Integrate-and-fire models with an almost periodic input function

Piotr Kasprzak [a,*], Adam Nawrocki [a], Justyna Signerska-Rynkowska [b]

[a] *Faculty of Mathematics and Computer Science, Adam Mickiewicz University in Poznań, ul. Umultowska 87, 61-614 Poznań, Poland*
[b] *Faculty of Applied Physics and Mathematics, Gdańsk University of Technology, ul. G. Narutowicza 11/12, 80-233 Gdańsk, Poland*

Abstract

We investigate leaky integrate-and-fire models (LIF models for short) driven by Stepanov and μ-almost periodic functions. Special attention is paid to the properties of the firing map and its displacement, which give information about the spiking behavior of the considered system. We provide conditions under which such maps are well-defined and are uniformly continuous. We show that the LIF models with Stepanov almost periodic inputs have uniformly almost periodic displacements. We also show that in the case of μ-almost periodic drives it may happen that the displacement map is uniformly continuous, but is not μ-almost periodic (and thus cannot be Stepanov or uniformly almost periodic). By allowing discontinuous inputs, we extend some previous results, showing, for example, that the firing rate for the LIF models with Stepanov almost periodic input exists and is unique. This is a starting point for the investigation of the dynamics of almost-periodically driven integrate-and-fire systems.
© 2017 Elsevier Inc. All rights reserved.

---

* Corresponding author.
E-mail addresses: kasp@amu.edu.pl (P. Kasprzak), adam.nawrocki@amu.edu.pl (A. Nawrocki), justyna.signerska@pg.edu.pl (J. Signerska-Rynkowska).

1. Introduction

Integrate-and-fire models are commonly used for modeling the activity of neuronal cells (see for example [17,20,22,37]). Although they are not able to capture all the electrophysiological phenomena, as a part of a big neural network, they are computationally more efficient than, for example, the classical Hodgkin–Huxley model (see for example [17,19]), and their biological relevance is in some cases satisfactory. In particular, the so-called *leaky integrate-and-fire model*[1]

$$\dot{x}(t) = -\sigma x(t) + f(t) \qquad \text{for a.e. } t \in \mathbb{R}, \tag{1}$$

where $\sigma \geq 0$, is one of the models in the center of interest of neuroscientists. This model dates back to Lapicque (see [25]), who discovered that the voltage x across the cell membrane decays exponentially to its resting state x_r and only an external input f, which might be current injected via an electrode or the impulse from a pre-synaptic neuron, might cause the increase of the voltage. Spiking (or firing), that is emitting an action potential, is introduced to this simple dynamics by adding the resetting condition

$$x(s) = x_\vartheta \quad \implies \quad \lim_{t \to s^+} x(t) = x_r, \tag{2}$$

which says that after the dynamical variable reaches a certain threshold x_ϑ at some time s, it is immediately reset to its resting value x_r and the dynamics continues again from the point (t, x_r). Although usually constant thresholds and resets are studied, it is also possible to consider integrate-and-fire models with varying bounds, allowing thus to incorporate into the model additional biological phenomena such as refractory periods and threshold modulation (see [16]). Such models with time-dependent thresholds and/or resets in some cases can be reduced by an appropriate change of variables to those with constant x_r and x_ϑ (for more details see for example [7, Section 2.5]). Therefore, for simplicity, in this paper we will always assume that $x_\vartheta = 1$ and $x_r = 0$.

Since isolated spikes of a given neuron look alike, often it is assumed that the form of the action potential does not carry any information, but rather it is the structure of the spike train[2] which matters. Therefore, the idea is to study the properties (and, in particular, dynamics) of two special maps associated with the integrate-and-fire models, which carry the information about the distribution of spikings in time, namely, the firing map Φ and its displacement Ψ (cf. Definitions 4.1 and 4.10). It turns out, for example, that the rotation number of a given point with respect to the mapping Φ corresponds to the average interspike interval, whereas its multiplicative inverse describes the firing rate (for more details see Section 6).

Despite the fact that integrate-and-fire models are commonly used, their dynamical behavior has been investigated rigorously in only few papers (see for instance [7,11,16]) and usually under the assumption that the forcing term f (or, in the case of general integrate-and-fire models of the form $\dot{x}(t) = F(t, x(t))$, the function $t \mapsto F(t, x)$) is periodic or uniformly almost periodic (see Definition 2.4 below). Recently, the broader class of stimulating processes than the periodic

---

[1] In the special case $\sigma = 0$, the LIF model is usually referred to as the *perfect integrator model* (or the PI model for short).

[2] A *spike train* is a chain of action potentials emitted by a single neuron.

ones have been analyzed in [21]. On the other hand, the dynamics of bidimensional adaptive integrate-and-fire models under constant input was studied e.g. in [38]. Our aim in this paper is to continue the study of one-dimensional integrate-and-fire models with (almost) periodic inputs carried out in the articles [27,28,30], and to establish several basic properties of the firing map and its displacement corresponding to the model (1)–(2) driven by Stepanov or μ-almost periodic functions.

It should be emphasized that in this context it seems quite natural to consider almost periodic forcing terms. Indeed, many neurons will respond to the current step with a spike train with (eventually) steady state of periodic firing, and therefore, even if the action potential is generated periodically by each neuron at the pre-synaptic level, the signal a given post-synaptic neuron receives may be no longer periodic as a sum of periodic inputs with incommensurable periods; clearly, such a signal is almost periodic (for more details see [17]). One of the ways to qualify neuron's response to such signals, is to measure the so-called firing rate and the regularity of the interspike intervals along the generated post-synaptic signal.

In the paper we work mainly with Stepanov and μ-almost periodic functions, because such functions satisfy quite general regularity conditions (for more details see Section 2), and thus we are able to cover quite wide range of LIF models, including models with discontinuous inputs. On the other hand, it seems that without any assumption involving some kind of (almost) periodicity of the forcing term, it would be very hard (if not impossible) to provide a complete description of the behavior of the firing map and its displacement, since, as we mentioned above, these models are (to some extent) periodic in nature.

The paper is organized as follows. In Section 2 we recall some basic definitions and facts concerning almost periodic functions. Special attention is paid to Stepanov and μ-almost periodic functions. Section 3 is devoted to the study of the mean value of μ-almost periodic functions. In particular, it is shown that in general the mean value of a μ-almost periodic function may not exist. In Section 4 firing map and its displacement are investigated. We begin with presenting some general properties of such maps. For example, we provide the conditions under which Φ and Ψ are well-defined for every $t \in \mathbb{R}$ and are (uniformly) continuous (see Proposition 4.4 and Theorem 4.11). Then we move to the discussion of the firing map and its displacement for LIF models driven by almost periodic functions. Among other things, we show that the LIF model with a Stepanov almost periodic input has a uniformly almost periodic displacement map (see Theorem 4.18). We also show that there are LIF models driven by μ-almost periodic functions with uniformly continuous and bounded displacement maps which fail to be μ-almost periodic (see Example 4.22). In Section 5 an application of the above-mentioned results to the theory of almost periodic functions is indicated. In Section 6 we discuss some properties of the firing rate. In particular, we prove a theorem on the existence and uniqueness of the firing rate for LIF models (see Theorem 6.14 below), which extends analogous results for PI models from [7] and [27]. Let us add that although it seems that the transition from $\sigma = 0$ to $\sigma \geq 0$ should be straightforward, it is not and the proof of Theorem 6.14 is based on the recent result concerning the dynamics of mappings of the real line with almost periodic displacements established by J. Kwapisz in [24]. We end the main part of the paper with Section 7 dealing with the approximation of Stepanov almost periodic functions with Haar wavelets. In the Appendix we return to the mean value of (uniformly) almost periodic functions and present some remarks on its relationship with the antiderivative of such functions. Therefore, apart from our interest in integrate-and-fire models, this work provides also a few contributions to the theory of almost periodic functions.

2. Limit-periodic and almost periodic functions

In this section we fix notation and recall some basic definitions and facts concerning limit-periodic and almost periodic functions, which will be needed in the sequel.

Notation. Throughout the paper by $L^0(\mathbb{R})$ we will denote the family of all equivalence classes of real-valued Lebesgue measurable functions defined on $\mathbb{R}$. Furthermore, by $L^p_{\mathrm{loc}}(\mathbb{R})$, where $p \in [1, +\infty)$, we will denote the family of all equivalence classes of real-valued functions defined on $\mathbb{R}$ which are locally Lebesgue integrable with p-th power. Very often, by abuse of notation, we will refer to elements of the families $L^0(\mathbb{R})$ and $L^p_{\mathrm{loc}}(\mathbb{R})$ as functions and we will simply write $f \in L^0(\mathbb{R})$ or $f \in L^p_{\mathrm{loc}}(\mathbb{R})$. The Lebesgue measure on $\mathbb{R}$ will be denoted by μ. Given a function $f : \mathbb{R} \to \mathbb{R}$ by f^τ, where $\tau \in \mathbb{R}$, we will denote the function $f^\tau : \mathbb{R} \to \mathbb{R}$ defined by the formula $f^\tau(t) = f(t + \tau)$ for $t \in \mathbb{R}$. Moreover, by $[\,\cdot\,]$ we will denote the *entier* function.

At the beginning of this section let us recall the notion of a limit-periodic function.

Definition 2.1. A function $f : \mathbb{R} \to \mathbb{R}$ is called *limit-periodic* (in the sense of Bohr) if it is the limit of a uniformly convergent sequence $(f_n)_{n \in \mathbb{N}}$ of continuous periodic functions.[3]

Remark 2.2. Clearly, every continuous periodic function is limit-periodic. On the other hand, the function $f : \mathbb{R} \to \mathbb{R}$ defined by the formula

$$f(t) = \sum_{k=1}^{\infty} \frac{1}{k^2} \cos\left(\frac{2\pi t}{2^k}\right) \qquad \text{for } t \in \mathbb{R},$$

is an example of a limit-periodic function which is not periodic (cf. [5]).

Although there are various classes of almost periodic functions known in the literature (see for example [3,8,10] and the references therein), in this paper we are going to deal with only three of them; namely, we are going to consider functions almost periodic in the sense of Bohr and Stepanov as well as functions almost periodic in the Lebesgue measure. Let us begin with the definition of a relatively dense set.

Definition 2.3. A set $A \subseteq \mathbb{R}$ is said to be *relatively dense* if there exists a positive number l such that the intersection $A \cap (x, x + l)$ is non-empty for every $x \in \mathbb{R}$. In this case, any number l with this property is said to characterize the relative density of the set A.

Definition 2.4. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called *almost periodic in the sense of Bohr* (or *uniformly almost periodic*) if for any $\varepsilon > 0$ the set of all ε-almost periods of f, defined as

$$E\{\varepsilon, f\} := \Big\{\tau \in \mathbb{R} : \sup_{t \in \mathbb{R}} |f^\tau(t) - f(t)| < \varepsilon \Big\},$$

is relatively dense.

---

[3] Of course, by definition, we assume that all the functions f_n are defined on $\mathbb{R}$.

Remark 2.5. Equivalently, Bohr almost periodic functions can be defined as uniform limits of sequences of generalized trigonometric polynomials $P_n(t) = \sum_{j=1}^{k(n)} \big(a_j \sin(\lambda_j t) + b_j \cos(\lambda_j t)\big)$, where $a_j, b_j \in \mathbb{R}$ and $\lambda_j \in \mathbb{R}$ (see [13]). Let us also add that Bohr almost periodic functions are uniformly continuous and bounded (see [6,13]).

Remark 2.6. The vector space $AP(\mathbb{R})$ of all uniformly almost periodic functions is a Banach space when endowed with the supremum norm $\|\cdot\|_\infty$ (see for example [3,13,14]).

Remark 2.7. Any limit-periodic function is uniformly almost periodic. On the other hand, the class of all limit periodic functions is identical with the class of all uniformly almost periodic functions all whose Fourier exponents are rational multiples of the same number (see [6, Theorem, p. 34]). Thus, for instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by the formula $f(t) = \cos(\pi t) + \cos(\sqrt{5} t)$, $t \in \mathbb{R}$, is an example of a Bohr almost periodic function which is not limit-periodic.

Now, let us pass to the definition of an almost periodic function in the sense of Stepanov.

Notation. Given $r > 0$ and $p \in [1, +\infty)$, for a function $f \in L^p_{\text{loc}}(\mathbb{R})$ let

$$\|f\|_{S^p_r} := \sup_{t \in \mathbb{R}} \left(\frac{1}{r} \int\limits_{t}^{t+r} |f(u)|^p \, \mathrm{d}u \right)^{1/p}.$$

Let us also observe that for every $r_1, r_2 > 0$ there exist $a, b > 0$ such that $a \|f\|_{S^p_{r_1}} \le \|f\|_{S^p_{r_2}} \le b \|f\|_{S^p_{r_1}}$, and therefore in the sequel we will assume that $r = 1$.

Definition 2.8. Let $p \in [1, +\infty)$. A function $f \in L^p_{\text{loc}}(\mathbb{R})$ is called S^p-*almost periodic* if for any $\varepsilon > 0$ the set of all (S^p, ε)-almost periods of f, defined as $S^p E\{\varepsilon, f\} := \big\{\tau \in \mathbb{R} : \|f^\tau - f\|_{S^p_1} < \varepsilon\big\}$, is relatively dense.

Remark 2.9. Clearly, any uniformly almost periodic function is S^p-almost periodic for every $p \in [1, +\infty)$. However, the continuous function $f : \mathbb{R} \to \mathbb{R}$ defined by the formula

$$f(t) = \sin\left(\frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)} \right) \qquad \text{for } t \in \mathbb{R},$$

is an example of a S^1-almost periodic function which is not uniformly almost periodic (see for example [26, p. 212] or [35, Example 2.2, p. 56]). Another simple example of a S^p-almost periodic function with $p \in [1, +\infty)$, which is not uniformly almost periodic, is given by the following formula $f(t) = a(-1)^{[\lambda t - \gamma]}$, where $a, \lambda > 0$, $\gamma \in \mathbb{R}$; let us recall that $[\cdot]$ denotes the *entier* function. Finally, let us add that if a S^p-almost periodic function is uniformly continuous, then it is Bohr almost periodic (see [13, Theorem 6.16, p. 174] or [6, Theorem, p. 81]).

Definition 2.10. Let $p \in [1, +\infty)$. A function $f \in L^p_{\text{loc}}(\mathbb{R})$ is said to be S^p-*bounded* if $\|f\|_{S^p_1} < +\infty$.

Remark 2.11. It is known that every S^p-almost periodic function, where $p \in [1, +\infty)$, is S^p-bounded (see [35, Theorem 2.1] or [26, Theorem 5.2.2]).

Remark 2.12. The space $S^p(\mathbb{R})$ of all S^p-almost periodic functions (equivalence classes) endowed with the norm $\|\cdot\|_{S_1^p}$ is a Banach space (see [3]). This space can also be obtained as the closure of the set of all generalized trigonometric polynomials in the Banach space $\{f \in L^p_{\text{loc}}(\mathbb{R}) : \|f\|_{S_1^p} < +\infty\}$ with respect to the norm $\|\cdot\|_{S_1^p}$. Let us add that different values of $r > 0$ give rise to different norms $\|\cdot\|_{S_r^p}$ on $S^p(\mathbb{R})$, but all of them generate the same topology. Finally, let us recall that if f is a S^{p_1}-almost periodic function, then it is also S^{p_2}-almost periodic for $p_2 \leq p_1$ (for more details see for example [3,6]).

Before we recall the definition of a function almost periodic in the Lebesgue measure we need to introduce the following

Notation. For $\eta > 0$ and $f, g \in L^0(\mathbb{R})$ let $D(\eta; f, g) := \sup_{u \in \mathbb{R}} \mu(\{t \in [u, u+1] : |f(t) - g(t)| \geq \eta\})$.

Remark 2.13. Clearly, if $f, g \in L^0(\mathbb{R})$, then for every $\eta > 0$ we have

$$D(\eta; f, g) \leq 2 \sup_{z \in \mathbb{Z}} \mu(\{t \in [z, z+1] : |f(t) - g(t)| \geq \eta\}) \leq 2D(\eta; f, g).$$

Definition 2.14 *([31,32]).* A function $f \in L^0(\mathbb{R})$ is said to be *almost periodic in the Lebesgue measure* μ (or *μ-almost periodic*) if for arbitrary numbers $\varepsilon, \eta > 0$ the set of all (ε, η)-almost periods of f, defined as $\mu E\{\varepsilon, \eta, f\} := \{\tau \in \mathbb{R} : D(\eta; f^\tau, f) \leq \varepsilon\}$, is relatively dense.

It turns out that the set of all (ε, η)-almost periods of a μ-almost periodic function is, in a sense, 'quite big' as evidenced by the following proposition, whose proof we omit, since it is technical and follows from [32, Lemma, p. 195] along the same lines as, for example, [26, Theorem 5.2.4] or [35, Theorem 2.2].

Proposition 2.15. *If f is μ-almost periodic, then for every $\varepsilon, \eta > 0$ the set $\mu E\{\varepsilon, \eta, f\} \cap \mathbb{Z}$ is relatively dense.*

Remark 2.16. A result analogous to Proposition 2.15 in the case of S^1-almost periodic functions is also true.

Remark 2.17. It can be easily shown that every S^p-almost periodic function is μ-almost periodic. On the other hand, the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = \frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)} \qquad \text{for } t \in \mathbb{R},$$

is an example of a continuous (and thus locally integrable) μ-almost periodic function which is not S^p-almost periodic for any $p \in [1, +\infty)$ (for more details see [9]). However, if a μ-almost periodic function is (essentially) bounded, then it is S^p-almost periodic for every $p \in [1, +\infty)$ (see [32, Theorem 7] or [35, Theorem 4.11]).

In the sequel we will also need the notion of a *D*-convergence.

Definition 2.18 *([32]).* A sequence $(f_n)_{n \in \mathbb{N}}$, where $f_n \in L^0(\mathbb{R})$ for $n \in \mathbb{N}$, is said to be *D-convergent* to a function $f \in L^0(\mathbb{R})$, if the following condition is satisfied

$$\forall \varepsilon > 0 \quad \forall \eta > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad D(\eta; f_n, f) < \varepsilon.$$

The function f is said to be the *D-limit* of the sequence $(f_n)_{n \in \mathbb{N}}$.

Remark 2.19. If $f \in L^0(\mathbb{R})$ is the *D*-limit of a *D*-convergent sequence of μ-almost periodic functions, then f is μ-almost periodic (see [32, Theorem 6]).

Remark 2.20. It turns out that the *D*-convergence is metrizable. Indeed, in [32] Stoiński proved that a sequence $(f_n)_{n \in \mathbb{N}}$, where $f_n \in L^0(\mathbb{R})$ for $n \in \mathbb{N}$, is *D*-convergent to $f \in L^0(\mathbb{R})$ if and only if $\lim_{n \to \infty} |f_n - f| = 0$, where

$$|f| := \sup_{u \in \mathbb{R}} \int_u^{u+1} \frac{|f(s)|}{1 + |f(s)|} \, ds.$$

In the same paper it was also shown (although not explicitly stated) that the functional $|\cdot|$ is a complete F-norm on the vector space $M(\mathbb{R})$ of all μ-almost periodic functions.

3. Mean value

In this section, we recall the concept of the mean value of an almost periodic function and we discuss its properties. Special attention is paid to the mean value of μ-almost periodic functions.

Before passing to further considerations let us recall the following

Definition 3.1. Let $f \in L^1_{\text{loc}}(\mathbb{R})$. The limit

$$\mathcal{M}\{f\} := \lim_{T \to +\infty} \frac{1}{T} \int_0^T f(s) \, ds$$

(whenever it exists) is called the *mean value* of the function f.

Remark 3.2. If f is an almost periodic function in the sense of Bohr or Stepanov (with any $1 \le p < +\infty$), then the mean value $\mathcal{M}\{f\}$ exists and is finite (see for example [42, Theorem 1, p. 85] or [6, Theorem on p. 12 and Corollary 1 on p. 93]).

The following example shows that the mean value of a continuous μ-almost periodic function may not exist.

Example 3.3. Let $A_n = 4^n \mathbb{Z} + 2^n$, $B_n = A_{2^n}$ and $a_n = (n+1)^2 \cdot 2^{2^n}$ for $n \in \mathbb{N}$. Moreover, for a given $n \in \mathbb{N}$ let the function $f_n \colon \mathbb{R} \to \mathbb{R}$ be defined by the following formula

$$f_n(x) = \begin{cases} a_n x - a_n\left(z + \frac{1}{2} - \frac{1}{n+1}\right) & \text{for } x \in \left[z + \frac{1}{2} - \frac{1}{n+1}, z + \frac{1}{2}\right), z \in B_n, \\ -a_n x + a_n\left(z + \frac{1}{2} + \frac{1}{n+1}\right) & \text{for } x \in \left[z + \frac{1}{2}, z + \frac{1}{2} + \frac{1}{n+1}\right), z \in B_n, \\ 0 & \text{for other } x \in \mathbb{R}. \end{cases}$$

It can be shown that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad \text{for } x \in \mathbb{R}$$

is continuous and μ-almost periodic (cf. [9, Example 8] and Example 4.22 below).

For every $n \in \mathbb{N}$ we have

$$\frac{1}{2^{2^n}} \int_0^{2^{2^n}} f_n(x)\mathrm{d}x = 0 \quad \text{and} \quad \frac{1}{2^{2^n} + 1} \int_0^{2^{2^n}+1} f_n(x)\mathrm{d}x = \frac{2^{2^n}}{2^{2^n} + 1}.$$

On the other hand, since the function f_n is 4^{2^n}-periodic, for $T \geq 4^{2^n}$ we have

$$\frac{1}{T} \int_0^T f_n(x)\mathrm{d}x \leq \frac{1}{[\frac{T}{4^{2^n}}] \cdot 4^{2^n}} \int_0^{([\frac{T}{4^{2^n}}]+1)\cdot 4^{2^n}} f_n(x)\mathrm{d}x = \frac{[\frac{T}{4^{2^n}}] + 1}{[\frac{T}{4^{2^n}}]} \cdot \frac{2^{2^n}}{4^{2^n}} \leq \frac{2}{2^{2^n}}.$$

If $1 \leq k \leq n - 1$, then $2^{2^n} \geq 4^{2^k}$. Therefore,

$$\frac{1}{2^{2^n}} \int_0^{2^{2^n}} f(x)\mathrm{d}x = \sum_{k=1}^{n-1} \frac{1}{2^{2^n}} \int_0^{2^{2^n}} f_k(x)\mathrm{d}x \leq \sum_{k=1}^{n-1} \frac{2}{2^{2^k}} \leq 2\sum_{k=1}^{\infty} \frac{1}{2^{2k}} = \frac{2}{3} \quad \text{for } n \geq 2.$$

Finally, for $n \in \mathbb{N}$ we also have

$$\frac{1}{2^{2^n} + 1} \int_0^{2^{2^n}+1} f(x)\mathrm{d}x \geq \frac{1}{2^{2^n} + 1} \int_0^{2^{2^n}+1} f_n(x)\mathrm{d}x = \frac{2^{2^n}}{2^{2^n} + 1}.$$

Hence the mean value $\mathcal{M}\{f\}$ does not exist.

Observe that the function f from Example 3.3 is not S^1-bounded. Let us also add that it is possible to construct an example of a continuous μ-almost periodic function f which is not S^1-bounded and whose mean value exists but is infinite, that is, $\mathcal{M}\{f\} = +\infty$. Therefore, a natural question arises whether every locally integrable μ-almost periodic function which is not S^1-bounded fails to have finite mean value. The answer to this question is provided by the following example.

Example 3.4. Let $A_n = 2 \cdot 3^n \mathbb{Z} - 3^n$ for $n \in \mathbb{N}$. Let us put

$$
f_n(x) = \begin{cases} n^2 & \text{for } x \in [z, z + \frac{1}{n}), \, z \in A_n, \\ 0 & \text{for other } x \in \mathbb{R}. \end{cases}
$$

As in Example 3.3 define a locally integrable μ-almost periodic function $f : \mathbb{R} \to \mathbb{R}$ by the formula

$$
f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad \text{for } x \in \mathbb{R}
$$

(cf. also [9, Example 8] and Example 4.22 below). Note that, since

$$
\int_{z}^{z+1} f(x)\mathrm{d}x \geq \int_{z}^{z+1} f_n(x)\mathrm{d}x = n \quad \text{for } z \in A_n,
$$

the function f is not S^1-bounded, and hence cannot be S^1-almost periodic (see Remark 2.11). However, as we will show below, its mean value exists and

$$
\mathcal{M}\{f\} = \lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} f(x)\mathrm{d}x = \sum_{n=1}^{\infty} \frac{n}{2 \cdot 3^n}.
$$

First, observe that for every $n \in \mathbb{N}$ the function f_n is $(2 \cdot 3^n)$-periodic, and therefore

$$
\lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} f_n(x)\mathrm{d}x = \frac{1}{2 \cdot 3^n} \int_{0}^{2 \cdot 3^n} f_n(x)\mathrm{d}x = \frac{n}{2 \cdot 3^n}
$$

(cf. [42, Remark, p. 88]). Furthermore, for every $T > 0$ we have

$$
\frac{1}{T} \int_{0}^{T} f_n(x)\mathrm{d}x \leq \frac{n}{3^n}.
$$

Indeed, if $0 < T \leq 3^n$, then

$$
\frac{1}{T} \int_{0}^{T} f_n(x)\mathrm{d}x = 0 \leq \frac{n}{3^n};
$$

if $3^n < T < 2 \cdot 3^n$, then

$$\frac{1}{T}\int_0^T f_n(x)\mathrm{d}x \le \frac{1}{3^n}\int_0^{2\cdot 3^n} f_n(x)\mathrm{d}x = \frac{n}{3^n};$$

and finally, if $T \ge 2\cdot 3^n$, then

$$\frac{1}{T}\int_0^T f_n(x)\mathrm{d}x \le \frac{1}{2\cdot 3^n\left[\frac{T}{2\cdot 3^n}\right]}\int_0^{2\cdot 3^n\left(\left[\frac{T}{2\cdot 3^n}\right]+1\right)} f_n(x)\mathrm{d}x = \frac{\left[\frac{T}{2\cdot 3^n}\right]+1}{\left[\frac{T}{2\cdot 3^n}\right]}\frac{n}{2\cdot 3^n} \le \frac{n}{3^n}.$$

Given $\varepsilon > 0$ let $k \in \mathbb{N}$ be such that

$$\sum_{n=k+1}^{\infty}\frac{n}{3^n} < \frac{1}{3}\varepsilon,$$

and choose $T_0 > 0$ such that for $T \ge T_0$ we have

$$\left|\frac{1}{T}\int_0^T\sum_{n=1}^k f_n(x)\mathrm{d}x - \sum_{n=1}^k\frac{n}{2\cdot 3^n}\right| < \frac{1}{3}\varepsilon.$$

Then, for $T \ge T_0$ we have

$$\left|\frac{1}{T}\int_0^T f(x)\mathrm{d}x - \sum_{n=1}^{\infty}\frac{n}{2\cdot 3^n}\right|$$

$$= \left|\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}f_n(x)\mathrm{d}x - \sum_{n=1}^{\infty}\frac{n}{2\cdot 3^n}\right| \le$$

$$\le \left|\frac{1}{T}\int_0^T\sum_{n=1}^k f_n(x)\mathrm{d}x - \sum_{n=1}^k\frac{n}{2\cdot 3^n}\right| + \frac{1}{T}\int_0^T\sum_{n=k+1}^{\infty}f_n(x)\mathrm{d}x + \sum_{n=k+1}^{\infty}\frac{n}{2\cdot 3^n} \le$$

$$\le \frac{1}{3}\varepsilon + \sum_{n=k+1}^{\infty}\frac{n}{3^n} + \sum_{n=k+1}^{\infty}\frac{n}{3^n} \le \varepsilon$$

(let us note that we could change the order of summation and integration in the above estimate, since on each interval $[0, T]$ only finitely many functions f_n do not vanish; cf. also [18, Theorem 12.21]). This shows that

$$\mathcal{M}\{f\} = \sum_{n=1}^{\infty}\frac{n}{2\cdot 3^n}.$$

Now we are going to provide a sufficient condition guaranteeing that the mean value of a locally integrable μ-almost periodic functions exists. However, first we need the following

Definition 3.5. For $f : \mathbb{R} \to \mathbb{R}$ and $N > 0$ we define the *truncated function* f_N corresponding to the function f by the formula $f_N(x) = \max\big(-N, \min(f(x), N)\big)$ for $x \in \mathbb{R}$.

Theorem 3.6. *Let $f \in L^1_{loc}(\mathbb{R})$ be a non-negative* (*almost everywhere*) *μ-almost periodic function. If for every $\varepsilon > 0$ there exist positive numbers T_0 and N_0 such that*

$$\frac{1}{T} \int_0^T \big(f(x) - f_{N_0}(x)\big)dx < \varepsilon \qquad \text{for } T \geq T_0,$$

then $\mathcal{M}\{f\}$ exists and is finite, and moreover $\mathcal{M}\{f\} = \lim_{N \to +\infty} \mathcal{M}\{f_N\}$.

Before we proceed to the proof of Theorem 3.6, several remarks are in order.

Remark 3.7. Let us observe that if f is a μ-almost periodic function, then the truncated function f_N is μ-almost periodic for every $N > 0$ (cf. [32, the proof of Theorem 9] and see [33, p. 172]), and since it is also bounded, we infer that f_N is S^p-almost periodic for every $p \in [1, +\infty)$ (cf. Remark 2.17). In particular, the mean value $\mathcal{M}\{f_N\}$ exists and is finite (see Remark 3.2). If, in addition, the function f is assumed to be almost everywhere non-negative, then the mapping $N \mapsto \mathcal{M}\{f_N\}$ is non-decreasing and the limit $\lim_{N \to +\infty} \mathcal{M}\{f_N\}$ exists (we do not exclude here that the limit is equal to $+\infty$).

Proof of Theorem 3.6. In view of the assumption, for $\varepsilon = 1$ and $N \geq N_0$ we have

$$\frac{1}{T} \int_0^T \big(f_N(x) - f_{N_0}(x)\big)dx \leq \frac{1}{T} \int_0^T \big(f(x) - f_{N_0}(x)\big)dx < 1, \qquad \text{whenever } T \geq T_0.$$

Thus, by Remark 3.7, we infer that the limit $m := \lim_{N \to +\infty} \mathcal{M}\{f_N\}$ exists and is finite.

Now, we will show that $\mathcal{M}\{f\} = m$. Given $\varepsilon > 0$ there exists $N_1 > 0$ such that $0 \leq m - \mathcal{M}\{f_N\} < \frac{1}{3}\varepsilon$ for $N \geq N_1$. Furthermore, in view of the assumption, there exist $T_1, N_2 > 0$ such that

$$\frac{1}{T} \int_0^T \big(f(x) - f_N(x)\big)dx \leq \frac{1}{T} \int_0^T \big(f(x) - f_{N_2}(x)\big)dx < \frac{1}{3}\varepsilon \qquad \text{for } T \geq T_1 \text{ and } N \geq N_2.$$

Let $N_3 = \max(N_1, N_2)$. Then, since the function f_{N_3} is S^1-almost periodic (cf. Remark 3.7), the mean value $\mathcal{M}\{f_{N_3}\}$ exists and is finite, and therefore there is $T_2 > 0$ such that

$$\left| \frac{1}{T} \int_0^T f_{N_3}(x)dx - \mathcal{M}\{f_{N_3}\} \right| < \frac{1}{3}\varepsilon \qquad \text{for } T \geq T_2.$$

Hence, for $T \geq \max(T_1, T_2)$ we have

$$\left| \frac{1}{T} \int_0^T f(x)\mathrm{d}x - m \right| \leq \frac{1}{T} \int_0^T \big(f(x) - f_{N_3}(x) \big)\mathrm{d}x + \left| \frac{1}{T} \int_0^T f_{N_3}(x)\mathrm{d}x - \mathcal{M}\{f_{N_3}\} \right|$$
$$+ m - \mathcal{M}\{f_{N_3}\} < \varepsilon,$$

which shows that $\mathcal{M}\{f\} = m$ and ends the proof. $\quad\square$

Remark 3.8. Let us note that if a locally integrable function $f : \mathbb{R} \to \mathbb{R}$ has finite mean value $\mathcal{M}\{f\}$, then for every $\alpha \in \mathbb{R}$ we have

$$\lim_{T \to +\infty} \frac{1}{T} \int_\alpha^{\alpha+T} f(x)\mathrm{d}x = \mathcal{M}\{f\}. \tag{3}$$

It is also known that for functions almost periodic in the sense of Bohr or Stepanov (with any $p \in [1, +\infty)$), the limit in (3) exists uniformly in $\alpha \in \mathbb{R}$ (see [26, Theorem 1.3.2 and Theorem 5.6.2] or [35, Theorem 1.9 and Theorem 2.16]). However, this result is no longer true, if we consider functions which are μ-almost periodic; to see this it suffices to consider the function f defined in Example 3.4.

Now, we will proceed to the study of further properties of the mean value, which will be needed in the sequel.

Proposition 3.9 (cf. [27, Lemma 3.7]). *Let $f \in S^1(\mathbb{R})$ be almost everywhere non-negative. Then $\mathcal{M}\{f\} = 0$ if and only if $f(t) = 0$ for almost all $t \in \mathbb{R}$.*

Proof. It is obvious that $f \equiv 0$ a.e. implies $\mathcal{M}\{f\} = 0$. The remaining part was proved in [27]. $\quad\square$

Clearly, if in Proposition 3.9 the function f is uniformly almost periodic, then the equality $\mathcal{M}\{f\} = 0$ implies that $f(t) = 0$ for every $t \in \mathbb{R}$.

In the case of μ-almost periodic functions we have the following

Theorem 3.10. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally integrable μ-almost periodic function which is almost everywhere non-negative. Then $\mathcal{M}\{f\} = 0$ if and only if $f(t) = 0$ for almost all $t \in \mathbb{R}$.*

Proof. Since the sufficiency part is obvious, let us assume that $\mathcal{M}\{f\} = 0$ and suppose that f does not vanish almost everywhere. Then, there exist a point $u \in \mathbb{R}$, together with positive numbers ε, η and a Lebesgue measurable set $A \subseteq [u, u+1]$ with $\mu(A) = \varepsilon$ such that $f(x) \geq \eta$ for a.e. $x \in A$. For every $n \in \mathbb{N}$ choose $\tau_n \in (2(n-1)\omega - u, 2(n-1)\omega - u + \omega) \cap \mu E\{\frac{\varepsilon}{2}, \frac{\eta}{2}, f\}$, where ω is a number which characterizes the relative density of the set $\mu E\{\frac{\varepsilon}{2}, \frac{\eta}{2}, f\}$ (clearly, we may assume that $\omega > 1$). Then

$$\mu\big(\{x \in [u, u+1] : |f(x + \tau_n) - f(x)| < \tfrac{\eta}{2}\} \big) \geq 1 - \frac{\varepsilon}{2},$$

and thus

$$\mu\big(\{x \in A : f(x + \tau_n) \geq \tfrac{\eta}{2}\}\big) \geq \mu\big(\{x \in A : |f(x + \tau_n) - f(x)| < \tfrac{\eta}{2}\}\big) \geq \frac{\varepsilon}{2}.$$

So, we have

$$\int\limits_{2(n-1)\omega}^{2n\omega} f(x)\mathrm{d}x \geq \frac{\varepsilon\eta}{4} \qquad \text{for } n \in \mathbb{N},$$

and hence

$$\frac{1}{2n\omega} \int\limits_{0}^{2n\omega} f(x)\mathrm{d}x \geq \frac{\varepsilon\eta}{8\omega} \qquad \text{for } n \in \mathbb{N}.$$

This contradicts the fact that $\mathcal{M}\{f\} = 0$. $\quad\square$

Remark 3.11. The difference between non-negative μ-almost periodic functions and non-negative S^p-almost periodic functions is the following: for S^p-almost periodic functions we have $\mathcal{M}\{f\} = 0$ or $\mathcal{M}\{f\} > 0$, while for μ-almost periodic functions from the negation of the condition $\mathcal{M}\{f\} = 0$ it does not follow that $\mathcal{M}\{f\} > 0$, since the mean value may not exist.

Example 3.12. Let us note that, in general, $\mathcal{M}\{f\} = 0$ does not imply that $f(t) = 0$ a.e. on $\mathbb{R}$, even if f is non-negative. Let us take for instance the function $f : \mathbb{R} \to \mathbb{R}$ defined by the formula $f(t) = \mathrm{e}^{-t^2/2}$ for $t \in \mathbb{R}$. Then, for every $T > 0$, we have

$$\frac{1}{T} \int\limits_{0}^{T} f(u)\,\mathrm{d}u \leq \frac{1}{T} \int\limits_{\mathbb{R}} f(u)\,\mathrm{d}u = \frac{\sqrt{2\pi}}{T},$$

which shows that $\mathcal{M}\{f\} = 0$, even though $f(t) > 0$ for every $t \in \mathbb{R}$.

Remark 3.13. Let us observe that the mean value $\mathcal{M}$ is a continuous functional on the spacer $AP(\mathbb{R})$ and $S^p(\mathbb{R})$ with $p \in [1, +\infty)$, which means that $\mathcal{M}\{f_n\} \to \mathcal{M}\{f\}$ if $f_n \to f$ in the corresponding almost-periodic norm, since $|\mathcal{M}\{f - g\}| \leq \mathcal{M}\{|f - g|\} \leq \|f - g\|_\infty$ for $f, g \in AP(\mathbb{R})$ and $|\mathcal{M}\{f - g\}| \leq \mathcal{M}\{|f - g|\} \leq \|f - g\|_{S_1^p}$ for $f, g \in S^p(\mathbb{R})$.

The next example shows, however, that, if a sequence $(f_n)_{n \in \mathbb{N}}$ of μ-almost periodic functions is D-convergent to f, then it may happen that $\mathcal{M}\{f_n\} \not\to \mathcal{M}\{f\}$, even if $\mathcal{M}\{f\}$ exists and is finite.

Example 3.14. For every $n \in \mathbb{N}$ let

$$f_n(x) = \begin{cases} n & \text{for } x \in [z, z + \tfrac{1}{n}), z \in \mathbb{Z}, \\ 0 & \text{for other } x \in \mathbb{R}, \end{cases}$$

and let us note that the functions f_n are locally integrable and 1-periodic, and thus μ-almost periodic.

It is easy to show that the sequence $(f_n)_{n\in\mathbb{N}}$ is D-convergent to the zero function f. However, $\mathcal{M}\{f_n\} \not\to \mathcal{M}\{f\}$, since $\mathcal{M}\{f_n\} = 1$ for $n \in \mathbb{N}$ and $\mathcal{M}\{f\} = 0$.

4. Firing map and its displacement

The following section is devoted to the study of the firing map and the displacement map corresponding to the LIF model (1)–(2). Starting in Subsection 4.1 with the discussion of some general properties of the above-mentioned maps, we then move to the investigation of the firing map and its displacement for the LIF model driven by Stepanov and μ-almost periodic functions.

4.1. General properties of the firing map and its displacement

Since throughout the rest of the paper we will consider leaky integrate-and-fire models with a locally integrable almost periodic input, first we need to rewrite the definition of the firing map for that setting.

Definition 4.1. Let $f \in L^1_{\text{loc}}(\mathbb{R})$. The *firing map* Φ corresponding to the system (1)–(2) is defined as

$$\Phi(t) := \inf\left\{t_* > t : e^{\sigma t} \leq \int_t^{t_*} \big(f(u) - \sigma\big)e^{\sigma u}\,du\right\}, \qquad t \in \mathbb{R}.$$

Remark 4.2. Let us observe that if by $x(\cdot\,; t, 0)$ we denote the solution of (1) originating from the point $(t, 0)$, then, equivalently, the value of the firing map at t may be defined by the formula $\Phi(t) = \inf\{t_* > t : x(t_*; t, 0) \geq 1\}$. Therefore, roughly speaking, the firing map assigns to each point $t \in \mathbb{R}$ the time $\Phi(t)$ at which the trajectory of (1) originating from $(t, 0)$ reaches the threshold.

Example 4.3. Let $\sigma = 1$ and let us consider the LIF model (1)–(2) driven by the locally integrable 2-periodic function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = \begin{cases} 2 & \text{for } t \in [2k, 2k + 1), \, k \in \mathbb{Z}, \\ 1 & \text{for } t \in [2k + 1, 2k + 2), \, k \in \mathbb{Z}. \end{cases}$$

It can be checked that the firing map Φ corresponding to such a model has the formula

$$\Phi(t) = \begin{cases} \ln(2e^t) & \text{for } t \in [2k, 2k + 1 - \ln 2], \, k \in \mathbb{Z}, \\ \ln(2e^t + e^{2k+2} - e^{2k+1}) & \text{for } t \in (2k + 1 - \ln 2, 2k + 1), \, k \in \mathbb{Z}, \\ \ln(e^t + e^{2k+2}) & \text{for } t \in [2k + 1, 2k + 2), \, k \in \mathbb{Z}. \end{cases}$$

It may happen that the firing map Φ is not well-defined for every $t \in \mathbb{R}$, meaning that for some $\tau \in \mathbb{R}$ the set appearing in the definition of $\Phi(\tau)$ is empty. However, we have the following known result which describes necessary and sufficient conditions for Φ to be well-defined for

every $t \in \mathbb{R}$; for Readers' convenience we will provide its proof, since the proof presented in [28] contains a minor gap.

Proposition 4.4 *([28, Lemma 2.2]). Let $f \in L^1_{loc}(\mathbb{R})$. The firing map Φ corresponding to the system* (1)–(2) *is well-defined for every $t \in \mathbb{R}$ if and only if*

$$\limsup_{t \to +\infty} \int_0^t \big(f(u) - \sigma\big)e^{\sigma u} \, du = +\infty. \tag{4}$$

Before we pass to the proof of Proposition 4.4, let us observe that if the value $\Phi(t)$ is well-defined for some $t \in \mathbb{R}$, then it has to satisfy the implicit equation

$$e^{\sigma t} = \int_t^{\Phi(t)} \big(f(u) - \sigma\big)e^{\sigma u} \, du. \tag{5}$$

Proof of Proposition 4.4. Suppose that the condition (4) is satisfied and fix $t_0 \in \mathbb{R}$. Then $\limsup_{t \to +\infty} \int_{t_0}^t (f(u) - \sigma)e^{\sigma u} \, du = +\infty$, and hence there exists $t_* > t_0$ such that $\int_{t_0}^{t_*} \big(f(u) - \sigma\big)e^{\sigma u} \, du \geq e^{\sigma t_0}$. Consequently, $\Phi(t_0)$ is defined.

Now, let us assume that $\Phi \colon \mathbb{R} \to \mathbb{R}$ is well-defined for every $t \in \mathbb{R}$. In particular, by (5), for $t = 0$ and every $n \in \mathbb{N}$ we have

$$\int_0^{\Phi^n(0)} \big(f(u) - \sigma\big)e^{\sigma u} \, du = \sum_{i=1}^n \int_{\Phi^{i-1}(0)}^{\Phi^i(0)} \big(f(u) - \sigma\big)e^{\sigma u} \, du = \sum_{i=1}^n e^{\sigma \Phi^{i-1}(0)} \geq n;$$

here Φ^n denotes the n-th iterate of Φ and, by definition, we set $\Phi^0(0) = 0$. Moreover, let us observe that the increasing sequence $\big(\Phi^n(0)\big)_{n \in \mathbb{N}}$ is unbounded, since otherwise we would have $n \leq \int_0^a |f(u) - \sigma|e^{\sigma u} \, du < +\infty$ for $n \in \mathbb{N}$ and some $a \in (0, +\infty)$ which is independent of n, and a contradiction with the local integrability of f would follow. Thus $\lim_{n \to \infty} \Phi^n(0) = +\infty$ and $\lim_{n \to \infty} \int_0^{\Phi^n(0)} \big(f(u) - \sigma\big)e^{\sigma u} \, du = +\infty$, which proves the claim. $\square$

Remark 4.5. Let us note that a slight modification of the proof of Proposition 4.4 shows that if $f \in L^1_{loc}(\mathbb{R})$, then the firing map Φ corresponding to the LIF model (1)–(2) is well-defined on $\mathbb{R}$ if and only if for some $t \in \mathbb{R}$ all the iterates $\Phi^n(t)$ are well-defined.

From Proposition 4.4 we get the following corollary, which in the case of the PI model was stated in [27].

Corollary 4.6. *Let $f \in L^1_{loc}(\mathbb{R})$ and suppose that the mean value $\mathcal{M}\{f\}$ exists.[4] If $\mathcal{M}\{f\} > \sigma$, then the firing map Φ corresponding to the LIF model* (1)–(2) *is well-defined for every $t \in \mathbb{R}$.*

---

[4] We do not exclude the case when the mean value $\mathcal{M}\{f\}$ is infinite.

Proof. For simplicity, let us set

$$\mathcal{N}(t) = \int_0^t \big(f(u) - \sigma\big)e^{\sigma u}\,du \qquad \text{for } t \geq 0.$$

It is easy to see that

$$\mathcal{N}(t) = e^{\sigma t}\int_0^t \big(f(u) - \sigma\big)du - \sigma\int_0^t\left(\int_0^u \big(f(w) - \sigma\big)dw\right)e^{\sigma u}\,du; \tag{6}$$

indeed, it suffices to change the order of integration in the second integral on the right-hand side of (6).

Now, we will show that for every $n \in \mathbb{N}$ there exists $t_n \geq n$ such that $\mathcal{N}(t_n) > n$. Suppose on the contrary that there is some n such that

$$e^{\sigma t}\int_0^t \big(f(u) - \sigma\big)du \leq n + \sigma\int_0^t\left(\int_0^u \big(f(w) - \sigma\big)dw\right)e^{\sigma u}\,du \qquad \text{for all } t \geq n. \tag{7}$$

Let $g(t) := e^{\sigma t}\int_0^t\big(f(u) - \sigma\big)du$. Then (7) can be equivalently rewritten as

$$g(t) \leq n + \sigma\int_0^n g(u)\,du + \sigma\int_n^t g(u)\,du \qquad \text{for all } t \geq n.$$

Applying Gronwall's inequality (see, for example, [4, Corollary 1.4]), we infer that

$$g(t) \leq \left(n + \sigma\int_0^n g(u)\,du\right)e^{\sigma(t-n)} \qquad \text{for } t \geq n.$$

Therefore,

$$\frac{1}{t}\int_0^t \big(f(u) - \sigma\big)du \leq \frac{1}{t}e^{-\sigma n}\left(n + \sigma\int_0^n\left(\int_0^u \big(f(w) - \sigma\big)dw\right)e^{\sigma u}\,du\right) \qquad \text{for } t \geq n. \tag{8}$$

Passing to the limit in (8) with $t \to +\infty$, yields $\mathcal{M}\{f\} - \sigma \leq 0$. This, however, leads to a contradiction.

This shows that for every $n \in \mathbb{N}$ there exists $t_n \geq n$ such that $\mathcal{N}(t_n) > n$, and thus

$$\limsup_{t \to +\infty}\int_0^t \big(f(u) - \sigma\big)e^{\sigma u}\,du = +\infty.$$

To end the proof it suffices now to apply Proposition 4.4. $\square$

Remark 4.7. Let us note that the firing map Φ corresponding to the system (1)–(2) may be not well-defined for every $t \in \mathbb{R}$ even if the function $f \in L^1_{\text{loc}}(\mathbb{R})$ is such that $f - \sigma > 0$ a.e. on $\mathbb{R}$; to see this it suffices to consider the function $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(t) = \sigma + e^{-(\sigma+1)t}$. However, it should be observed that if $f \in L^1_{\text{loc}}(\mathbb{R})$ is such that $f(t) - \sigma \geq 0$ for a.e. $t \in \mathbb{R}$ and $\Phi(t_0)$ is defined for some $t_0 \in \mathbb{R}$, then $\Phi(t)$ is defined for every $t \leq t_0$. For functions f such that the difference $f - \sigma$ is negative on some set of positive Lebesgue measure, the above claim may not hold; to see this it suffices to consider the PI model and the function $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(t) = \frac{1}{2}\sin t$ for $t \in \mathbb{R}$, since then we have $\Phi(2\pi) = 3\pi$, but $\int_{\frac{\pi}{2}}^{t} f(u)\mathrm{d}u < 1$ for every $t > \frac{\pi}{2}$, which shows that $\Phi(\frac{\pi}{2})$ is not well-defined.

In our further considerations we will also need the following simple result on the monotonicity of the firing map (for a similar result for the PI model see [27]).

Lemma 4.8. *Suppose that* $f \in L^1_{loc}(\mathbb{R})$ *is such that* $f(t) - \sigma > 0$ *for a.e.* $t \in \mathbb{R}$ *and suppose that* $\Phi(t_0)$ *is defined for some* $t_0 \in \mathbb{R}$, *where* Φ *is the firing map corresponding to the LIF model* (1)–(2). *Then* $\Phi(s)$ *is well-defined for every* $s < t_0$ *and* $\Phi(s) < \Phi(t_0)$.

Proof. First, let us note that in view of Remark 4.7 $\Phi(s)$ is defined for every $s < t_0$.

Now, on the contrary, let us suppose that $\Phi(t_0) \leq \Phi(s)$. Then

$$\int\limits_{t_0}^{\Phi(t_0)} \big(f(u) - \sigma\big)e^{\sigma(u-t_0)}\mathrm{d}u = \int\limits_{s}^{\Phi(s)} \big(f(u) - \sigma\big)e^{\sigma(u-s)}\mathrm{d}u,$$

and thus

$$0 \leq \int\limits_{\Phi(t_0)}^{\Phi(s)} \big(f(u) - \sigma\big)e^{\sigma(u-s)}\mathrm{d}u = \int\limits_{t_0}^{\Phi(t_0)} \big(f(u) - \sigma\big)e^{\sigma(u-t_0)}\mathrm{d}u - \int\limits_{s}^{\Phi(t_0)} \big(f(u) - \sigma\big)e^{\sigma(u-s)}\mathrm{d}u$$

$$\leq -\int\limits_{s}^{t_0} \big(f(u) - \sigma\big)e^{\sigma(u-s)}\mathrm{d}u < 0.$$

This leads to a contradiction. Therefore, $\Phi(t_0) > \Phi(s)$. $\quad\square$

Remark 4.9. Similar result to Lemma 4.8 holds also in the case when $f(t) - \sigma \geq 0$ for a.e. $t \in \mathbb{R}$. Then, in the claim one needs to replace the strict inequality $\Phi(s) < \Phi(t_0)$ with $\Phi(s) \leq \Phi(t_0)$.

Since it will be easier to formulate our next result in terms of the displacement map, we proceed now with the following

Definition 4.10. Let $f \in L^1_{\text{loc}}(\mathbb{R})$ and let Φ be the firing map corresponding to the system (1)–(2). The *displacement map* Ψ of the map Φ is defined as $\Psi(t):=\Phi(t) - t$.

Clearly, the displacement map is well-defined only for those t's for which the value $\Phi(t)$ is well-defined. Then of course $\Psi(t) \geq 0$, since $\Phi(t) \geq t$ by definition. Roughly speaking, the value $\Psi(t)$ says how long we have to wait for the next firing if the previous firing was at time t.

Our next result describes sufficient conditions for the displacement map to be uniformly continuous.

Theorem 4.11. *Let $\varsigma > 0$ and assume that $f \in L^1_{loc}(\mathbb{R})$ satisfies the following conditions:*

(i) $\displaystyle \sup_{s \in \mathbb{R}} \int_s^{s+\delta} f(u)\mathrm{d}u \to 0$ *as $\delta \to 0^+$;*

(ii) $f(t) - \sigma \geq \varsigma$ *for a.e. $t \in \mathbb{R}$.*

Then the displacement map $\Psi \colon \mathbb{R} \to \mathbb{R}$ corresponding to the LIF model (1)–(2) is bounded and uniformly continuous. Moreover, $\inf_{t \in \mathbb{R}} \Psi(t) > 0$.

Proof. First, let us note that in view of the assumption (ii) and Proposition 4.4 the firing map Φ and the displacement map Ψ are well-defined for every $t \in \mathbb{R}$.

The boundedness of the displacement map Ψ follows from the following estimate

$$e^{\sigma t} = \int_t^{\Phi(t)} \big(f(u) - \sigma\big)e^{\sigma u}\mathrm{d}u \geq \varsigma e^{\sigma t}\big(\Phi(t) - t\big) = \varsigma e^{\sigma t}\Psi(t) \qquad \text{for every } t \in \mathbb{R}.$$

Now, we are going to prove that Ψ is uniformly continuous. We start with observing that, in view of the assumption (i), for every $t \in \mathbb{R}$ the integral $\int_t^{t+1/\varsigma} f(u)\mathrm{d}u$ can be estimated from above by a constant independent of t. Indeed, there exists $\delta_0 > 0$ such that

$$\sup_{s \in \mathbb{R}} \int_s^{s+\delta_0} f(u)\mathrm{d}u \leq 1,$$

and so, if $k \in \mathbb{N}$ is the smallest number such that $1/\varsigma \leq k\delta_0$, then

$$\int_t^{t+1/\varsigma} f(u)\mathrm{d}u \leq \sum_{i=0}^{k-1} \int_{t+i\delta_0}^{t+(i+1)\delta_0} f(u)\mathrm{d}u \leq k \cdot \sup_{s \in \mathbb{R}} \int_s^{s+\delta_0} f(u)\mathrm{d}u \leq k \qquad \text{for every } t \in \mathbb{R}.$$

Now, for a given $\varepsilon > 0$ choose $\delta > 0$ such that

$$\frac{1}{\varsigma}e^{\sigma\delta}\sup_{s \in \mathbb{R}} \int_s^{s+\delta} f(u)\mathrm{d}u \leq \frac{1}{2}\varepsilon \qquad \text{and} \qquad \frac{1}{\varsigma}(e^{\sigma\delta} - 1)k \leq \frac{1}{2}\varepsilon.$$

Take arbitrary points $t, \tau \in \mathbb{R}$ such that $|t - \tau| \leq \delta$. Without loss of generality, we may assume that $t \leq \tau$. Then $\Phi(t) \leq \Phi(\tau)$ and

$$\int_t^{\Phi(t)} \big(f(u) - \sigma\big)e^{\sigma(u-t)}\mathrm{d}u = \int_\tau^{\Phi(\tau)} \big(f(u) - \sigma\big)e^{\sigma(u-\tau)}\mathrm{d}u.$$

Thus

$$\int\limits_{\Phi(t)}^{\Phi(\tau)} \big(f(u) - \sigma\big)e^{\sigma(u-\tau)}\mathrm{d}u = \int\limits_{t}^{\tau} \big(f(u) - \sigma\big)e^{\sigma(u-\tau)}\mathrm{d}u + \int\limits_{t}^{\Phi(t)} \big(f(u) - \sigma\big)\big(e^{\sigma(u-t)} - e^{\sigma(u-\tau)}\big)\mathrm{d}u.$$

Our aim is to estimate the above integrals. For the sake of simplicity let us denote them (starting from the left) by I_1, I_2 and I_3. Then

$$I_1 \geq \varsigma e^{\sigma(\Phi(t)-\tau)}\big(\Phi(\tau) - \Phi(t)\big)$$

and

$$I_2 \leq \int\limits_{t}^{\tau} f(u)\mathrm{d}u \leq \sup_{s\in\mathbb{R}} \int\limits_{s}^{s+\delta} f(u)\mathrm{d}u.$$

For the integral I_3 we have

$$I_3 = \int\limits_{t}^{\Phi(t)} \big(f(u) - \sigma\big)\big(e^{\sigma(u-t)} - e^{\sigma(u-\tau)}\big)\mathrm{d}u \leq \big(e^{\sigma(\Phi(t)-t)} - e^{\sigma(\Phi(t)-\tau)}\big) \cdot \int\limits_{t}^{\Phi(t)} f(u)\mathrm{d}u$$

$$\leq \big(e^{\sigma(\Phi(t)-t)} - e^{\sigma(\Phi(t)-\tau)}\big)k,$$

because $0 \leq \Phi(t) - t \leq 1/\varsigma$. Since $\Phi(t) - \tau \geq t - \tau \geq -\delta$, taking into account all the above estimates, we get

$$\Phi(\tau) - \Phi(t) \leq \frac{1}{\varsigma}e^{\sigma\delta} \sup_{s\in\mathbb{R}} \int\limits_{s}^{s+\delta} f(u)\mathrm{d}u + \frac{1}{\varsigma}(e^{\sigma\delta} - 1)k \leq \varepsilon.$$

This shows that the firing map Φ is uniformly continuous. Then, of course, the displacement map Ψ is also uniformly continuous.

Finally, we will show that $\inf_{t\in\mathbb{R}} \Psi(t) > 0$. Let us observe that

$$1 = \int\limits_{t}^{t+\Psi(t)} \big(f(u) - \sigma\big)e^{\sigma(u-t)}\mathrm{d}u \leq e^{\sigma\Psi(t)} \int\limits_{t}^{t+\Psi(t)} \big(f(u) - \sigma\big)\mathrm{d}u \leq e^{\sigma/\varsigma} \int\limits_{t}^{t+\Psi(t)} f(u)\mathrm{d}u,$$

and therefore

$$0 < e^{-\sigma/\varsigma} \leq \int\limits_{t}^{t+\Psi(t)} f(u)\mathrm{d}u \qquad \text{for every } t \in \mathbb{R}.$$

Moreover, in view of the assumption (i), there is $\delta > 0$ such that

$$\sup_{t \in \mathbb{R}} \int_t^{t+\delta} f(u)\mathrm{d}u \le \frac{1}{2}e^{-\sigma/\varsigma}.$$

If $\inf_{t \in \mathbb{R}} \Psi(t) = 0$, then there would exist a real number t_0 such that $\Psi(t_0) \le \delta$. So we would have

$$0 < e^{-\sigma/\varsigma} \le \int_{t_0}^{t_0+\Psi(t_0)} f(u)\mathrm{d}u \le \int_{t_0}^{t_0+\delta} f(u)\mathrm{d}u \le \frac{1}{2}e^{-\sigma/\varsigma},$$

which, clearly, is impossible. This shows that $\inf_{t \in \mathbb{R}} \Psi(t) > 0$. □

Let us add that the continuity of the firing map for the PI and LIF models with locally integrable forcing term was investigated, for example, in [27] and [28], respectively. In particular, in [28] the following result was established (here we added the missing assumption that the firing map Φ is well-defined)

Proposition 4.12 *([28, Lemma 2.11 (b)]). Let $f \in L^1_{loc}(\mathbb{R})$ and assume that the firing map Φ corresponding to the LIF model* (1)–(2) *is defined for every $t \in \mathbb{R}$. If $f(t) - \sigma > 0$ for a.e. $t \in \mathbb{R}$, then the firing map is continuous on $\mathbb{R}$.*

Remark 4.13. Let us note that in Proposition 4.12 the assumption that $f(t) - \sigma > 0$ for a.e. $t \in \mathbb{R}$ cannot be replaced (even if f is periodic) with a weaker condition: $f(t) - \sigma > 0$ for t belonging to a set of positive Lebesgue measure. To see this it suffices to consider the PI model driven by the locally integrable 2-periodic function $f \colon \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = \begin{cases} 1 & \text{for } t \in [2k, 2k+1], k \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

since then $\Phi(0) = 1$, but $\lim_{t \to 0^+} \Phi(t) = 2$ (see also Example 4.3).

4.2. Firing map and its displacement for LIF models with (almost) periodic input

In this subsection we investigate the properties of the firing map and its displacement for the LIF models driven by (almost) periodic functions. Special attention is paid to μ-almost periodic inputs and the (somewhat unexpected) behavior of such models.

We start with the following

Proposition 4.14. *Let us assume that $f \in L^1_{loc}(\mathbb{R})$ is μ-almost periodic and such that $f(t) - \sigma \ge 0$ for a.e. $t \in \mathbb{R}$. The firing map Φ corresponding to the LIF model* (1)–(2) *is well-defined for every $t \in \mathbb{R}$ if and only if there exists a Lebesgue measurable set $A \subseteq \mathbb{R}$ with $\mu(A) > 0$ such that $f(t) - \sigma > 0$ for a.e. $t \in A$.*

Proof. The proof of necessity is obvious, so let us proceed to the sufficiency part. From the assumptions it follows that there exist a point $u \in \mathbb{R}$, together with two positive numbers ε, η

and a set $B \subseteq A \cap [u, u + 1]$ with $\mu(B) = \varepsilon$ such that $f(t) - \sigma \geq \eta$ for a.e. $t \in B$. Reasoning analogous to that in the proof of Theorem 3.10 leads to the following estimate

$$\int\limits_0^{2n\omega} \big(f(u) - \sigma\big)\mathrm{d}u \geq \frac{\varepsilon\eta}{4}n \qquad \text{for } n \in \mathbb{N},$$

where the number ω characterizes the relative density of the set $\mu E\{\frac{\varepsilon}{2}, \frac{\eta}{2}, f - \sigma\}$. Thus, we get

$$\limsup_{t \to +\infty} \int\limits_0^t \big(f(u) - \sigma\big)e^{\sigma u}\mathrm{d}u \geq \lim_{n \to +\infty} \int\limits_0^{2n\omega} \big(f(u) - \sigma\big)\mathrm{d}u = +\infty.$$

This, in view of Proposition 4.4, implies that the firing map Φ is well-defined for every $t \in \mathbb{R}$. □

Remark 4.15. Let us note that Proposition 4.14 extends Proposition 3 from [27] to LIF models and μ-almost periodic forcing terms, since for a S^1-almost periodic function such that $f - \sigma \geq 0$ a.e. on $\mathbb{R}$, the existence of a set A with the requested properties is equivalent with the condition: $\mathcal{M}\{f\} > \sigma$.

Moreover, let us also add that Proposition 4.14 does not follow from Corollary 4.6, since for μ-almost periodic functions the mean value may not exist (cf. Example 3.3).

Now, let us recall a well-known result concerning the periodically driven models; the idea behind this results can be traced to the paper of J. P. Keener, F. C. Hoppensteadt and J. Rinzel (see [22]), although no rigorous formulation (i.e. similar to the following one) can be found there.

Proposition 4.16. *Let $f \in L^1_{loc}(\mathbb{R})$ be a ω-periodic function (with $\omega > 0$) and let us assume that the firing map Φ corresponding to the LIF model (1)–(2) is well-defined for every $t \in \mathbb{R}$. Then $\Phi(t + \omega) = \Phi(t) + \omega$ for $t \in \mathbb{R}$, and thus the displacement map of Φ is ω-periodic.*

Proof. The proof follows easily from the properties of definite integrals of periodic functions and the fact that $\omega + \inf A = \inf(\omega + A)$ for non-empty sets $A \subseteq \mathbb{R}$. □

As observed in the above proposition, periodically driven LIF models have periodic displacement maps (which, in particular, allows to view Φ as a lift of a degree one circle map, and therefore to explore its dynamics and the properties of the spike trains by tools of circle maps theory; see e.g. [16,22,28,30]). Therefore, a natural question arises whether for an almost periodic input f the corresponding firing map has always almost periodic displacement.

Before we address this question, let us, firstly, consider the case of a limit-periodic forcing term.

Theorem 4.17. *Let $f : \mathbb{R} \to \mathbb{R}$ be a limit-periodic function. Moreover, assume that there exists $\varsigma > 0$ such that $f(t) - \sigma > \varsigma$ for $t \in \mathbb{R}$. Then the firing map corresponding to the LIF model (1)–(2) has limit-periodic displacement.*

Proof. Since f is limit-periodic, there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of continuous periodic functions uniformly convergent to f on $\mathbb{R}$. Clearly, we may assume that $f_n(t) - \sigma > \frac{1}{2}\varsigma$ for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$.

By Proposition 4.4 the firing maps Φ and Φ_n (and their displacements Ψ and Ψ_n), which correspond to the LIF model (1)–(2) driven by f and f_n, respectively, are well-defined for every $t \in \mathbb{R}$. Moreover, in view of Theorem 4.11 (or Proposition 4.12) and Proposition 4.16, we infer that the displacement maps Ψ_n are continuous and periodic.

Let us also observe that in order to show that the displacement map Ψ is limit-periodic, it suffices to show that the sequence $(\Phi_n)_{n \in \mathbb{N}}$ is uniformly convergent to Φ on $\mathbb{R}$, since $\sup_{t \in \mathbb{R}} |\Psi(t) - \Psi_n(t)| = \sup_{t \in \mathbb{R}} |\Phi(t) - \Phi_n(t)|$.

Given $\varepsilon > 0$ choose $N \in \mathbb{N}$ such that $\sup_{t \in \mathbb{R}} |f_n(t) - f(t)| < \frac{1}{2}\varsigma^2\varepsilon$ for all $n \geq N$. Let $n \geq N$ and fix $t \in \mathbb{R}$. Suppose that $\Phi_n(t) > \Phi(t)$. We calculate that

$$\int_{\Phi(t)}^{\Phi_n(t)} \big(f_n(u) - \sigma\big)e^{\sigma u} \, du = \int_{t}^{\Phi(t)} \big(f(u) - f_n(u)\big)e^{\sigma u} \, du \leq \frac{1}{2}\varsigma^2\varepsilon e^{\sigma \Phi(t)}(\Phi(t) - t) \leq \frac{1}{2}\varsigma\varepsilon e^{\sigma \Phi(t)};$$

the last inequality in the above chain of inequalities follows from the fact that $\Phi(t) - t \leq 1/\varsigma$ for $t \in \mathbb{R}$ (cf. the proof of Theorem 4.11). Simultaneously,

$$\int_{\Phi(t)}^{\Phi_n(t)} \big(f_n(u) - \sigma\big)e^{\sigma u} \, du \geq \frac{1}{2}\varsigma e^{\sigma \Phi(t)}|\Phi_n(t) - \Phi(t)|.$$

Therefore, $|\Phi_n(t) - \Phi(t)| \leq \varepsilon$ for all $n \geq N$. When $\Phi(t) > \Phi_n(t)$ we arrive at the same conclusion. As a result, Φ is the uniform limit of $(\Phi_n)_{n \in \mathbb{N}}$. This ends the proof. $\quad\square$

Now we will pass to the case of an almost periodic input.

Theorem 4.18. *Let $f : \mathbb{R} \to \mathbb{R}$ be a S^1-almost periodic function. Moreover, assume that there exists $\varsigma > 0$ such that $f(t) - \sigma > \varsigma$ for a.e. $t \in \mathbb{R}$. Then the firing map Φ corresponding to the LIF model (1)–(2) has uniformly almost periodic displacement Ψ.*

Remark 4.19. Let us note that in the above result we can also assume that f is S^p-almost periodic for some $p \in [1, +\infty)$ or even uniformly almost periodic, since $AP(\mathbb{R}) \subseteq S^p(\mathbb{R}) \subseteq S^1(\mathbb{R})$ (see Remark 2.9 and Remark 2.12).

Proof of Theorem 4.18. Clearly, the firing map Φ and the displacement map Ψ are defined for every $t \in \mathbb{R}$. Furthermore, the maps Φ and Ψ are continuous in view of Proposition 4.12.

Without loss of generality we may assume that $\varsigma < 1$. Given $\varepsilon > 0$ let $\tau \in S^1 E\{\frac{\varsigma^2\varepsilon}{4}, f\}$. Then $\|f^\tau - f\|_{S^1_1} < \frac{\varsigma^2\varepsilon}{4}$. Now, fix $t \in \mathbb{R}$ and suppose that $\min\{\Phi(t), \Phi(t + \tau) - \tau\} = \Phi(t)$. From the definition of the firing map the following equality can be easily derived:

$$\left| \int_{t}^{\Phi(t)} \big(f(u + \tau) - f(u)\big)e^{\sigma(u+\tau)} \, du \right| = \left| \int_{\Phi(t)}^{\Phi(t+\tau)-\tau} \big(f(u + \tau) - \sigma\big)e^{\sigma(u+\tau)} \, du \right|. \tag{9}$$

Since $\tau \in S^1 E\{\frac{\varsigma^2 \varepsilon}{4}, f\}$, elementary calculations show that

$$\left| \int_t^{\Phi(t)} \big(f(u+\tau) - f(u)\big) e^{\sigma(u+\tau)} du \right| < e^{\sigma(\Phi(t)+\tau)} \frac{k\varsigma^2 \varepsilon}{4}, \tag{10}$$

where $k \in \mathbb{N}$ is the smallest integer such that $\Phi(t) \leq t + k$. Furthermore, since $f(u) - \sigma > \varsigma$ for a.e. $u \in \mathbb{R}$, we infer that

$$\int_{\Phi(t)}^{\Phi(t+\tau)-\tau} \big(f(u+\tau) - \sigma\big) e^{\sigma(u+\tau)} du \geq e^{\sigma(\Phi(t)+\tau)} \varsigma \big(\Phi(t+\tau) - \tau - \Phi(t)\big). \tag{11}$$

Let us note that $\Phi(u) - u \leq 1/\varsigma$ for any $u \in \mathbb{R}$, and thus $k \leq (1/\varsigma + 1)$. So by (9)–(11) and the above observation, we get

$$\Phi(t+\tau) - \tau - \Phi(t) < \frac{k\varsigma\varepsilon}{4} \leq \frac{(\frac{1}{\varsigma} + 1)\varsigma\varepsilon}{4} < \frac{\varepsilon}{2}.$$

The case $\min\{\Phi(t), \Phi(t+\tau) - \tau\} = \Phi(t+\tau) - \tau$ treated similarly yields $\Phi(t) - \Phi(t+\tau) + \tau < \frac{\varepsilon}{2}$. Thus $|\Phi(t+\tau) - \Phi(t) - \tau| < \frac{\varepsilon}{2}$ for every $t \in \mathbb{R}$, which means that the set $E\{\varepsilon, \Psi\}$ is relatively dense as it contains the relatively dense set $S^1 E\{\frac{\varsigma^2 \varepsilon}{4}, f\}$. This, in turn, shows that Ψ is uniformly almost periodic and ends the proof. $\quad\square$

Since a uniformly almost periodic function is uniformly continuous we have the following

Corollary 4.20. *Let $f : \mathbb{R} \to \mathbb{R}$ be a S^1-almost periodic function. Moreover, assume that there exists $\varsigma > 0$ such that $f(t) - \sigma > \varsigma$ for a.e. $t \in \mathbb{R}$. Then the firing map Φ corresponding to the LIF model (1)–(2) and its displacement Ψ are uniformly continuous.*

Remark 4.21. Corollary 4.20 is also a consequence of Theorem 4.11, since it can be shown that every almost everywhere non-negative S^1-almost periodic function satisfies the condition (i) of Theorem 4.11 (for more details see [9,31]; cf. also Theorem 5.1 below).

Now, we move to the investigation of the firing map and its displacement for LIF models driven by a locally integrable μ-almost periodic functions. We begin with an example showing that in the 'μ-almost periodic setting' it may happen that the displacement map, although uniformly continuous and bounded, fails to be almost periodic in any sense considered in this paper. This somewhat unexpected phenomenon has interesting consequences, since, for example, it allows to establish some results in the theory of almost periodicity (for more details see Section 5 below).

Example 4.22. Let us consider the following sets: $A_n = 2^n \mathbb{Z} + 2^{n-1}$ for $n \in \mathbb{N}$ and let us observe that

$$A_n \cap A_m = \emptyset \text{ if } n \neq m \qquad \text{and} \qquad \bigcup_{n \in \mathbb{N}} A_n = \mathbb{Z} \setminus \{0\}.$$

Furthermore, for the sake of simplicity, let us put

$$B_{n,k}(z) = \left(z + \frac{k}{2^{n-1}}, z + \frac{k}{2^{n-1}} + \frac{1}{2} \cdot \frac{1}{4^{n-1}}\right) \qquad \text{and} \qquad C_n = \bigcup_{z \in A_n} \bigcup_{k=0}^{2^{n-1}-1} B_{n,k}(z). \qquad (12)$$

Note that for a given $n \in \mathbb{N}$ and $z \in A_n$ we have $B_{n,k}(z) \cap B_{n,l}(z) = \emptyset$ for distinct $k, l \in \{0, \ldots, 2^{n-1} - 1\}$, and moreover $B_{n,k}(z) \subseteq (z, z+1)$ for $k \in \{0, \ldots, 2^{n-1} - 1\}$. Observe also that the sets C_n are pairwise disjoint.

Now, let us define the locally integrable functions $f_n \colon \mathbb{R} \to \mathbb{R}$ by the formulas:

$$f_n(x) = \begin{cases} 2^n & \text{for } x \in C_n, \\ 0 & \text{for } x \in \mathbb{R} \setminus C_n, \end{cases}$$

and let us note that for each $n \in \mathbb{N}$ the function f_n is 2^n-periodic, since $x \in C_n$ if and only if $x \pm 2^n \in C_n$.

We shall show that the locally integrable function $f \colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = 1 + \sum_{n=1}^{\infty} f_n(x) \qquad \text{for } x \in \mathbb{R}, \qquad (13)$$

is μ-almost periodic. In fact, we are going to show that f is the D-limit of the sequence $(g_k)_{k \in \mathbb{N}}$ of locally integrable periodic functions, where

$$g_k(x) = 1 + \sum_{n=1}^{k} f_n(x) \qquad \text{for } x \in \mathbb{R}.$$

Fix $\varepsilon > 0$ and $\eta \in (0, 1)$, and choose $N \in \mathbb{N}$ such that $2^{-(N+1)} < \varepsilon$. Then, for every $k \geq N$ and every $z \in \mathbb{Z} \setminus \{0\}$, we have

$$\{x \in [z, z+1] : |f(x) - g_k(x)| \geq \eta\} \subseteq [z, z+1] \cap C_m$$

for some $m \geq k + 1$. Hence

$$\mu\big(\{x \in [z, z+1] : |f(x) - g_k(x)| \geq \eta\}\big) \leq \mu\big([z, z+1] \cap C_m\big) \leq \frac{1}{2^m} \leq \frac{1}{2^{N+1}} \leq \varepsilon.$$

Since $\{x \in [0, 1] : |f(x) - g_k(x)| \geq \eta\} = \emptyset$ for every $k \in \mathbb{N}$, in view of Remark 2.13, we obtain that $D(\eta; f, g_k) \leq 2\varepsilon$ for $k \geq N$. This proves that the sequence $(g_k)_{k \in \mathbb{N}}$ is D-convergent to f, and thus the function f is μ-almost periodic (see Remark 2.19).

Now, we would like to show that the displacement map Ψ corresponding to the PI model driven by the function f is uniformly continuous and bounded. To this end we are going to apply Theorem 4.11. Since the assumption (ii) of Theorem 4.11 is satisfied with $\varsigma = 1$, it suffices to show that f satisfies the assumption (i). We claim that

$$\sup_{u \in \mathbb{R}} \int_{u}^{u + \frac{1}{2^{m-1}}} f(s) \, ds \le \frac{1}{2^{\frac{1}{2}m - 3}} \qquad \text{for } m \in \mathbb{N}. \tag{14}$$

First, observe that given any $u \in \mathbb{R}$ and $m \in \mathbb{N}$, there exist $z \in \mathbb{Z}$ and $k \in \{0, \dots, 2^{m-1} - 1\}$ such that

$$\left(u, u + \frac{1}{2^{m-1}} \right) \subseteq \left(z + \frac{k}{2^{m-1}}, z + \frac{k+2}{2^{m-1}} \right), \tag{15}$$

and therefore to obtain (14) it suffices to estimate the integral of the function f on every interval of the form

$$\left(z + \frac{k}{2^{m-1}}, z + \frac{k+1}{2^{m-1}} \right), \qquad \text{where } z \in \mathbb{Z} \text{ and } k \in \{0, \dots, 2^{m-1} - 1\}.$$

Let us consider the following five cases.

Case 1: If $z = 0$, then

$$\int_{z + \frac{k}{2^{m-1}}}^{z + \frac{k+1}{2^{m-1}}} f(s) \, ds = \frac{1}{2^{m-1}} \le \frac{1}{2^{\frac{1}{2}m - 2}}.$$

Case 2: If $z \in A_n$ for some $n \in \mathbb{N}$ and $n > m$, then there exists $l \in \{0, \dots, 2^{n-1} - 2^{n-m}\}$ such that

$$\frac{k}{2^{m-1}} = \frac{l}{2^{n-1}} < \frac{l+1}{2^{n-1}} < \dots < \frac{l + 2^{n-m} - 1}{2^{n-1}} < \frac{l + 2^{n-m}}{2^{n-1}} = \frac{k+1}{2^{m-1}}.$$

Moreover, for $r \in \{0, \dots, 2^{n-m} - 1\}$ we have

$$\left(\frac{l+r}{2^{n-1}}, \frac{l+r}{2^{n-1}} + \frac{1}{2} \cdot \frac{1}{4^{n-1}} \right) \subseteq \left(\frac{k}{2^{m-1}}, \frac{k+1}{2^{m-1}} \right),$$

and the intervals

$$\left(\frac{l+r}{2^{n-1}}, \frac{l+r}{2^{n-1}} + \frac{1}{2} \cdot \frac{1}{4^{n-1}} \right), \qquad r \in \{0, \dots, 2^{n-m} - 1\},$$

are pairwise disjoint. So

$$\int_{z + \frac{k}{2^{m-1}}}^{z + \frac{k+1}{2^{m-1}}} f(s) \, ds = \frac{1}{2^{m-1}} + \sum_{i=0}^{2^{n-m}-1} \int_{B_{n,l+i}(z)} f_n(s) \, ds = \frac{1}{2^{m-1}} + 2^{n-m} \cdot \frac{1}{2} \cdot \frac{1}{4^{n-1}} \cdot 2^n$$

$$= \frac{1}{2^{m-2}} \le \frac{1}{2^{\frac{1}{2}m - 2}}.$$

Case 3: If $z \in A_n$ for some $n \in \mathbb{N}$ and $n = m$, then

$$\int_{z+\frac{k}{2m-1}}^{z+\frac{k+1}{2m-1}} f(s)ds = \frac{1}{2^{m-1}} + \int_{z+\frac{k}{2m-1}}^{z+\frac{k+1}{2m-1}} f_m(s)ds = \frac{1}{2^{m-1}} + \int_{B_{m,k}(z)} f_m(s) = \frac{1}{2^{m-2}} \le \frac{1}{2^{\frac{1}{2}m-2}}.$$

Case 4: Suppose that $z \in A_n$ for some $n \in \mathbb{N}$ and $n < m \le 2n$. Then, it is easy to show that there exists exactly one number $l \in \{0, \ldots, 2^{n-1} - 1\}$ such that

$$\left(z + \frac{k}{2^{m-1}}, z + \frac{k+1}{2^{m-1}} \right) \subseteq \left(z + \frac{l}{2^{n-1}}, z + \frac{l+1}{2^{n-1}} \right).$$

And so

$$\int_{z+\frac{k}{2m-1}}^{z+\frac{k+1}{2m-1}} f(s)ds \le \frac{1}{2^{m-1}} + \int_{z+\frac{l}{2n-1}}^{z+\frac{l+1}{2n-1}} f_n(s)ds = \frac{1}{2^{m-1}} + \int_{B_{n,l}(z)} f_n(s)ds$$

$$= \frac{1}{2^{m-1}} + \frac{1}{2^{n-1}} \le \frac{1}{2^{m-1}} + \frac{1}{2^{\frac{1}{2}m-1}} \le \frac{1}{2^{\frac{1}{2}m-2}}.$$

Case 5: Suppose that $z \in A_n$ for some $n \in \mathbb{N}$ and $2n < m$. Then

$$\int_{z+\frac{k}{2m-1}}^{z+\frac{k+1}{2m-1}} f(s)ds = \frac{1}{2^{m-1}} + \int_{z+\frac{k}{2m-1}}^{z+\frac{k+1}{2m-1}} f_n(s)ds \le \frac{1}{2^{m-1}} + \frac{1}{2^{m-n-1}} \le \frac{1}{2^{m-1}} + \frac{1}{2^{\frac{1}{2}m-1}}$$

$$\le \frac{1}{2^{\frac{1}{2}m-2}}.$$

Summarizing, in each case, we have

$$\int_{z+\frac{k}{2m-1}}^{z+\frac{k+1}{2m-1}} f(s)ds \le \frac{1}{2^{\frac{1}{2}m-2}},$$

which in connection with (15) proves (14). This, in turn, implies that the function f satisfies the assumption (i) of Theorem 4.11, and therefore the displacement map Ψ corresponding to the PI model driven by the function f is bounded and uniformly continuous.

Finally, we will show that the displacement map Ψ is not μ-almost periodic. First, observe that

$$\int_z^{z+\frac{1}{2}} f(s)ds \ge 1 \qquad \text{for } z \in \mathbb{Z} \setminus \{0\}.$$

Indeed, if $z \in A_1$, then we have $B_{1,0}(z) = \left(z, z + \frac{1}{2}\right)$ and

$$\int_z^{z+\frac{1}{2}} f(s)\,ds \geq \int_z^{z+\frac{1}{2}} f_1(s)\,ds = 1.$$

If $z \in A_n$ for some $n \geq 2$, then for $k \in \{0, \ldots, 2^{n-2} - 1\}$ we have

$$\left(z + \frac{k}{2^{n-1}}, z + \frac{k}{2^{n-1}} + \frac{1}{2} \cdot \frac{1}{4^{n-1}}\right) \subseteq \left[z, z + \frac{1}{2}\right]$$

(for $k \in \{2^{n-2}, \ldots, 2^{n-1} - 1\}$ the above inclusion does not hold), and thus

$$\int_z^{z+\frac{1}{2}} f(s)\,ds = \frac{1}{2} + \int_z^{z+\frac{1}{2}} f_n(s)\,ds = \frac{1}{2} + 2^{n-2} \cdot \frac{1}{2^{2n-1}} \cdot 2^n = 1.$$

Moreover, let us note that $\Phi(t) = 1 + t$ for $t \in [-\frac{1}{4}, 0]$. Besides, for $t \in [z - \frac{1}{4}, z]$, where $z \in \mathbb{Z} \setminus \{0\}$, we have $\Phi(t) - t \leq \frac{3}{4}$. In fact,

$$\int_t^{z+\frac{1}{2}} f(s)\,ds \geq \int_z^{z+\frac{1}{2}} f(s)\,ds \geq 1,$$

and hence $\Phi(t) \leq z + \frac{1}{2}$, which shows that $\Phi(t) - t \leq \frac{3}{4}$ for every t in the considered interval.

If the function Ψ were μ-almost periodic, then for any non-zero $\tau \in \mu E\{\frac{1}{8}, \frac{1}{4}, \Psi\} \cap \mathbb{Z}$ (the existence of such a number follows from Proposition 2.15), we would have

$$\mu\left(\{x \in [-1, 0] : |\Psi(x) - \Psi(x + \tau)| \geq \tfrac{1}{4}\}\right) \leq \frac{1}{8}.$$

However, on the other hand, since $\tau \in \mathbb{Z} \setminus \{0\}$, we have

$$[-\tfrac{1}{4}, 0] \subseteq \{x \in [-1, 0] : |\Psi(x) - \Psi(x + \tau)| \geq \tfrac{1}{4}\},$$

whence

$$\mu\left(\{x \in [-1, 0] : |\Psi(x) - \Psi(x + \tau)| \geq \tfrac{1}{4}\}\right) \geq \frac{1}{4}.$$

The obtained contradiction shows that Ψ is not μ-almost periodic.

Our next example shows that the impression, which one might have, that every LIF (or PI) model driven by a locally integrable μ-almost periodic function which is not S^1-almost periodic fails to have almost periodic (in some sense) displacement map is wrong. We construct a purely μ-almost periodic input which gives rise to a S^p-almost periodic displacement map.

Example 4.23. Let us consider the following sets: $A_n = 2^n \mathbb{Z} + s_n$, where $s_n = \frac{1}{3}\left[(-2)^{n-1} - 1\right]$ for $n \in \mathbb{N}$, and let us note that

$$A_n \cap A_m = \emptyset \text{ if } n \neq m \qquad \text{and} \qquad \bigcup_{n=1}^{\infty} A_n = \mathbb{Z}. \tag{16}$$

Moreover, for $n \in \mathbb{N}$ and $z \in \mathbb{Z}$ put $B_n(z) = (z + 1 - \frac{1}{n+1}, z + 1)$, and for each fixed $n \in \mathbb{N}$ let us define the locally integrable 2^n-periodic function $f_n : \mathbb{R} \to \mathbb{R}$ by the following formula

$$f_n(x) = \begin{cases} (n+1)^2, & \text{if } x \in \bigcup_{z \in A_n} B_n(z), \\ 0, & \text{otherwise.} \end{cases}$$

Similarly to Example 4.22, it can be shown that the locally integrable function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = 2 + \sum_{n=1}^{\infty} f_n(x) \qquad \text{for } x \in \mathbb{R},$$

is μ-almost periodic as the D-limit of a sequence of periodic functions. Let us note that f is well-defined, since $\left(\bigcup_{z \in A_n} B_n(z)\right) \cap \left(\bigcup_{w \in A_m} B_m(w)\right) = \emptyset$ if $n \neq m$.

It is easy to see that the function f is not S^1-bounded, and hence it cannot be S^1-almost periodic (cf. Remark 2.11).

Our aim is now to show that the displacement map Ψ corresponding to the PI model driven by the μ-almost periodic function f, which clearly is well-defined for every $t \in \mathbb{R}$ and Lebesgue measurable, is also μ-almost periodic. In fact, we are going to show that for a given $m \in \{2, 3, \ldots\}$ we have

$$\sup_{z \in \mathbb{Z}} \mu\left(\{t \in [z, z+1] : |\Psi(t + 2^m w) - \Psi(t)| \geq \tfrac{2}{m+1}\}\right) \leq \frac{2}{m+1} \qquad \text{for every } w \in \mathbb{Z}, \tag{17}$$

which would imply that $2^m \mathbb{Z} \subseteq \mu E\{\frac{4}{m+1}, \frac{2}{m+1}, \Psi\}$ (cf. Remark 2.13). In particular, this would mean that for every $\varepsilon, \eta > 0$ the set $\mu E\{\varepsilon, \eta, \Psi\}$ is relatively dense.

Observe also that in order to prove (17) with fixed $m \geq 2$ and $w \in \mathbb{Z}$, it suffices to show that

$$\text{for every } z \in \mathbb{Z} \text{ we have } |\Psi(t + 2^m w) - \Psi(t)| \leq \frac{1}{m+1},$$

$$\text{whenever } t \in [z, z + \tfrac{1}{2} - \tfrac{1}{m+1}] \cup [z + \tfrac{1}{2}, z + 1 - \tfrac{1}{m+1}), \tag{18}$$

since then

$$\mu\left(\{t \in [z, z+1] : |\Psi(t + 2^m w) - \Psi(t)| \geq \tfrac{2}{m+1}\}\right) \leq$$

$$\leq \mu\left(\{t \in [z, z+1] : |\Psi(t + 2^m w) - \Psi(t)| > \tfrac{1}{m+1}\}\right) \leq \frac{2}{m+1}.$$

So let us fix $m \in \{2, 3, \ldots\}$ and $w \in \mathbb{Z}$, and take arbitrary $z \in \mathbb{Z}$. By (16) there exists exactly one $n \in \mathbb{N}$ such that the integer z belongs to A_n.

Case 1: Suppose that $n \le m$. Then, because $2^n | 2^m$, we infer that $f(t + 2^m w) = f(t)$ for every $t \in [z, z + \frac{3}{2}]$. Moreover, in view of the fact that

$$\int_{k}^{k+\frac{1}{2}} f(x)\mathrm{d}x = 1 \qquad \text{for } k \in \mathbb{Z}, \tag{19}$$

we get $\Phi(t) \le \Phi(z+1) \le z + \frac{3}{2}$ for every $t \in [z, z+1]$. Hence,

$$\int_{t+2^m w}^{\Phi(t)+2^m w} f(x)\mathrm{d}x = \int_{t}^{\Phi(t)} f(x + 2^m w)\mathrm{d}x = \int_{t}^{\Phi(t)} f(x)\mathrm{d}x = 1 \quad \text{for } t \in [z, z+1],$$

which proves that $\Phi(t + 2^m w) = \Phi(t) + 2^m w$ for $t \in [z, z+1]$. This, in turn, shows that the condition in (18) is satisfied if $z \in A_n$ with $n \le m$.

Case 2: Suppose now that $n > m$. First, observe that $\Phi(t) = t + \frac{1}{2}$ for $t \in [z, z + \frac{1}{2} - \frac{1}{n+1}]$, since on the interval $[z, z + 1 - \frac{1}{n+1}]$ the function f is identically equal to 2. Furthermore, $\Phi(t) \in [z + 1 - \frac{1}{n+1}, z + 1]$ for $t \in [z + \frac{1}{2} - \frac{1}{n+1}, z + 1 - \frac{1}{n+1})$, which is a consequence of the following estimates:

$$\int_{t}^{z+1-\frac{1}{n+1}-\gamma} f(x)\mathrm{d}x \le \int_{z+\frac{1}{2}-\frac{1}{n+1}}^{z+1-\frac{1}{n+1}-\gamma} f(x)\mathrm{d}x = 1 - 2\gamma < 1 \qquad \text{for every } \gamma \in (0, \tfrac{1}{2})$$

and

$$\int_{t}^{z+1} f(x)\mathrm{d}x \ge \int_{z+1-\frac{1}{n+1}}^{z+1} f(x)\mathrm{d}x = n + 1 + \frac{2}{n+1} > 1. \tag{20}$$

An easy computation shows that $z + 2^m w \notin \bigcup_{i=1}^{m} A_i$, and thus there exists a positive integer $j > m$ such that $z + 2^m w \in A_j$. Therefore, by the above reasoning, we obtain that

$$\Phi(t + 2^m w) = t + 2^m w + \frac{1}{2} \qquad \text{for } t \in [z, z + \tfrac{1}{2} - \tfrac{1}{j+1}]$$

and

$$\Phi(t + 2^m w) - 2^m w \in [z + 1 - \tfrac{1}{j+1}, z + 1] \qquad \text{for } t \in [z + \tfrac{1}{2} - \tfrac{1}{j+1}, z + 1 - \tfrac{1}{j+1}).$$

In conclusion, for $t \in [z, z + \frac{1}{2} - \frac{1}{m+1}]$ we have $\Phi(t + 2^m w) - 2^m w - \Phi(t) = 0$, and moreover, for $t \in [z + \frac{1}{2}, z + 1 - \frac{1}{m+1})$ we have $|\Phi(t + 2^m w) - 2^m w - \Phi(t)| \le \frac{1}{m+1}$. This proves that in the considered case the condition (18) is satisfied.

Therefore, the displacement map Ψ is μ-almost periodic, and since Ψ is bounded (cf. the proof of Theorem 4.11), by Remark 2.17, we see that Ψ is S^p-almost periodic for every $p \in [1, +\infty)$.

Finally, we are going to show that the function Ψ is not uniformly continuous, and hence it cannot be uniformly almost periodic. If $z \in A_n$, then $\Phi(z+1-\frac{1}{n+1}) \leq z+1$ and $\Phi(z+1) = z+\frac{3}{2}$ (cf. the formulas (19) and (20)), and thus

$$\left|\Phi\left(z + 1 - \tfrac{1}{n+1}\right) - \Phi(z+1)\right| \geq \frac{1}{2},$$

which clearly shows that Ψ is not uniformly continuous.

Remark 4.24. Let us add that the characterization of those locally integrable μ-almost periodic inputs for which the LIF model (1)–(2) has μ-almost periodic (or S^p-almost periodic) displacement map is still open.

We end this section with a continuity result for displacement maps corresponding to LIF models driven by S^1-almost periodic functions.

For simplicity, let us put

$$C = \{f \in S^1(\mathbb{R}) : \text{there exists } a_f > 0 \text{ such that } f(t) - \sigma > a_f \text{ for a.e. } t \in \mathbb{R}\},$$

and let us observe that C is a convex set. Clearly, C is a metric space with the metric induced by the S^1-norm.

Theorem 4.25. *The mapping $T : C \to AP(\mathbb{R})$, which to every S^1-almost periodic function $f \in C$ assigns the displacement Ψ of the firing map Φ corresponding to the LIF model (1)–(2) driven by f, is continuous.*

Proof. First, let us observe that T is well-defined thanks to Theorem 4.18.

Given $\varepsilon \in (0, 1)$ and $f \in C$ let $\delta = \varepsilon a_f\left(\left[\frac{1}{a_f}\right] + 2\right)^{-1} e^{-\sigma(1+1/a_f)}$, and suppose that $\widehat{f}$ is a function in C such that $\left\|f - \widehat{f}\right\|_{S^1_1} \leq \delta$. If $t \in \mathbb{R}$ is fixed, then by the definition of the firing map, we have

$$\int\limits_{t}^{\Phi(t)} \left(f(u) - \sigma\right)e^{\sigma u}\, du = \int\limits_{t}^{\widehat{\Phi}(t)} \left(\widehat{f}(u) - \sigma\right)e^{\sigma u}\, du \tag{21}$$

(cf. the formula (5)); here $\widehat{\Phi}$ denotes the firing map corresponding to the LIF model driven by $\widehat{f}$.

Suppose that $\Phi(t) \geq \widehat{\Phi}(t)$. Then, since $\widehat{\Phi}(t) - t \leq \Phi(t) - t \leq 1/a_f$ (cf. the first part of the proof of Theorem 4.11), from (21) it follows that

$$\int\limits_{\widehat{\Phi}(t)}^{\Phi(t)} \left(f(u) - \sigma\right)e^{\sigma u}\, du = \int\limits_{t}^{\widehat{\Phi}(t)} \left(\widehat{f}(u) - f(u)\right)e^{\sigma u}\, du$$

$$\leq e^{\sigma\widehat{\Phi}(t)} \int\limits_{t}^{t+\left[\frac{1}{a_f}\right]+1} \left|\widehat{f}(u) - f(u)\right|\, du \leq e^{\sigma\widehat{\Phi}(t)}\left(\left[\tfrac{1}{a_f}\right] + 1\right)\delta.$$

Simultaneously,

$$\int_{\widehat{\Phi}(t)}^{\Phi(t)} \big(f(u) - \sigma\big)e^{\sigma u}\, du \geq a_f e^{\sigma\widehat{\Phi}(t)}\big(\Phi(t) - \widehat{\Phi}(t)\big).$$

Thus, $\Phi(t) - \widehat{\Phi}(t) \leq \frac{1}{a_f}\big(\big[\frac{1}{a_f}\big] + 2\big)\delta \leq \varepsilon$.

Now, let us assume that $\widehat{\Phi}(t) \geq \Phi(t)$. Then

$$\int_{t}^{\Phi(t)+\varepsilon} \big(\widehat{f}(u) - \sigma\big)e^{\sigma u}du = \int_{t}^{\Phi(t)+\varepsilon} \big(\widehat{f}(u) - f(u)\big)e^{\sigma u}du + e^{\sigma t} + \int_{\Phi(t)}^{\Phi(t)+\varepsilon} \big(f(u) - \sigma\big)e^{\sigma u}du$$

$$\geq - \int_{t}^{\Phi(t)+1} |\widehat{f}(u) - f(u)|e^{\sigma u}du + e^{\sigma t} + a_f \varepsilon e^{\sigma\Phi(t)}$$

$$\geq -e^{\sigma(\Phi(t)+1)} \int_{t}^{\Phi(t)+1} |\widehat{f}(u) - f(u)|du + e^{\sigma t} + a_f \varepsilon e^{\sigma\Phi(t)}$$

$$\geq -e^{\sigma(t+1+1/a_f)} \int_{t}^{t+[\frac{1}{a_f}]+2} |\widehat{f}(u) - f(u)|du + e^{\sigma t} + a_f \varepsilon e^{\sigma t}$$

$$\geq e^{\sigma t}\Big(1 + a_f\varepsilon - \big(\big[\tfrac{1}{a_f}\big] + 2\big)e^{\sigma(1+1/a_f)}\delta\Big) = e^{\sigma t},$$

which shows that $\widehat{\Phi}(t) \leq \Phi(t) + \varepsilon$.

Summing up, we have shown that for a given $t \in \mathbb{R}$ we have $|\Phi(t) - \widehat{\Phi}(t)| \leq \varepsilon$. However, let us observe that the chosen δ does not depend on t, and therefore, we may conclude that $\sup_{t\in\mathbb{R}}|\Phi(t) - \widehat{\Phi}(t)| \leq \varepsilon$. To end the proof it suffices to note that $\sup_{t\in\mathbb{R}}|\Phi(t) - \widehat{\Phi}(t)| = \sup_{t\in\mathbb{R}}|\Psi(t) - \widehat{\Psi}(t)|$, where Ψ and $\widehat{\Psi}$ denote the displacements of the firing maps Φ and $\widehat{\Phi}$, respectively. $\square$

5. Applications to the theory of almost periodicity

In this very short section we give an example of an application of the results established in Section 4.2 to the theory of almost periodic functions.

Let us recall the following characterization of S^1-almost periodic functions.

Theorem 5.1 (see [9,31]). *Suppose that $f \in L^1_{loc}(\mathbb{R})$ is a μ-almost periodic function. Then f is S^1-almost periodic if and only if*

$$\sup_{u\in\mathbb{R}} \sup_{\substack{A\subseteq[u,u+1]\\ \mu(A)\leq\delta}} \int_A |f(t)|dt \to 0 \qquad as\ \delta \to 0^+. \tag{22}$$

Remark 5.2. Obviously, in (22) we can replace the interval $[u, u + 1]$ with any closed interval of arbitrary (but fixed) length.

To the best of our knowledge, so far it has not been known whether the condition (22) can be simplified, that is, whether it is possible to consider only intervals $[u, u + \delta]$ instead of any Lebesgue measurable set $A \subseteq [u, u + 1]$ with $\mu(A) \leq \delta$. Our next result gives the answer to this question.

Theorem 5.3. *There exists a locally integrable μ-almost periodic function satisfying the condition*

$$\sup_{u \in \mathbb{R}} \int_u^{u+\delta} |f(t)| \mathrm{d}t \to 0 \qquad \text{as } \delta \to 0^+ \tag{23}$$

which is not S^1-almost periodic.

Proof. Let us consider the locally integrable μ-almost periodic function $f : \mathbb{R} \to \mathbb{R}$ defined in Example 4.22. It was shown that f satisfies the condition (23). If f were S^1-almost periodic, then due to Theorem 4.18, the displacement map Ψ corresponding to the PI model driven by the function f would be uniformly almost periodic. However, it was shown in Example 4.22 that Ψ is not μ-almost periodic, and hence it cannot be uniformly almost periodic. The obtained contradiction proves our claim. $\quad\square$

6. Firing rate for the LIF model

The following section is devoted to the study of the firing rate for the LIF model. We start with the following

Definition 6.1. Let $t \in \mathbb{R}$. The limit

$$\mathrm{Fr}(t) = \lim_{n \to \infty} \frac{n}{\Phi^n(t)}$$

(whenever it exists) is called the *firing rate* of the LIF model (1)–(2); here Φ^n denotes the n-th iterate of the firing map corresponding to the LIF model.

Remark 6.2. Let us note that in the definition of the firing rate $\mathrm{Fr}(t)$ we implicitly assume that all the iterates $\Phi^n(t)$ are well-defined. In particular, this means that the firing map Φ must be well-defined for every $t \in \mathbb{R}$ (cf. Remark 4.5).

Before we proceed further, let us recall a result on the existence of the firing rate for the PI model.

Theorem 6.3 (*cf. [27, Theorem 3.2] and [7, Theorem 4]*). *Suppose that $\sigma = 0$ and that $f \in L^1_{loc}(\mathbb{R})$ is such that the firing map Φ corresponding to the PI model is well-defined for every $t \in \mathbb{R}$. If $\mathcal{M}\{f\}$ exists, then for every $t \in \mathbb{R}$ the firing rate $\mathrm{Fr}(t)$ for the PI model also exists, and moreover $\mathrm{Fr}(t) = \mathcal{M}\{f\} \in [0, +\infty]$ for $t \in \mathbb{R}$.*

Let us consider a simple example illustrating Theorem 6.3.

Example 6.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly almost periodic function given by the formula $f(t) = 2 + \cos(t) + \cos(\sqrt{2}t)$. Then the firing rate for the PI model driven by the function f equals

$$\mathrm{Fr}(t) = \mathcal{M}\{f\} = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \left(2 + \cos(u) + \cos(\sqrt{2}u)\right) du = 2 \quad \text{for } t \in \mathbb{R}.$$

In some situations the existence of the firing rate implies also the existence of the mean value.

Proposition 6.5. *Suppose that $\sigma = 0$ and that $f \in L^1_{loc}(\mathbb{R})$ is such that $f(t) \geq 0$ for a.e. $t \in \mathbb{R}$. If for some $s \in \mathbb{R}$ the firing rate $\mathrm{Fr}(s)$ for the PI model exists,[5] then also the mean value $\mathcal{M}\{f\}$ exists, and moreover $\mathcal{M}\{f\} = \mathrm{Fr}(s)$.*

Proof. First, let us observe that $(\Phi^n(s))_{n \in \mathbb{N}}$ is an increasing sequence such that $\lim_{n \to \infty} \Phi^n(s) = +\infty$ (cf. the proof of Proposition 4.4). In particular, $\mathrm{Fr}(s) \geq 0$.

Suppose that $\mathrm{Fr}(s)$ is finite. Given $\varepsilon > 0$ let $k_0 \in \mathbb{N}$ be such that for all $k \geq k_0$ we have

$$0 < \Phi^k(s), \qquad \left| \mathrm{Fr}(s) - \frac{k}{\Phi^k(s)} \right| \leq \frac{1}{3}\varepsilon, \qquad \frac{k}{\Phi^k(s)} - \frac{k}{\Phi^{k+1}(s)} \leq \frac{1}{3}\varepsilon, \qquad \frac{1}{\Phi^k(s)} \leq \frac{1}{3}\varepsilon.$$

Then, for $T \geq \Phi^{k_0}(s)$ we have

$$\left| \mathrm{Fr}(s) - \frac{1}{T} \int_s^T f(u) du \right| \leq \varepsilon. \tag{24}$$

Indeed, because the sequence $(\Phi^n(s))_{n \in \mathbb{N}}$ is increasing and $\lim_{n \to \infty} \Phi^n(s) = +\infty$, for every $T \geq \Phi^{k_0}(s)$ there exists $k \geq k_0$ such that $\Phi^k(s) \leq T \leq \Phi^{k+1}(s)$, and then

$$\left| \mathrm{Fr}(s) - \frac{1}{T} \int_s^T f(u) du \right| \leq \left| \mathrm{Fr}(s) - \frac{k}{\Phi^k(s)} \right| + \left| \frac{k}{\Phi^k(s)} - \frac{1}{T} \int_s^T f(u) du \right|$$

$$\leq \frac{1}{3}\varepsilon + \left| \frac{k}{\Phi^k(s)} - \frac{1}{T} \int_s^{\Phi^k(s)} f(u) du - \frac{1}{T} \int_{\Phi^k(s)}^T f(u) du \right|$$

$$\leq \frac{1}{3}\varepsilon + \left| \frac{k}{\Phi^k(s)} - \frac{k}{T} \right| + \frac{1}{\Phi^k(s)} \int_{\Phi^k(s)}^{\Phi^{k+1}(s)} f(u) du$$

---

[5] We do not exclude the case $\mathrm{Fr}(s) = +\infty$.

$$\leq \frac{1}{3}\varepsilon + \frac{k}{\Phi^k(s)} - \frac{k}{\Phi^{k+1}(s)} + \frac{1}{\Phi^k(s)} \leq \varepsilon.$$

This proves (24), and shows that $\mathcal{M}\{f\} = \mathrm{Fr}(s)$.

Now, let us assume that $\mathrm{Fr}(s) = +\infty$. Given $N > 0$, there exists $k_0 \in \mathbb{N}$ such that $\Phi^{k_0}(s) > 0$ and

$$\frac{k}{\Phi^{k+1}(s)} \geq N \quad \text{for } k \geq k_0.$$

Then for $T \geq \Phi^{k_0}(s)$ we have

$$\frac{1}{T} \int_s^T f(u)\mathrm{d}u \geq N.$$

Indeed, for every $T \geq \Phi^{k_0}(s)$ there exists $k \geq k_0$ such that $\Phi^k(s) \leq T \leq \Phi^{k+1}(s)$, and then

$$\frac{1}{T} \int_s^T f(u)\mathrm{d}u \geq \frac{1}{\Phi^{k+1}(s)} \int_s^{\Phi^k(s)} f(u)\mathrm{d}u = \frac{k}{\Phi^{k+1}(s)} \geq N.$$

This clearly shows that $\mathcal{M}\{f\} = +\infty$. $\quad\square$

From Theorem 6.3 and Proposition 6.5 we get the following

Corollary 6.6. *Suppose that $\sigma = 0$ and that $f \in L^1_{loc}(\mathbb{R})$ is such that $f(t) \geq 0$ for a.e. $t \in \mathbb{R}$. If for some $s \in \mathbb{R}$ the firing rate $\mathrm{Fr}(s)$ exists, then the mean value $\mathcal{M}\{f\}$ and the firing rates $\mathrm{Fr}(t)$, $t \in \mathbb{R}$, exist, and moreover $\mathrm{Fr}(t) = \mathcal{M}\{f\} \in [0, +\infty]$ for $t \in \mathbb{R}$.*

Corollary 6.6 can be viewed as a result on the invariance of the firing rate closely related with the following known

Proposition 6.7 *(cf. [7, Proposition 1]). Let $f \in L^1_{loc}(\mathbb{R})$ be such that $f(t) - \sigma \geq 0$ for a.e. $t \in \mathbb{R}$. If the firing rate $\mathrm{Fr}(t)$ for the LIF model (1)–(2) exists for some $t \in \mathbb{R}$, then it exists for every $s \in \mathbb{R}$ and $\mathrm{Fr}(s) = \mathrm{Fr}(t)$.*

Remark 6.8. Let us add that the fact that the firing rate for integrate-and-fire models with right-hand side of the form $F(x, t)$, if exists, does not depend on the initial point was observed in [7]. The function F was assumed to be sufficiently regular so that the corresponding differential equation had a unique solution starting from any initial condition and had to be either decreasing in x for all t, or satisfy the condition: $F(0, t) > 0$ for every $t \in \mathbb{R}$. However, no general conditions on F (apart from the very simple ones corresponding to PI models) guaranteeing the existence of the firing rate were given.

Now, we would like to find a result similar to Theorem 6.3 in the 'LIF setting'. To this end, let us recall a few facts from the rotation theory.

Definition 6.9. The *rotation number* of $t \in \mathbb{R}$ with respect to $f : \mathbb{R} \to \mathbb{R}$ is defined as the limiting average displacement

$$\varrho(f, t) := \lim_{n \to \infty} \frac{f^n(t) - t}{n},$$

provided the limit exists.

Remark 6.10. In the theory of integrate-and-fire models the rotation number of a point $t \in \mathbb{R}$ with respect to the firing map Φ is called the *average interspike interval*.

Definition 6.11. The *pointwise rotation set* of $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$\varrho_p(f) := \{\varrho(f, t) : t \in \mathbb{R} \text{ for which } \varrho(f, t) \text{ exists}\}.$$

Theorem 6.12 (*cf. [24, Theorem 1 and Theorem 2]*). *Suppose that a non-decreasing function $f : \mathbb{R} \to \mathbb{R}$ admits a decomposition $f(t) = t + g(t)$, where $g : \mathbb{R} \to \mathbb{R}$ is uniformly is almost periodic and $\inf_{t \in \mathbb{R}} g(t) > 0$. Then $\varrho_p(f) = \{r\}$ for some $r \in \mathbb{R}$.*

Remark 6.13. Let us add that J. Kwapisz in [24] gave a thorough description of rotation sets and rotation numbers for functions with uniformly almost periodic displacements.

Now, we can move to the main result of this subsection.

Theorem 6.14. *Let $f : \mathbb{R} \to \mathbb{R}$ be a S^1-almost periodic function and assume that there exists $\varsigma > 0$ such that $f(t) - \sigma > \varsigma$ for a.e. $t \in \mathbb{R}$. Then for every $t \in \mathbb{R}$ the firing rate for the LIF model (1)–(2) driven by f exists, and moreover $\mathrm{Fr}(t) = \mathrm{Fr}(0) \in (0, +\infty)$ for $t \in \mathbb{R}$.*

Proof. Since f is a S^1-almost periodic function and $f(t) - \sigma > \varsigma > 0$ a.e. on $\mathbb{R}$, the displacement map Ψ corresponding to the LIF model driven by f is uniformly almost periodic (see Theorem 4.18). Moreover, in view of Lemma 4.8 and Theorem 4.11 (cf. also Remark 4.21), the firing map Φ satisfies the assumptions of Theorem 6.12, and so there is a number $r \in \mathbb{R}$ such that $\varrho_p(\Phi) = \{r\}$, that is,

$$\lim_{n \to \infty} \frac{\Phi^n(t) - t}{n} = r \qquad \text{for some } t \in \mathbb{R}.$$

Now, we will show that $r > 0$. For every $n \in \mathbb{N}$ we have

$$\Phi^n(t) - t = \sum_{i=1}^{n} \left(\Phi^i(t) - \Phi^{i-1}(t)\right) = \sum_{i=1}^{n} \Psi\left(\Phi^{i-1}(t)\right) \geq n \cdot \psi,$$

where $\psi := \inf_{t \in \mathbb{R}} \Psi(t) > 0$ and $\Phi^0(t) := t$, and thus $r \geq \psi > 0$. This shows that the firing rate $\mathrm{Fr}(t)$, which in this case is the multiplicative inverse of the rotation number $\varrho(\Phi, t)$ exists and is positive. To end the proof it suffices to apply Proposition 6.7. $\square$

Remark 6.15. Let us observe that if $\Psi(\mathbb{R}) \subseteq [a, b]$ for some positive real numbers a, b, then $\mathrm{Fr}(t) \in [1/b, 1/a]$, provided $\mathrm{Fr}(t)$ exists.

The question whether for LIF models the firing rate can be expressed in terms of the mean value of the drive (as it is for PI models) still remains open. The following example shows, however, that if such an expression exists it must depend not only on the mean value of the input. We are going to define two S^1-almost periodic functions with the same mean value which give rise to LIF models with different firing rates.

Example 6.16. Let us consider two locally integrable periodic (and thus S^1-almost periodic) functions $f, g \colon \mathbb{R} \to \mathbb{R}$ given by

$$
f(t) = \begin{cases} 2 & \text{for } t \in \big[k \ln 3, k \ln 3 + \ln 2\big), k \in \mathbb{Z}, \\ 3 & \text{for } t \in \big[k \ln 3 + \ln 2, (k+1) \ln 3\big), k \in \mathbb{Z} \end{cases} \quad \text{and} \quad g(t) = 3 - \log_3 2.
$$

It is easy to see that $\mathcal{M}\{f\} = \mathcal{M}\{g\} = 3 - \log_3 2$.

Now, we will show that the firing rates Fr_f and Fr_g for the LIF model (1)–(2) with $\sigma = 1$ driven by f and g, respectively, are different. First, let us note that, in view of Theorem 6.14, the firing rates $\mathrm{Fr}_f(t)$ and $\mathrm{Fr}_g(t)$ exist for every $t \in \mathbb{R}$ and $\mathrm{Fr}_f(t) = \mathrm{Fr}_f(0)$, $\mathrm{Fr}_g(t) = \mathrm{Fr}_g(0)$.

By Φ_f and Φ_g let us denote the firing maps corresponding to the inputs f and g. Since $\Phi_f(0) = \ln 2$, $\Phi_f^2(0) = \ln 3$ and $\Phi_f^2(t + \ln 3) = \Phi_f^2(t) + \ln 3$ for $t \in \mathbb{R}$ (cf. Proposition 4.16), we infer that $\Phi_f^{2n}(0) = n \ln 3$ for $n \in \mathbb{N}$. Thus $\mathrm{Fr}_f(0) = 2/\ln 3$. Similarly,

$$
\Phi_g^n(t) = t + n \ln\left(1 + \frac{1}{2 - \log_3 2} \right) \qquad \text{for } t \in \mathbb{R} \text{ and } n \in \mathbb{N},
$$

and hence

$$
\mathrm{Fr}_g(0) = \left[\ln\left(1 + \frac{1}{2 - \log_3 2} \right) \right]^{-1}.
$$

Finally, it suffices to note that $\mathrm{Fr}_f(0) \neq \mathrm{Fr}_g(0)$.

At the end of this section, let us point out that establishing the existence and uniqueness of the firing rate is an important issue, since it allows to qualitatively characterize the spike train of a given model via the average frequency of firing, independently of the initial condition. This result for the LIF model in both cases of (locally integrable) periodic and Stepanov almost periodic drive was achieved by studying the displacement map Ψ and showing that it is periodic or, correspondingly, uniformly almost periodic. In the situation of a sufficiently regular periodic forcing (see [28]), periodic displacement allowed to project the firing map onto the circle and view it as a lift of the orientation-preserving circle homeomorphism, for which the existence and uniqueness of the rotation number follows immediately from the classical Poincaré rotation theory. However, in the almost periodic case we needed to rely on the much more recent results of [24] on the maps of the real line with almost periodic displacements. Investigating the dynamics of such maps, which would lead to more detailed description of the corresponding spike trains, is also much more challenging. However, asserting the almost periodicity of the displacement map Ψ opens up the possibility of using some results on different almost periodic structures (see e.g. [2]).

7. Approximation of S^p-almost periodic functions by Haar wavelets

As we have seen in Section 4, giving an exact arithmetic formula for the firing map Φ is in general not an easy task. However, it is much easier to obtain the formula for Φ, when the forcing term f is a piece-wise constant function (see Example 4.3). Thus, in the case of almost periodic inputs, it seems that it would be better if we could use Haar series rather than Fourier series to approximate such functions.

Our aim in this section is to provide the answer to the following question: Is it possible to approximate Stepanov almost periodic functions by Haar wavelets with any desired accuracy in the appropriate almost periodic norm?

Before we proceed further, let us recall some facts from the wavelet theory.

Definition 7.1. The *Haar wavelet* is the function $h: \mathbb{R} \to \mathbb{R}$ defined by the formula $h(t) = \chi_{[0, \frac{1}{2})}(t) - \chi_{[\frac{1}{2}, 1)}(t)$, where χ_A denotes the characteristic function of the set $A \subseteq \mathbb{R}$.

To simplify the notation for $k \in \mathbb{Z}$ and $j = 2^m + r$, where m is a non-negative integer and $r = 1, 2, \ldots, 2^m$, let

$$h_{k,1}(t) = \chi_{[k, k+1)}(t) \quad \text{and} \quad h_{k,j}(t) = 2^{\frac{1}{2}m} h\big(2^m(t - k) - r + 1\big).$$

The collection $\{h_{k,j} : k \in \mathbb{Z}, \ j \in \mathbb{N}\}$ is called the *Haar system*. Since $h_{k,j} \in L^q(\mathbb{R})$ for every $1 \leq q \leq +\infty$ and $\operatorname{supp} h_{k,j} \subseteq [k, k+1]$, given a S^p-almost periodic function $f: \mathbb{R} \to \mathbb{R}$ (where $1 \leq p < +\infty$), we can define the *Haar–Fourier coefficients*

$$a_{k,j} := \int_{\mathbb{R}} f(u) h_{k,j}(u) \, \mathrm{d}u.$$

Let us also define the projection operators P_n, $n \in \mathbb{N}$, on $S^p(\mathbb{R})$ by

$$(P_n f)(t) = \sum_{k=-\infty}^{+\infty} \sum_{j=1}^{n} a_{k,j} h_{k,j}(t) \qquad \text{for } t \in \mathbb{R}. \tag{25}$$

Observe that for each $t \in \mathbb{R}$ only finitely many values $h_{k,j}(t)$ are non-zero, and so the sum in (25) is in fact finite, which means that the projections $P_n f$ are well-defined on $\mathbb{R}$. Moreover, it should be clear that $P_n f \in L^p_{\mathrm{loc}}(\mathbb{R})$.

For a thorough treatment of the wavelet theory we refer the Reader to [40,41].

Now, let us pass to the main part of this section. We begin with the following simple

Lemma 7.2. *Let* $[a, b), [c, d) \subseteq [0, 1)$. *If* f *is a* S^1-almost periodic function, then the function $P: \mathbb{R} \to \mathbb{R}$ defined by the formula*

$$P(t) = \sum_{k=-\infty}^{+\infty} \left(\int_a^b f(u + k) \, \mathrm{d}u \right) \chi_{[c,d)}(t - k), \quad t \in \mathbb{R},$$

is S^p-almost periodic for every $p \in [1, +\infty)$.

Proof. First, let us note that the function P is well-defined, since for every $t \in \mathbb{R}$ the value of $\chi_{[c,d)}(t - k)$ is non-zero for at most one $k \in \mathbb{Z}$. Moreover, $P \in L^p_{\mathrm{loc}}(\mathbb{R})$ for every $p \in [1, +\infty)$.

Let us fix an arbitrary $\varepsilon > 0$ and let $\tau \in S^1 E\{\frac{1}{2}\varepsilon, f\} \cap \mathbb{Z}$. Given $t \in \mathbb{R}$, if $t - [t] \in [0, 1) \setminus [c, d)$, then $P(t + \tau) = P(t)$. On the other hand, if $t - [t] \in [c, d)$, then

$$|P(t+\tau) - P(t)| = \left| \int_a^b f(u + [t] + \tau)\mathrm{d}u - \int_a^b f(u + [t])\mathrm{d}u \right| \leq \int_{[t]}^{[t]+1} |f(u + \tau) - f(u)|\mathrm{d}u \leq \frac{1}{2}\varepsilon.$$

Therefore,

$$\sup_{s \in \mathbb{R}} \left(\int_s^{s+1} |P(u + \tau) - P(u)|^p \mathrm{d}u \right)^{1/p} < \varepsilon,$$

which means that $S^1 E\{\frac{1}{2}\varepsilon, f\} \cap \mathbb{Z} \subseteq S^p E\{\varepsilon, P\}$. Thus, P is S^p-almost periodic (cf. Proposition 2.15 and Remark 2.16). $\quad\square$

Remark 7.3. Let us note that from the proof of Lemma 7.2 it follows that for every $\varepsilon > 0$ the set $E\{\varepsilon, P\}$ is relatively dense. However, in general, the function P is not uniformly almost periodic, since usually P is not continuous.

Corollary 7.4. *If f is S^1-almost periodic, then for every $n \in \mathbb{N}$ the projection $P_n f$ is S^p-almost periodic with any $p \in [1, +\infty)$.*

Proof. From Lemma 7.2 we know that the function $h_1 : \mathbb{R} \to \mathbb{R}$ defined as

$$h_1(t) := \sum_{k=-\infty}^{+\infty} a_{k,1} h_{k,1}(t) = \sum_{k=-\infty}^{+\infty} \left(\int_0^1 f(u + k)\mathrm{d}u \right) \chi_{[0,1)}(t - k),$$

is S^p-almost periodic for any $p \in [1, +\infty)$.

Now, let $j \geq 2$ be of the form $j = 2^m + r$, where m is a non-negative integer and $r \in \{1, \ldots, 2^m\}$. Since for every $k \in \mathbb{Z}$ and $t \in \mathbb{R}$ we have

$$a_{k,j} h_{k,j}(t) = 2^m \left(\int_{\frac{r-1}{2^m}}^{\frac{2r-1}{2^{m+1}}} f(u + k)\mathrm{d}u \right) \chi_{[\frac{r-1}{2^m}, \frac{2r-1}{2^{m+1}})}(t - k)$$

$$- 2^m \left(\int_{\frac{r-1}{2^m}}^{\frac{2r-1}{2^{m+1}}} f(u + k)\mathrm{d}u \right) \chi_{[\frac{2r-1}{2^{m+1}}, \frac{r}{2^m})}(t - k)$$

$$-2^m\left(\int_{\frac{2r-1}{2^{m+1}}}^{\frac{r}{2^m}} f(u+k)\mathrm{d}u\right)\chi_{[\frac{r-1}{2^m},\frac{2r-1}{2^{m+1}})}(t-k)$$

$$+2^m\left(\int_{\frac{2r-1}{2^{m+1}}}^{\frac{r}{2^m}} f(u+k)\mathrm{d}u\right)\chi_{[\frac{2r-1}{2^{m+1}},\frac{r}{2^m})}(t-k),$$

the function $h_j: \mathbb{R} \to \mathbb{R}$ defined as

$$h_j(t) := \sum_{k=-\infty}^{+\infty} a_{k,j} h_{k,j}(t),$$

is S^p-almost periodic with any $p \in [1, +\infty)$.

To end the proof it suffices to note that $(P_n f)(t) = \sum_{j=1}^n h_j(t)$ for $t \in \mathbb{R}$. □

Now, we are in position to prove the main theorem of this section.

Theorem 7.5. *Let $p \in [1, +\infty)$. Given a S^p-almost periodic function $f: \mathbb{R} \to \mathbb{R}$, the sequence of projections $(P_n f)_{n\in\mathbb{N}}$ converges to f with respect to the $\|\cdot\|_{S_1^p}$-norm.*

Proof. First, let us note that the norm $\|\cdot\|_{S_1^p}$ is equivalent to the norm $|\cdot|_{S_1^p}$ given by

$$|f|_{S_1^p} = \sup_{l\in\mathbb{Z}}\left(\int_l^{l+1} |f(u)|^p \mathrm{d}u\right)^{1/p}$$

(cf. Remark 2.13). Therefore, for a given $l \in \mathbb{Z}$, in view of [39, Theorem 11] (see also [12, Theorem 7]), we have

$$\left(\int_l^{l+1} |(P_n f)(u) - f(u)|^p \mathrm{d}u\right)^{1/p} = \left(\int_l^{l+1} \left|\sum_{j=1}^n a_{l,j} h_{l,j}(u) - f(u)\right|^p \mathrm{d}u\right)^{1/p}$$

$$\leq 24 \sup_{0\leq h\leq\frac{1}{n}} \left(\int_l^{l+1-h} |f(u+h) - f(u)|^p\right)^{1/p}$$

$$\leq 24 \sup_{0\leq h\leq\frac{1}{n}} \sup_{l\in\mathbb{Z}} \left(\int_l^{l+1} |f(u+h) - f(u)|^p\right)^{1/p}$$

$$= 24 \sup_{0\leq h\leq\frac{1}{n}} \left|f - f^h\right|_{S_1^p}.$$

But S^p-almost periodic functions are S^p-continuous, which means that $|f - f^h|_{S^p_1} \to 0$ as $h \to 0$ (see for example [35, Theorem 2.1] or [26, Theorem 5.2.3]). Thus

$$|P_n f - f|_{S^p_1} \leq 24 \sup_{0 \leq h \leq \frac{1}{n}} |f - f^h|_{S^p_1} \to 0 \qquad \text{as } n \to +\infty.$$

This shows our claim and ends the proof. □

Remark 7.6. Let us point out that in our approach we have used the same basic Haar system for all the intervals $[k, k + 1)$, although the Haar–Fourier coefficients had to be calculated separately for each interval.

For completeness, let us also add that the problem of wavelet approximations of certain almost periodic functions was investigated, for example, in [15,23,29].

Acknowledgments

Appendix A. Some remarks on the mean value

In this appendix we would like to address very briefly a problem which is not directly connected with the study of integrate-and-fire models, but which seems interesting from the point of view of the general theory of ordinary differential equations in spaces of almost periodic functions.

It is known that if $f \in AP(\mathbb{R})$, then its antiderivative, that is the function $F(t) := \int_0^t f(s)\mathrm{d}s$, is uniformly almost periodic if and only if it is bounded (see for example [13, Theorem 4.1]). For this reason there are plenty of examples of uniformly almost periodic functions whose antiderivatives fail to be uniformly almost periodic (take for example the function $f(t) = \frac{1}{2} + \sin t$). As a consequence, it is not so straightforward (in general) to apply fixed point approach in order to establish existence (and uniqueness) results for differential and integral equations in classes of almost periodic functions.

Given a uniformly almost periodic function $f : \mathbb{R} \to \mathbb{R}$ we would like to find a real number m such that the function $t \mapsto f(t) - m$, where $t \in \mathbb{R}$, has a uniformly almost periodic antiderivative. Let us observe that from the fact that a uniformly almost periodic function with bounded antiderivative must have zero mean value, it follows that the sole candidate for the number m is the mean value $\mathcal{M}\{f\}$.

First, let us consider the case when f is a locally integrable periodic function, although the class of such functions is not contained in $AP(\mathbb{R})$.

Proposition A.1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a locally integrable periodic function with a period* $\omega > 0$. *Then the antiderivative* F *of the function* $t \mapsto f(t) - \mathcal{M}\{f\}$ *is bounded.*

Proof. Let us recall that in the case of locally integrable periodic functions, the mean value can be expressed in a simpler form, namely

$$\mathcal{M}\{f\} = \frac{1}{\omega} \int_0^\omega f(s) \mathrm{d}s$$

(cf. [42, Remark, p. 88]). Fix $t \geq 0$. Then $t = k\omega + h$ for some $k \in \mathbb{N} \cup \{0\}$ and $h \in [0, \omega)$. Since

$$\int_0^t f(s) \mathrm{d}s = \sum_{n=1}^k \int_{(n-1)\omega}^{n\omega} f(s) \mathrm{d}s + \int_{k\omega}^t f(s) \mathrm{d}s = k \int_0^\omega f(s) \mathrm{d}s + \int_0^h f(s) \mathrm{d}s,$$

we get

$$\left| \int_0^t f(s) \mathrm{d}s - \mathcal{M}\{f\} t \right| = \left| k \int_0^\omega f(s) \mathrm{d}s + \int_0^h f(s) \mathrm{d}s - \frac{t}{\omega} \int_0^\omega f(s) \mathrm{d}s \right|$$

$$= \left| \int_0^h f(s) \mathrm{d}s - \frac{h}{\omega} \int_0^\omega f(s) \mathrm{d}s \right| \leq 2 \int_0^\omega |f(s)| \mathrm{d}s.$$

The proof for $t < 0$ is analogous. $\square$

From Proposition A.1 we immediately get the following

Corollary A.2. *If* $f : \mathbb{R} \to \mathbb{R}$ *is a generalized trigonometric polynomial, that is,* $f(t) = \sum_{j=1}^n \left(a_j \sin(\lambda_j t) + b_j \cos(\lambda_j t) \right)$, *where* $a_j, b_j, \lambda_j \in \mathbb{R}$, *then the antiderivative of the function* $t \mapsto f(t) - \mathcal{M}\{f\}$ *is bounded.*

However, in general, the antiderivative of the function $t \mapsto f(t) - \mathcal{M}\{f\}$, where $f \in AP(\mathbb{R})$, may be unbounded as is shown by the following

Example A.3 *(cf. [35, pp. 39-40] or [13, Corollary 2, p. 31]).* Let us consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by the following formula

$$f(t) = -\sum_{n=1}^\infty \frac{1}{n^2} \sin\left(\frac{t}{n^2}\right) \qquad \text{for } t \in \mathbb{R}.$$

Since the above series is uniformly convergent on $\mathbb{R}$, the function f is uniformly almost periodic. Moreover, $\mathcal{M}\{f\} = 0$.

However, the antiderivative F of f is not uniformly almost periodic, since if it were, then its Fourier series would be of the form

$$c + \sum_{n=1}^{\infty} \cos\left(\frac{t}{n^2}\right)$$

for some $c \in \mathbb{R}$, which is impossible due to the fact that the sequence of coefficients of the Fourier series of a uniformly almost periodic function is convergent to zero (see [35, p. 37] or cf. [13, Theorem 1.18]).

Remark A.4. It turns out that the problem addressed in this appendix is closely connected with the study of the class of the so-called (IC)-almost periodic functions, since the function $t \mapsto f(t) - \mathcal{M}\{f\}$, where $f : \mathbb{R} \to \mathbb{R}$ is uniformly almost periodic, admits bounded antiderivative if and only if $f - \mathcal{M}\{f\}$ is (IC)-almost periodic.

We refer the Reader interested in (IC)-almost periodic functions to the paper [1] or to the monograph [35, Section 3.2]. The discussion concerning the relationship between almost periodic functions and their antiderivatives can be also found in [34,36].

References

[1] M. Adamczak, Some remarks on almost periodic functions, Fasc. Math. (33) (2002) 5–19.

[2] J. Aliste-Prieto, T. Jäger, Almost periodic structures and the semiconjugacy problem, J. Differential Equations 252 (2012) 4988–5001.

[3] J. Andres, A.M. Bersani, R.F. Grande, Hierarchy of almost-periodic function spaces, Rend. Mat. Appl. 26 (2006) 121–188.

[4] D. Bainov, P. Simeonov, Integral Inequalities and Applications, Kluwer Academic Publshers, Dordrecht, Boston, London, 1992.

[5] H. Bell, K.R. Meyer, Limit periodic functions, adding machines, and solenoids, J. Dynam. Differential Equations 7 (1995) 409–422.

[6] A.S. Besicovitch, Almost Periodic Functions, Dover Publications, Inc., New York, 1955.

[7] R. Brette, Dynamics of one-dimensional spiking neuron model, J. Math. Biol. 48 (2004) 38–56.

[8] D. Bugajewski, T. Diagana, Almost automorphy of the convolution operator and applications to differential and functional differential equations, Nonlinear Stud. 13 (2) (2006) 129–140.

[9] D. Bugajewski, A. Nawrocki, Some remarks on almost periodic functions in view of the Lebesgue measure with applications to linear differential equations, Ann. Acad. Sci. Fenn. Math. 42 (2017) 809–836, https://doi.org/10.5186/aasfm.2017.4250.

[10] D. Bugajewski, G.M. N'Guérékata, On some classes of almost periodic functions in abstract spaces, Int. J. Math. Math. Sci. (61) (2004) 3237–3247.

[11] H. Carrillo, F.A. Ongay, On the firing maps of a general class of forced integrate-and-fire neurons, Math. Biosci. 172 (2001) 33–53.

[12] Z. Ciesielski, Properties of the orthonormal Franklin system. II, Studia Math. 27 (1966) 289–323.

[13] C. Corduneanu, Almost Periodic Functions, 2nd Edition, Chelsea Publishing Company, New York, 1989, with the collaboration of N. Gheorghiu and V. Barbu.

[14] A.M. Fink, Almost Periodic Differential Equations, Lecture Notes in Mathematics, vol. 377, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[15] F. Galindo, Some remarks on "On the windowed Fourier transform and wavelet transform of almost periodic functions," by J.R. Partington and B. Ünalmiş, Appl. Comput. Harmon. Anal. 16 (2004) 174–181.

[16] T. Gedeon, M. Holzer, Phase locking in integrate-and-fire models with refractory periods and modulation, J. Math. Biol. 49 (2004) 577–603.

[17] W. Gerstner, W.M. Kistler, R. Naud, L. Paninski, Neuronal Dynamics: From Single Neurons to Networks and Models of Cognition, Cambridge University Press, 2014.

[18] E. Hewitt, K. Stromberg, Real and Abstract Analysis: A Modern Treatment of the Theory of Functions of a Real Variable, Graduate Texts in Mathematics, vol. 25, Springer-Verlag, Berlin, Heidelberg, New York, 1975.

[19] A. Hodgkin, A. Huxley, A quantitative description of membrane current and its application to conduction and excitation in nerve, J. Physiol. 117 (1952) 500–544.

[20] E.M. Izhikevich, Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting, The MIT Press, Cambridge, MA, 2007.

[21] T. Jäger, Neuronal coding of pacemaker neurons—a random dynamical systems approach, Commun. Pure Appl. Anal. 10 (3) (2011) 995–1009.

[22] J.P. Keener, F.C. Hoppensteadt, J. Rinzel, Integrate-and-fire models of nerve membrane response to oscillatory input, SIAM J. Appl. Math. 41 (1981) 503–517.

[23] Y.H. Kim, A. Ron, Time frequency representations of almost periodic functions, Constr. Approx. 29 (2009) 303–323.

[24] J. Kwapisz, Poincaré rotation number for maps of the real line with almost periodic displacement, Nonlinearity 13 (2000) 1841–1854.

[25] L. Lapicque, Recherches quantitatives sur l'excitation électrique des nerfs traitée comme une polarisation, J. Physiol. Pathol. Gén. 9 (1907) 620–635.

[26] B.M. Levitan, Počti-periodičeskie funkcii, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953 (in Russian).

[27] W. Marzantowicz, J. Signerska, Firing map of an almost periodic input function, Discrete Contin. Dyn. Syst. Suppl. 2 (2011) 1032–1041.

[28] W. Marzantowicz, J. Signerska, On the interspike-intervals of periodically-driven integrate-and-fire models, J. Math. Anal. Appl. 423 (1) (2015) 456–479.

[29] J.R. Partington, B. Unalmis, On the windowed Fourier transform and wavelet transform of almost periodic functions, Appl. Comput. Harmon. Anal. 10 (2001) 45–60.

[30] J. Signerska-Rynkowska, Analysis of interspike-intervals for the general class of integrate-and-fire models with periodic drive, Math. Model. Anal. 20 (5) (2015) 529–551.

[31] W. Stepanoff, Über einige Verallgemeinerungen der fast periodischen Funktionen, Math. Ann. 95 (1) (1926) 473–498 (in German).

[32] S. Stoiński, Almost periodic function in the Lebesgue measure, Comment. Math. Prace Mat. 34 (1994) 189–198.

[33] S. Stoiński, On compactness of almost periodic functions in the Lebesgue measure, Fasc. Math. 30 (1999) 171–175.

[34] S. Stoiński, Some remarks on spaces of almost periodic functions, Fasc. Math. (31) (2001) 105–115.

[35] S. Stoiński, Almost Periodic Functions, Wydawnictwo Naukowe UAM, Poznań, 2008 (in Polish).

[36] S. Stoiński, A note on the primitive function of a Bohr almost periodic function, Comment. Math. 51 (1) (2011) 77–80.

[37] P.H.E. Tiesinga, Precision and reliability of periodically and quasiperiodically driven integrate-and-fire neurons, Phys. Rev. E 65 (2002) 041913.

[38] J. Touboul, R. Brette, Spiking dynamics of bidimensional integrate-and-fire neurons, SIAM J. Appl. Dyn. Syst. 8 (4) (2009) 1462–1506.

[39] P.L. Ul'janov, On Haar series, Mat. Sb. (N.S.) 63 (105) (3) (1964) 356–391 (in Russian).

[40] D.F. Walnut, An Introduction to Wavelet Analysis, Applied and Numerical Harmonic Analysis, Birkhäuser Boston, Inc., Boston, MA, 2002.

[41] P. Wojtaszczyk, A Mathematical Introduction to Wavelets, London Mathematical Society Student Texts, vol. 37, Cambridge University Press, Cambridge, 1997.

[42] S. Zaidman, Almost-Periodic Functions in Abstract Spaces, Research Notes in Mathematics, vol. 126, Pitman Advanced Publishing Program, Boston–London–Melbourne, 1985.