edge subdivision on the convex domination number

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Abstract

We study the influence of edge subdivision on the convex domination number. We show that in general an edge subdivision can arbitrarily increase and arbitrarily decrease the convex domination number. We also find some bounds for unicyclic graphs and we investigate graphs G for which the convex domination number changes after subdivision of any edge in G.

1 Introduction

Let $G = (V, E)$ be a connected undirected graph with $|V| = n$. The *neighbourhood* of a vertex $v \in V$ in G is the set $N_G(v)$ of all vertices adjacent to v in G. For a set $X \subseteq V$, *the open neighbourhood* $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$ and *the closed neighbourhood* $N_G[X] = N_G(X) \cup X$.

The *degree* $d_G(v) = d(v)$ of a vertex $v \in V$ is the number of edges incident to v; $d_G(v) = |N_G(v)|$. The minimum and maximum degrees among all vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A vertex u of degree $d(u) = 1$ we call an end-vertex. A *support* is a vertex adjacent to an end-vertex. A set of all end-vertices of a graph G we denote by $\Omega(G)$ and a set of all supports of G by $S(G)$. If G is connected and $\delta(G) = \Delta(G) = 2$, then G is a cycle and the cycle on n vertices is denoted by C_n. The length of a shortest cycle in G is the *girth* of G and is denoted by $g(G)$.

A set $D \subseteq V$ is a *dominating set* of G if $N_G[D] = V$. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set in G. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of the shortest $(u-v)$ path in G. A $(u-v)$ path of length $d_G(u, v)$ is called a $(u-v)$-*geodesic*. For unexplained terms and symbols see [6].

A set $X \subseteq V$ is *convex* in G if vertices from all $(a-b)$-geodesics belong to X for every two vertices $a, b \in X$. A set X is a *convex dominating set* if it is convex and dominating. The *convex domination number* $\gamma_{con}(G)$ of a graph G is the minimum cardinality of a convex dominating set. The convex domination number was first introduced by Jerzy Topp, Gdańsk University of Technology (2002) and studied in [1], [7], [8].

The *subdivision* of some edge $e = uv$ in a graph G yields a graph containing one new vertex w, and with an edge set replacing e by two new edges with endpoints uw and wv. Let us denote by G_{uv} or G_e the graph obtained from a graph G by subdivision of an edge $e = uv$ in G. The *domination subdivision number* $\mathrm{sd}_\gamma(G)$ of a graph G is the minimum number of edges that must be subdivided, where each edge in G can be subdivided at most once, in order to increase the domination number. The domination subdivision number was defined in [10] and has been studied also in [3], [4]. The similar concept related to total domination was defined in [5]. Moreover, the weakly connected domination subdivision number was defined in [9] and studied in [2]. In this paper we study similar concepts related to convex domination. Since an edge subdivision can arbitrarily increase and arbitrarily decrease the convex domination number (see Theorem 1), we do not define the subdivision number for convex domination, we just consider the influence of the edge subdivision on the convex domination number and we investigate graphs G for which the convex domination number changes after subdivision of any edge in G.

2 Results

We show that an edge subdivision can arbitrarily increase and arbitrarily decrease the convex domination number.

Theorem 1 *The difference between $\gamma_{con}(G)$ and $\gamma_{con}(G_{uv})$ and between $\gamma_{con}(G_{uv})$ and $\gamma_{con}(G)$ can be arbitrarily large.*

Proof.

First we show that $\gamma_{con}(G_{uv}) - \gamma_{con}(G)$ can be arbitrarily large. We show that for any positive integer k there exists a graph G such that $\gamma_{con}(G_{uv}) - \gamma_{con}(G) = k$. We shall construct a graph G in a following way: we begin with $\overline{K_{k-1}} + K_2$, where $\{u, v\}$ are two vertices belonging to K_2. Then we add two vertices x, y and two edges xu, yv. It is easy to observe that $\{u, v\}$ is a minimum convex dominating set of G and in such a graph G is $\gamma_{con}(G) = 2$.

If we subdivide an edge uv, then we obtain the graph G_{uv}, where $V(G_{uv}) - \Omega(G_{uv})$ is a minimum convex dominating set of G_{uv}. Thus $\gamma_{con}(G_{uv}) = k+2$ and $\gamma_{con}(G_{uv}) - \gamma_{con}(G) = k$.

Figure 1 shows G and G_{uv} for $k = 4$.

Now we show that the difference $\gamma_{con}(G) - \gamma_{con}(G_{uv})$ can be arbitrarily large. We construct a graph G in a following way: we begin with a cycle C_{2k+2} with $2k+2$ vertices and label the vertices of the cycle consecutively $v_1, \ldots, v_k, z, w_k, w_{k-1}, \ldots, w_1, u$.

Then we add the edges v_iw_i for $i = 1, \ldots, k$ and finally we add two vertices x, y and two edges xu, yz.

Again it is easy to observe that in such a graph G, $V(G) - \Omega(G)$ is a minimum convex dominating set of G and $\gamma_{con}(G) = 2k + 2$. The graph G_{uw_1} we obtain by subdivision of an edge uw_1. In this graph, the minimum convex dominating set is the set of supports of G_{uw_1} together with vertices $v_1 \ldots, v_k$. Thus $\gamma_{con}(G_{uw_1}) = k+2$ and $\gamma_{con}(G) - \gamma_{con}(G_{uw_1}) = k$. Figure 2 shows the graphs G and G_{uw_1} for $k = 3$.

■

We have shown that in general the differences between $\gamma_{con}(G)$ and $\gamma_{con}(G_{uv})$ and between $\gamma_{con}(G_{uv})$ and $\gamma_{con}(G)$ can be arbitrarily large. But there are some classes of graphs where we can find some bounds.

We begin with the following observation.

Observation 1 *If G is a unicyclic graph with the unique cycle C and if D is a minimum convex dominating set of G, then at most two vertices of C do not belong to D.*

Proof. Suppose to the contrary that three or more vertices of C do not belong to D; say $c_1, \ldots, c_k$ do not belong to $D, k \geq 3$. Since G is unicyclic and D is convex, $c_1, \ldots, c_k$ are consecutive vertices. Since D is dominating, every c_i has a neighbour in D. If two c_i have the same neighbour in D, then $C = C_3$ and $k = 2$, a contradiction. Thus every c_i has a different neighbour in D.

Let $c'_i \in N_G(c_i) \cap D$, $c'_{i+1} \in N_G(c_{i+1}) \cap D$, $1 \leq i \leq k-1$. Since D is convex, there is a $(c'_i - c'_{i+1})$-path in D, which produces a cycle in G. Thus $c'_i, c'_{i+1} \in C$ and every vertex of C except c_i, c_{i+1} belongs to D. ■

Theorem 2 *If G is a unicyclic graph with the only cycle C, then $\gamma_{con}(G) \leq \gamma_{con}(G_{uv}) \leq \gamma_{con}(G) + 3$.*

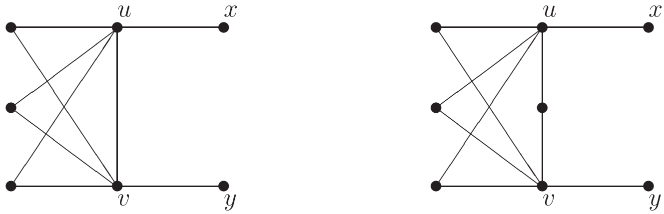

Figure 1: Graphs G and G_{uv} for $k = 4$.

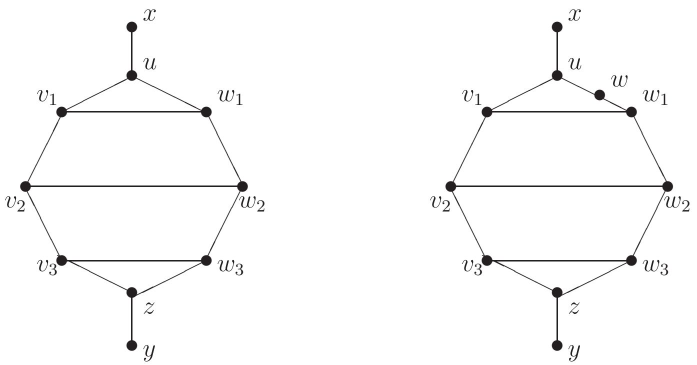

Figure 2: Graph G and G_{uw_1} for $k = 3$.

Proof. First we show that $\gamma_{con}(G_{uv}) \le \gamma_{con}(G) + 3$. Let D be a minimum convex dominating set of G and let uv be the subdivided edge. We consider three cases.

Case 1. Let $u \in D$ and $v \in D$. If uv is not a cyclic edge, then $D \cup \{w\}$ is a convex dominating set of G_{uv} and $\gamma_{con}(G_{uv}) \le |D| + 1 = \gamma_{con}(G) + 1 \le \gamma_{con}(G) + 3$.

Now let $uv \in E(C)$. From Observation 1, at most two vertices of C do not belong to D. If every vertex of C belongs to D, then $D \cup \{w\}$ is a convex dominating set of G_{uv} and we are done. Now let one vertex, say z, of C belong to $V - D$. Since D is convex, $C = C_3$. Then $D \cup \{w, z\}$ is a convex dominating set of G_{uv} and again we obtain the required inequality. If two vertices, let us say, z_1, z_2 of a cycle C belong to $V - D$, then, since D is convex, $C = C_4$ or $C = C_5$. If $C = C_4$, then $D \cup \{w\}$ is a convex dominating set of G_{uv}; if $C = C_5$, then $D \cup \{w, z_1, z_2\}$ is a convex dominating set of G_{uv} and we are done.

Case 2. Now let $|\{u, v\} \cap D| = 1$; without loss of generality let $u \in D, v \in V - D$. If $e = uv$ does not belong to C, then similarly to the previous case, $D \cup \{w\}$ is a convex dominating set of G_{uv} and we obtain the desired inequality.

If $e = uv$ belongs to the cycle C and $C = C_3$, then by Observation 1, two vertices or one vertex of the cycle are outside D. If exactly one vertex v of the cycle C_3 is outside D, then D is a convex dominating set of G_{uv} and $\gamma_{con}(G_{uv}) \le |D| = \gamma_{con}(G) \le \gamma_{con}(G) + 3$. If two vertices of C_3 do not belong to D, then $D \cup \{w\}$ is a convex dominating set of G_{uv} and we are done.

Now let $e = uv$ belong to the cycle $C = C_4$. From Observation 1, at most two vertices of the cycle are outside D. Since D is convex, no vertices or else exactly two vertices of C are in $V - D$, and then $D \cup \{w\}$ is a convex dominating set of G_{uv} and we obtain the required inequality.

If $e = uv$ belongs to the cycle C_5, then, since D is convex, no vertices from C_5 or exactly two vertices of C_5 belong to $V - D$. If no vertex of C_5 belongs to $V - D$, then $D \cup \{w\}$ is a convex dominating set of G_{uv} and we obtain the required inequality. If exactly two vertices of C_5 belong to $V - D$, let us say $v, x \in (V - D) \cap C_5$, then $D \cup \{w, v, x\}$ is a convex dominating set of G_{uv} and $\gamma_{con}(G_{uv}) \le \gamma_{con}(G) + 3$.

If $e = uv$ belongs to the cycle C_p, $p \geq 6$, then, since D is convex, every vertex of C_p belongs to D, which gives a contradiction with the fact that $|\{u, v\} \cap D| = 1$.

Case 3. Now let $u \in V - D$ and $v \in V - D$. Then, since u and v are dominated and since D is convex, both vertices u, v belong to C. Since D is a convex set and $u, v \notin D$, we have $C = C_p$, $3 \leq p \leq 5$. If $p = 3$ or $p = 4$, then $D \cup \{u\}$ is a convex dominating set of G_{uv} and we are done. If $p = 5$, then $D \cup \{u, v, w\}$ is a convex dominating set of G_{uv} and finally we obtain the required inequality.

Now we show that $\gamma_{con}(G) \leq \gamma_{con}(G_{uv})$. Let D_0 be a minimum convex dominating set of G_{uv}. Again we consider three cases.

Case 1. Let $u \in D_0, v \in D_0$. Then $D_0 - \{w\}$ is a convex dominating set of G and thus $\gamma_{con}(G) \leq \gamma_{con}(G_{uv}) - 1$, which gives $\gamma_{con}(G) \leq \gamma_{con}(G) + 1 \leq \gamma_{con}(G_{uv})$.

Case 2. Now let $u \in D_0, v \notin D_0$. If $w \in D_0$, then $D_0 - \{w\}$ is a convex dominating set of G and we are done. If $w \notin D_0$, then D_0 is a convex dominating set of G and $\gamma_{con}(G) \leq \gamma_{con}(G_{uv})$.

Case 3. If $u \notin D_0$ and $v \notin D_0$, then u, v, w must have neighbours in D_0, which produces more than one cycle, a contradiction. ■

Corollary 3 *Let G be a unicyclic graph with the only cycle C and let D be a minimum convex dominating set of G.*

- *If e is not a cyclic edge, then $\gamma_{con}(G_e) = \gamma_{con}(G) + 1$.*
- *If $V(C) \subseteq D$, then for any edge $e \in E(G)$ we obtain $\gamma_{con}(G_e) = \gamma_{con}(G) + 1$.*
- *If there is $v \in V(C)$ such that $v \notin D$, then $C = C_3$ or $C = C_4$ or $C = C_5$:*
  - *If $C = C_3$ and $|V(C) \cap D| = 1$, then for any edge $e \in E(G)$ we obtain $\gamma_{con}(G_e) = \gamma_{con}(G) + 1$. If $|V(C) \cap D| = 2$, let us say $x, y, z \in V(C)$ and $x, y \in D$, then $\gamma_{con}(G_e) = \gamma_{con}(G) + 1$ for $e = xy$ and $\gamma_{con}(G_e) = \gamma_{con}(G)$ for $e = xz$ and $e = yz$.*
  - *If $C = C_4$, then $|V(C) \cap D| = 2$ and for any edge $e \in E(G)$ we obtain $\gamma_{con}(G_e) = \gamma_{con}(G) + 1$.*
  - *If $C = C_5$, then $|V(C) \cap D| = 3$ and for any edge $e \in V(C)$ we obtain $\gamma_{con}(G_e) = \gamma_{con}(G) + 3$.*

Now we investigate graphs G for which the convex domination number increases by exactly one after subdividing an edge.

Proposition 4 *If T is a tree of order at least three, then for any edge uv of T is $\gamma_{con}(T_{uv}) = \gamma_{con}(T) + 1$.*

Proof. The only minimum convex dominating set of a tree T is $D = V(T) - \Omega(T)$. After subdividing any edge we obtain the tree T' such that $|V(T')| = |V(T)| + 1$ and $|\Omega(T')| = |\Omega(T)|$. Hence we have $\gamma_{con}(T_{uv}) = \gamma_{con}(T) + 1$ for any edge uv of T. ∎

Observation 2 *Let G be a connected graph with $\delta(G) = 1$ and let uv be an end-edge of G. Then $\gamma_{con}(G_{uv}) = \gamma_{con}(G) + 1$.*

Observation 3 *Let G be a connected graph with $\Delta(G) = n(G) - 1$. Then $\gamma_{con}(G_{uv}) = \gamma_{con}(G) + 1$ for any edge $uv \in E(G)$.*

Theorem 5 *[1] If G is a connected graph with $\delta(G) \geq 2$ and $g(G) \geq 6$, then $\gamma_{con} = n(G)$.*

Since subdividing an edge does not decrease the girth of the graph, we have the following corollary.

Corollary 6 *If G is a connected graph with $\delta(G) \geq 2$ and $g(G) \geq 6$, then $\gamma_{con}(G_{uv}) = \gamma_{con}(G) + 1$ for any edge $uv \in E(G)$.*

Theorem 7 *If G is a connected graph with $g(G) \geq 6$, then $\gamma_{con}(G) = n - |\Omega(G)|$.*

Proof. If $\delta(G) \geq 2$, then the result holds by Theorem 5. Let $\delta(G) = 1$. Let D be a minimum convex dominating set of G. Of course, no end-vertex belongs to D. Let v be any non-end-vertex of G. If v does not belong to a cycle, then v belongs to some $(a - b)$-geodesic, where we have the following possibilities for vertices a, b:

- $a \in V(C_1)$ and $b \in V(C_2)$, where C_1, C_2 are cycles in G;
- $a, b \in S(G)$;
- $a \in V(C)$ and $b \in S(G)$, where C is a cycle in G.

Thus $v \in D$.

Now let v belong to a non-induced cycle C with p vertices; of course $p \geq 6$. Suppose $v \notin D$. The vertex v has k neighbours $v_1, \ldots, v_k$, $k \geq 2$. Suppose more than one of $v_i, 1 \leq i \leq p$ belongs to D; without loss of generality let $v_1, v_1 \in D$.

Since $g(G) \geq 6$, we have $v_1 v_2 \notin E(G), |N_G(v_1) \cap N_G(v_2)| = 1$ and v is the only vertex belonging to $N_G(v_1) \cap N_G(v_2)$. But then v belongs to a $(v_1 - v_2)$-geodesic and $v \in D$, a contradiction. Thus v has exactly one neighbour in D; let $v_1 \in N_G(v) \cap D$. Then $v_2 \in V - D$.

Since v_2 is dominated, there exists $y \in D \cap N_G(v_2)$, $y \neq v_1$, since $g(G) \geq 6$. Since D is convex, vertices from every $(v_1 - y)$-geodesic belong to D. Since $v_1 v \in E(G)$, $vv_2 \in E(G)$ and $v_2 y \in E(G)$, a $(v_1 - y)$-geodesic has length at most 3. If it has length one or two, we obtain a cycle of length less than 6, a contradiction. If it has length 3, then v, v_2 belong to a $(v_1 - y)$-geodesic and $v, v_2 \in D$, a contradiction. Thus $v \in D$. Then $|D| \geq n - |\Omega(G)|$. On the other hand, $V(G) - \Omega(G)$ is a convex dominating set of G and thus $\gamma_{con}(G) \leq |V(G) - \Omega(G)| = n - |\Omega(G)|$. ■

Corollary 8 *If* $g(G) \geq 6$, *then* $\gamma_{con}(G_{uv}) = \gamma_{con}(G) + 1$.

Proof. Let e be a subdivided edge and let D' be a minimum convex dominating set of G_{uv}. If e belongs to a cycle C, then $g(G_{uv}) \geq 6$ and $|D'| = n(G_{uv}) - |\Omega(G_{uv})| = n(G) + 1 - |\Omega(G)| = \gamma_{con}(G) + 1$.

If e does not belong to C, then the new vertex w of G_{uv} lies on an $(a - b)$-geodesic, where a, b belong to a cycle or to $S(G)$. So $x \in D'$ and $|D'| = |D| + 1$, where $|D| = |V(G) - \Omega(G)|$ and $\gamma_{con}(G_{uv}) = \gamma_{con}(G) + 1$. ■

For an edge $e = uv \in E(G)$, let us define $\mathrm{diff}(e) = \gamma_{con}(G_{uv}) - \gamma_{con}(G)$ and for a graph G we consider $S'(G) = \sum_{e \in E(G)} \mathrm{diff}(e)$.

Now we show that for every integer k there exists a graph G such that $S'(G) = k$. We begin with the definition of the family of graphs $\mathcal{G}$.

Let $\mathcal{G}$ be the family of graphs G that can be obtained from a sequence $G_1, \ldots, G_j$ ($j \geq 1$) of graphs such that G_1 is a graph shown in Figure 3 and $G = G_j$, and, if $j \geq 2$, then G_{i+1} can be obtained from G_i by operation $\mathcal{Y}$ listed below.

We define the *status* of a vertex v denoted $\mathrm{sta}(v)$ to be A or B, where initially for G_1 we put $\mathrm{sta}(v) = A$ if v is an end-vertex or a support of G_1 and $\mathrm{sta}(v) = B$ if v is a vertex of degree two in G_1. Once a vertex is assigned a status, this status remains unchanged as the graph is recursively constructed.

- **Operation $\mathcal{Y}$.** The graph G_{i+1} is obtained from G_i by adding the graph H shown in Figure 3 and identifying an end-vertex of G_i with a vertex u of H. Then we let $\mathrm{sta}(u) = A$, $\mathrm{sta}(x) = A$ if $x \in V(H)$ is a support vertex or an end-vertex and $\mathrm{sta}(y) = B$ if y is a vertex of degree two in H and $y \neq u$.

If $G_k \in \mathcal{G}$, $k \geq 1$, is a graph obtained by using $k - 1$ times of operation $\mathcal{Y}$, then by G_k^t, $t \geq 0$, we denote a graph obtained from G_k by adding t end-vertices $u_1, \ldots, u_t$ and t pendant edges $w_k u_1, \ldots, w_k u_t$, where w_k is a support vertex of G_k. In particular $G_k^0 = G_k$. In Figure 4 we have G_k^t for $t = 2$ and $k = 2$. Every vertex u_i is assigned a status A. The edge $e = uv$ we call an $(A - A)$-edge if $\mathrm{sta}(u) = A$ and $\mathrm{sta}(v) = A$. In the other case we denote it $(A - B)$-edge.

Observation 4 *For a graph* $G_k \in \mathcal{G}$ *we have:*

- $\mathrm{diff}(e) = 1$ *if* e *is* $(A - A)$*-edge;*

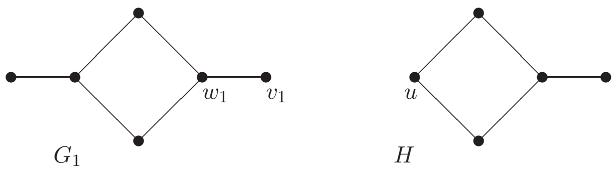

Figure 3: Graphs G_1 and H

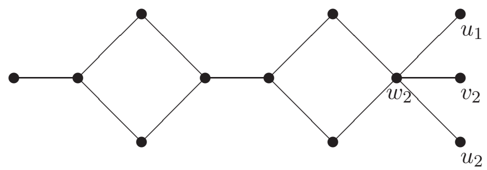

Figure 4: Graph G_2^2

- $\mathrm{diff}(e) = -1$ *if* e *is* $(A - B)$*-edge.*

Lemma 9 *For every integer* k *there exists a graph* G *such that*

$$S'(G) = \sum_{e \in E(G)} \mathrm{diff}(e) = k.$$

Proof. By Observation 4 we have that $S'(G_1) = -2$ and $S'(G_1^t) = -2 + t$, where $t \geq 0$. Hence for $k \geq -2$ there exists a graph $G = G_1^{k+2}$ such that $S'(G) = k$.

For $k \leq -3$ we consider three cases:

Case 1. Let $k \equiv 1 \pmod 3$. This gives $k = 3p + 1$ for an integer $p < -1$. Thus for $G = G_{-p}$ we have $S'(G) = 3p + 1 = k$.

Case 2. Now let $k \equiv 2 \pmod 3$. This gives $k = 3p + 2$ for an integer $p < -1$. Thus for $G = G_{-p}^1$ we have $S'(G) = 3p + 2 = k$.

Case 3. If $k \equiv 0 \pmod 3$. This gives $k = 3p$ for an integer $p < 0$. Thus for $G = G_{-p+1}^2$ we have $S'(G) = 3p = k$. ■

Lemma 10 *For any integer* $k \geq 3$ *there exists a graph* G *such that for any edge* $e \in E(G)$ *we have* $\mathrm{diff}(e) = k$.

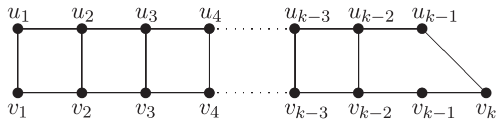

Figure 5: Graph H^k

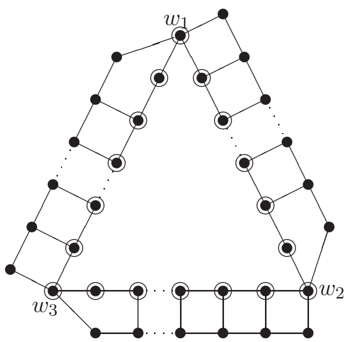

Figure 6: Graph G^k

Proof. We show the example of a graph such that subdivision of any edge of this graph increases the convex domination number by $k \geq 3$. First, let us consider a graph H^k, $k \geq 3$, constructed in the following way. We begin with a path $P_k = (v_1, v_2, \ldots, v_k)$ and a path $P_{k-1} = (u_1, u_2, \ldots, u_{k-1})$ and then we add edges $u_1v_1, u_2v_2, \ldots, u_{k-2}v_{k-2}, u_{k-1}v_k$ (see Fig. 5). Next, we take three copies of H^k: H^{k1}, H^{k2} and H^{k3}. For $j = 1, 2, 3$ we denote vertices of H^{kj} by $v_1^j, v_2^j, \ldots, v_k^j, u_1^j, u_2^j, \ldots, u_{k-1}^j$. Afterwards, we take the union of graphs H^{k1}, H^{k2} and H^{k3} and identify vertices v_1^1 and v_k^3 (which gives a vertex w_1), vertices v_k^1 and v_1^2 (which gives a vertex w_2) and vertices v_k^2 and v_1^3 (what gives a vertex w_3). In this way we obtain a graph G^k (see Fig. 6). Note that the graph G^k can be obtained also from a cycle C_3 with vertices w_1, w_2 and w_3 by an adequate replacement the edges of C_3 with the copy of H^k. Instead of C_3 we can also consider cycle C_p with vertices $w_1, \ldots, w_p$ for $p \geq 4$ to obtain more general example of G.

In G^k, vertices w_1, w_2 and w_3 and vertices belonging to $(v_2^1 - v_{k-1}^1)$, $(v_2^2 - v_{k-1}^2)$, $(v_2^3 - v_{k-1}^3)$-geodesics create the unique minimum convex dominating set (see Figure 6). Hence, $\gamma_{con}(G^k) = 3k - 3$. Because of the symmetry of the graph G^k, it suffices to consider subdivision of the edge from one copy of H^k, let us say H^1. We denote the minimum convex dominating set of G^k with subdivided edge e, G_e^k, by D' and, because of the convexity of D', vertices w_1, w_2, w_3 and $v_2^2, \ldots, v_{k-1}^2, v_2^3, \ldots, v_{k-1}^3$ belong to D'. Let e be the subdivided edge. We consider four cases:

Case 1. First let $e = v_i^1 v_j^1$, $1 \le i,j \le k$. Then $d_{G_e}(v_1^1, v_k^1) = k+1$ and vertices $w, v_2^1, v_3^1, \dots, v_{k-1}^1, u_1^1, u_2^1, \dots u_{k-1}^1$ belong to $(v_1^1 - v_k^1)$-geodesic, so they are in D'. Hence, $|D'| \ge 4k-3$. On the other hand, $\{w_1, w_2, w_3, w, v_2^1, v_3^1, \dots, v_{k-1}^1, u_1^1, u_2^1, \dots u_{k-1}^1, v_2^2, \dots, v_{k-1}^2, v_2^3, \dots, v_{k-1}^3\}$ is a convex dominating set of G_e^k. Hence, $|D'| \le 4k-3$. Finally, $\mathrm{diff}(v_i^1 v_j^1) = |D'| - |D| = 4k-3-(3k-3) = k$.

Case 2. Now let $e = v_1^1 u_1^1$. Then $d_{G_e}(v_1^1, v_k^1) = k$ and $v_1^1, v_2^1, v_3^1, \dots, v_{k-1}^1, v_k^1 \in D'$. Because $k \ge 3$, $w \in D'$ in order to dominate u_1^1 and $d_{G_e}(w, v_k^1) = k+1$. Hence, $u_1^1, u_2^1, \dots, u_{k-1}^1$ belong to a $(w - v_k^1)$-geodesic, so they are in D'. This gives $|D'| \ge 4k-3$. On the other hand, $\{w_1, w_2, w_3, w, v_2^1, v_3^1, \dots, v_{k-1}^1, u_1^1, u_2^1, \dots u_{k-1}^1, v_2^2, \dots, v_{k-1}^2, v_2^3, \dots, v_{k-1}^3\}$ is a convex dominating set of G_e^k. Hence, $|D'| \le 4k-3$. Finally, $\mathrm{diff}(v_1^1 u_1^1) = |D'| - |D| = 4k-3-(3k-3) = k$. Similarly we can show that $\mathrm{diff}(u_{k-1}^1 v_k^1) = k$.

Case 3. Let $e = u_i^1 u_j^1$, $1 \le i,j \le k-1$. Vertices $v_1^1, v_2^1, v_3^1, \dots v_{k-1}^1, v_k^1 \in D'$ and w is not dominated.

Case 3.1. Let $u_i^1 \in D'$. Then $d_{G_e}(v_1^1, u_i^1) = i$ and vertices $u_1^1, u_2^1, \dots, u_{i-1}^1$ belong to a $(u_i^1 - v_1^1)$-geodesic, so they are in D'. Moreover, $d_{G_e}(u_i^1, v_k^1) = k-i+1$ and $w, u_j^1, u_{j+1}^1, \dots, u_{k-1}^1$ belong to a $(u_i^1 - v_k^1)$-geodesic, which implies they are in D'. Then $|D'| \ge 4k-3$. On the other hand, $\{w_1, w_2, w_3, w, v_2^1, v_3^1, \dots, v_{k-1}^1, u_1^1, u_2^1, \dots u_{k-1}^1, v_2^2, \dots, v_{k-1}^2, v_2^3, \dots, v_{k-1}^3\}$ is a convex dominating set of G_e^k. Hence, $|D'| \le 4k-3$. Finally, $\mathrm{diff}(u_i^1 u_j^1) = |D'| - |D| = 4k-3-(3k-3) = k$.

Case 3.2. Let $u_j^1 \in D'$. Then $d_{G_e}(v_1^1, u_j^1) = j$ and vertices $u_1^1, u_2^1, \dots, u_i^1, w$ belong to a $(u_j^1 - v_1^1)$-geodesic, so they are in D'. Moreover, $d_{G_e}(u_j^1, v_k^1) = k-j+1$ and $u_{j+1}^1, u_{j+2}^1, \dots, u_{k-1}^1$ belong to a $(u_j^1 - v_k^1)$-geodesic, which implies they are in D'. Then $|D'| \ge 4k-3$. On the other hand, $\{w_1, w_2, w_3, w, v_2^1, v_3^1, \dots, v_{k-1}^1, u_1^1, u_2^1, \dots u_{k-1}^1, v_2^2, \dots, v_{k-1}^2, v_2^3, \dots, v_{k-1}^3\}$ is a convex dominating set of G_e^k. Hence, $|D'| \le 4k-3$. Finally, $\mathrm{diff}(u_i u_j) = |D'| - |D| = 4k-3-(3k-3) = k$.

Case 4. If $k \ge 4$, we subdivide an edge $u_i^1 v_i^1$, $2 \le i \le k-2$. Vertices $v_1^1, v_2^1, v_3^1, \dots v_{k-1}^1, v_k^1 \in D'$ and u_i^1 is not dominated.

Case 4.1. Let $w \in D'$. Then $d_{G_e}(w, v_k^1) = k-i+1$ and vertices $u_i^1, u_{i+1}^1, \dots, u_{k-1}^1$ belong to a $(w - v_k^1)$-geodesic, so they are in D'. Moreover, $d_{G_e}(u_i^1, v_1^1) = i+1$ and $u_1^1, u_2^1, \dots, u_{i-1}^1$ belong to $(u_i^1 - v_1^1)$-geodesic, which implies they are in D'. This gives $|D'| \ge 4k-3$. On the other hand, $\{w_1, w_2, w_3, w, v_2^1, v_3^1, \dots, v_{k-1}^1, u_1^1, u_2^1, \dots u_{k-1}^1, v_2^2, \dots, v_{k-1}^2, v_2^3, \dots, v_{k-1}^3\}$ is a convex dominating set of G_e^k. Hence, $|D'| \le 4k-3$. Finally, $\mathrm{diff}(u_i v_i) = |D'| - |D| = 4k-3-(3k_3) = k$.

Case 4.2. Let $u_{i+1}^1 \in D'$. Then $d_{G_e}(v_1^1, u_{i+1}^1) = i+1$ and vertices $u_1^1, u_2^1, \dots, u_i^1$ belong to a $(u_{i+1}^1 - v_1^1)$-geodesic, so they are in D'. Moreover, $d_{G_e}(u_{i+1}^1, v_k^1) = k-i-1$ and $u_{i+2}^1, u_{i+3}^1, \dots, u_{k-1}^1$ belong to a $(u_{i+1}^1 - v_k^1)$-geodesic, which implies they are in D'. Also $w \in D'$, because it belongs to a $(u_i^1 - v_i^1)$-geodesic. This gives $|D'| \ge 4k-3$. On the other hand, $\{w_1, w_2, w_3, w, v_2^1, v_3^1, \ldots, v_{k-1}^1,\ u_1^1, u_2^1, \ldots u_{k-1}^1,\ v_2^2, \ldots, v_{k-1}^2, v_2^3, \ldots, v_{k-1}^3\}$ is a convex dominating set of G_e^k. Hence, $|D'| \leq 4k-3$. Finally $\mathrm{diff}(u_i v_i) = |D'| - |D| = 4k-3-(3k-3) = k$.

Case 4.3. Let $u_{i-1}^1 \in D'$. Similarly as in *Case 4.2* we can show that $\mathrm{diff}(u_i^1 v_i^1) = k$. ■

In Figure 7 we have an example of a graph G such that any edge $e \in E(G)$ satisfies $\mathrm{diff}(e) = -1$. By the symmetry of G it suffices to subdivide only one edge. Minimum convex dominating sets of G and G_e are indicated in Figure 7.

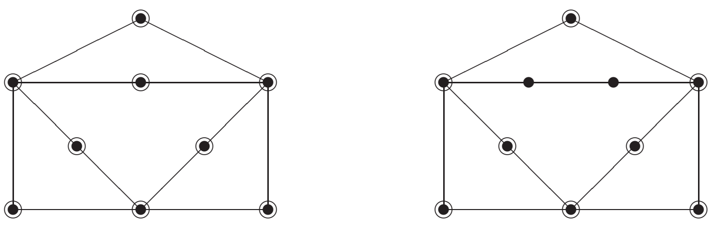

Figure 7: Graphs G and G_e

Proposition 4 gives an example of graphs for which for any edge e satisfies $\mathrm{diff}(e) = 1$. The existence of graphs G for which for any $k < -1$ and $k = 0, 2$ satisfies $\mathrm{diff}(e) = k$ remains an open problem.

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(Received 29 Dec 2010; revised 11 Jan 2012, 21 Mar 2012)