AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **46** (2010), Pages 3–12

Interpolation properties of domination parameters of a graph

Dorota Kuziak

Faculty of Applied Physics and Mathematics
Gdańsk University of Technology
Narutowicza 11/12, 80–952 Gdańsk
Poland
dkuziak@mif.pg.gda.pl

Abstract

An integer-valued graph function π is an interpolating function if a set $\pi(\mathcal{T}(G)) = \{\pi(T) : T \in \mathcal{T}(G)\}$ consists of consecutive integers, where $\mathcal{T}(G)$ is the set of all spanning trees of a connected graph G. We consider the interpolation properties of domination related parameters.

1 Introduction

The interpolation properties of different graph parameters were studied in a number of papers. In particular, the interpolating character of domination related parameters were investigated: domination number, lower and upper distance k-domination numbers [8, 14], global and total domination numbers and n-domination number [14] and (r, s)-domination number [15]. In this paper we establish the interpolation properties of other types of domination numbers of a graph.

In general, we use the terminology and notation of [9]. For the sake of completeness we now give a few definitions. Let G be a graph and v be a vertex of G. The *neighbourhood* of v in G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The *closed neighbourhood* of v in G is the set $N_G[v] = N_G(v) \cup \{v\}$. More generally, for a subset $S \subseteq V(G)$, $N_G(S)$ and $N_G[S]$ are defined to be $\bigcup_{v\in S} N_G(v)$ and $N_G(S) \cup S$, respectively. The degree of v in G is $d_G(v) = |N_G(v)|$.

A set $D \subseteq V(G)$ is said to be independent if no two vertices of D are adjacent. The *independence number* of G, denoted by $\alpha(G)$, is the maximum cardinality of an independent set in G.

A unicyclic graph is a connected graph containing precisely one cycle.

A well-known method of transforming one spanning tree into another spanning tree of a graph is very useful in our research. We describe this transformation here.

For a connected graph G, let $\mathcal{T}(G)$ be the set of all spanning trees of G and let $T \in \mathcal{T}(G)$. Let e be an edge of G which is not in T. Then $T+e$ is a graph which has a unique cycle. If f is an edge which belongs to the cycle of $T+e$, then $T+e-f$ is a spanning tree of G. The transformation of T to $T+e-f$ is called a *simple edge-exchange*. A simple edge-exchange of T to $T+e-f$ is called an *adjacent edge-exchange* if e and f are adjacent edges of G. If e and f are incident with a common end vertex of T (and then also of $T+e-f$), then the transformation of T to $T+e-f$ is called an *end edge-exchange*.

An integer-valued graph function π *interpolates over a connected graph* G if the set $\pi(\mathcal{T}(G)) = \{\pi(T) : T \in \mathcal{T}(G)\}$ consists of consecutive integers. The function π is called an *interpolating function* if π interpolates over each connected graph.

Now we present theorems and corollaries which are basic tools in our results.

Theorem 1 [6] *If G is a 2-connected graph, then any $T \in \mathcal{T}(G)$ can be transformed into any other $T' \in \mathcal{T}(G)$ by a sequence of end edge-exchanges.*

Theorem 2 [14] *An integer-valued graph function π is an interpolating function if and only if π interpolates over every unicyclic graph.*

Corollary 3 [7] *An integer-valued graph function π is an interpolating function if one of the conditions holds:*

(1) *for every unicyclic graph H and every edge uv belonging to the unique cycle of H, $\pi(H) \leqslant \pi(H-uv) \leqslant \pi(H)+1$;*

(2) *for every unicyclic graph H and every edge uv belonging to the unique cycle of H, $\pi(H)-1 \leqslant \pi(H-uv) \leqslant \pi(H)$.*

2 Results

This section is devoted to establishing the interpolating character of some variants of domination related parameters. We begin with a connected domination number of a graph introduced in [12]. Let G be a connected graph. A set D of vertices of G is a *connected dominating set* of G if D is dominating in G and a subgraph induced by D in G, denoted by $\langle D\rangle_G$, is connected. The cardinality of a minimum connected dominating set of G is called the *connected domination number* of G and is denoted by $\gamma_c(G)$. We now prove that the connected domination number is an interpolating function.

Observation 1 *In a unicyclic graph G a connected dominating set $S \subseteq V(G)$ contains all but at most two vertices from the unique cycle of G. A vertex on the cycle, but not in S, has valency two and two vertices on the cycle, but not in S, are adjacent.*

Theorem 4 *The connected domination number γ_c is an interpolating function.*

Proof. Let G be a unicyclic graph and let C be the unique cycle of G. Let S be a connected dominating set of G such that $|S| = a = \gamma_c(G)$. We obtain all spanning trees of G by successively removing each edge of C, $\mathcal{T}(G) = \{T = G - uv : uv \in E(C)\}$. On the basis of Observation 1 we consider three cases.

Case 1. $V(C) - S = \emptyset$. If we remove any edge $uv \in E(C)$, then S is also a dominating set of $G - uv$ and $\langle S\rangle_{G-uv}$ is connected, so S is a minimum connected dominating set of $G - uv$. Thus, $\gamma_c(T) = a$ for all spanning trees T of G.

Case 2. $|V(C) - S| = 1$. Assume $V(C) - S = \{x\}$. Observe that $d_G(x) = 2$. If $u = x$ or $v = x$, then S is a minimum connected dominating set of $G - uv$. Thus assume that $u \neq x$ and $v \neq x$. Then one of the sets $(S - \{v\}) \cup \{x\}$, $(S - \{u\}) \cup \{x\}$ or $S \cup \{x\}$ is a minimum connected dominating set of $G - uv$. Thus, $\gamma_c(\mathcal{T}(G)) = \{a\}$ or $\gamma_c(\mathcal{T}(G)) = \{a, a + 1\}$.

Case 3. $|V(C) - S| = 2$. Assume $V(C) - S = \{x, y\}$. In this case $d_G(x) = d_G(y) = 2$ and by Observation 1, we know that x and y are adjacent vertices of G. If $|\{u, v\} \cap \{x, y\}| = 2$, then S is a minimum connected dominating set of $G - uv$. Thus assume that $|\{u, v\} \cap \{x, y\}| = 1$, say $u = x$. Then $(S - \{v\}) \cup \{y\}$ or $S \cup \{y\}$ is a minimum connected dominating set of $G - uv$. Finally, assume that $\{u, v\} \cap \{x, y\} = \emptyset$. Then one of the sets $(S - \{u, v\}) \cup \{x, y\}$, $(S - \{u\}) \cup \{x, y\}$, $(S - \{v\}) \cup \{x, y\}$ or $S \cup \{x, y\}$ is a minimum connected dominating set of $G - uv$. Thus $\gamma_c(G - uv) \in \{a, a + 1, a + 2\}$ and we note that if $\gamma_c(G - uv) = \{a + 2\}$, then there exists other edge $wz \in E(C)$ such that $d_G(w) = 2$ and $d_G(z) > 2$ (because $d_G(x) = d_G(y) = 2$, $d_G(u) > 2$, $d_G(v) > 2$ and $xy, uv \in E(C)$). Hence $(S - \{w\}) \cup \{x, y\}$ is a minimum connected dominating set of $G - wz$ and $\gamma_c(G - wz) = \{a + 1\}$. Thus, $\gamma_c(\mathcal{T}(G)) = \{a\}$ or $\gamma_c(\mathcal{T}(G)) = \{a, a + 1\}$ or $\gamma_c(\mathcal{T}(G)) = \{a, a + 1, a + 2\}$.

Consequently, by Theorem 2, the number γ_c is an interpolating function. ■

The fact that γ_c is an interpolating function we may also prove in another way. We begin with an observation which is a consequence of the definition of the connected domination number.

Observation 2 *Let T be a non-trivial tree of order n. Then $\gamma_c(T) = n - n_1(T)$, where $n_1(T)$ is a number of end vertices of T.*

To prove that γ_c is an interpolating function, it is now enough to show the interpolating character of n_1. This fact was proved in [1, 2, 10, 13].

In [4], weakly connected domination was introduced. For a subset S of vertices of a connected graph G, the subgraph weakly induced by S in G, denoted by $\langle S\rangle_G^w$, is the graph $(N_G[S], E^{'})$, where $E^{'}$ consists of the set of all edges of G having at least one vertex in S. A set S is a *weakly connected dominating set* of G if S is dominating in G and $(N_G[S], E^{'})$ is connected. The *weakly connected domination number* of G, denoted by $\gamma_{wc}(G)$, is the minimum cardinality of a weakly connected dominating set of G. We now consider relationships between a weakly connected dominating set and a connected dominating set.

Lemma 5 *Let G be a connected graph and let D be a connected dominating set of G. Then D is a weakly connected dominating set of G.*

Proof. Since D is a connected dominating set of G, the subgraph $\langle D\rangle_G$ is connected, i.e. any two vertices $u, v \in D$ are linked by a path contained in D. Moreover, $N_G(x) \cap D \neq \emptyset$ for each $x \in V(G) - D = N_G[D] - D$. Therefore D is also a weakly connected dominating set of G. ∎

The converse implication of Lemma 5 is false. A weakly connected dominating set of graph does not have to be a connected dominating set of graph. The counterexample is a set D formed of two vertices adjacent to the end vertices of the path P_5 (this path has five vertices). In spite of this, the weakly connected domination number is an interpolating function. We now prove this fact. First we cite a relationship between the number γ_{wc} and the independence number α.

Proposition 6 [4] *If T is a tree of order $n \geqslant 2$, then $\gamma_{wc}(T) = n - \alpha(T)$.*

Theorem 7 *The weakly connected domination number γ_{wc} is an interpolating function.*

Proof. Let G be a unicyclic graph and let C be the unique cycle of G. Remove any edge $uv \in E(G)$, which belongs to C. Now we have a spanning tree $G - uv$. By Proposition 6, it suffices to show that the set $\alpha(\mathcal{T}(G)) = \{\alpha(T) : T = G - uv, uv \in E(C)\}$ consists of consecutive integers, because n is a constant value for a given graph. This fact was proved in [8, 14]. ∎

The fact that the weakly connected domination number γ_{wc} is an interpolating function is also an immediate consequence of the following lemma.

Lemma 8 *If H is a unicyclic graph and uv is an edge belonging to the unique cycle C of H, then $\gamma_{wc}(H) \leqslant \gamma_{wc}(H - uv) \leqslant \gamma_{wc}(H) + 1$.*

Proof. Since every weakly connected dominating set of $H - uv$ is also a weakly connected dominating set of H, so $\gamma_{wc}(H) \leqslant \gamma_{wc}(H - uv)$. On the other hand, let D

be a minimum weakly connected dominating set of H. If D is also a weakly connected dominating set of $H-uv$, then $\gamma_{wc}(H-uv) \leqslant |D| = \gamma_{wc}(H) \leqslant \gamma_{wc}(H)+1$. Assume now that D is not a weakly connected dominating set of $H-uv$. Then $D \cap \{u,v\} \neq \emptyset$. We consider two possible cases.

Case 1. $|D \cap \{u,v\}| = 1$, say $v \in D$ and $u \notin D$. Then D is not dominating in $H-uv$ or $\langle D \rangle^w_{H-uv}$, the subgraph weakly induced by D in $H-uv$, is not connected. If D is not dominating in $H-uv$ and $\langle D \rangle^w_{H-uv}$ is connected, then $D \cup \{u\}$ is a weakly connected dominating set of $H-uv$. Thus assume that D is dominating in $H-uv$ and $\langle D \rangle^w_{H-uv}$ is not connected. Then there exists exactly one edge $ab \in E(G)$ which belongs to C and $D \cap \{a,b\} = \emptyset$. Hence $D \cup \{a\}$ or $D \cup \{b\}$ is a weakly connected dominating set of $H-uv$. Finally, assume that D is not dominating in $H-uv$ and $\langle D \rangle^w_{H-uv}$ is not connected. Then there exists exactly one edge $ab \in E(G)$ which belongs to C and $D \cap \{a,b\} = \emptyset$ and $a = u$. Hence $D \cup \{u\}$ or $D \cup \{b\}$ is a weakly connected dominating set of $H-uv$. Thus, $\gamma_{wc}(H-uv) \leqslant |D| = \gamma_{wc}(H) \leqslant \gamma_{wc}(H)+1$.

Case 2. $|D \cap \{u,v\}| = 2$. It is easy to observe that D is dominating in $H-uv$ and $\langle D \rangle^w_{H-uv}$, the subgraph weakly induced by D in $H-uv$ is not connected. Then there exists exactly one edge $ab \in E(G)$ which belongs to C and $D \cap \{a,b\} = \emptyset$. Hence $D \cup \{a\}$ or $D \cup \{b\}$ is a weakly connected dominating set of $H-uv$ and $\gamma_{wc}(H-uv) \leqslant |D|+1 = \gamma_{wc}(H)+1$.

■

A set $D \subseteq V(G)$ is said to be a *double dominating set* of G if $|N_G[v] \cap D| \geqslant 2$ for every vertex $v \in V(G)$. The *double domination number* of G, denoted by $\gamma^{2d}(G)$, is the minimum cardinality of a double dominating set of G. This parameter is only defined for graphs without an isolated vertex. It is worth observing that every end vertex of G and its neighbour always belong to every double dominating set of G. Double dominating sets were characterized in [5]. We now study the interpolating character of the double domination number. First we show that the removal of a non-end edge from a graph may increase its domination number by at most 2.

Lemma 9 *Let G be a graph without an isolated vertex, and let uv be an edge of a graph G. If uv is not an end edge of G, then $\gamma^{2d}(G) \leqslant \gamma^{2d}(G-uv) \leqslant \gamma^{2d}(G)+2$.*

Proof. Since every double dominating set of $G-uv$ is also a double dominating set in G, the inequality $\gamma^{2d}(G) \leqslant \gamma^{2d}(G-uv)$ is obvious. In order to prove the remaining inequality, let D be a minimum double dominating set of G. If D is also a double dominating set of $G-uv$, then $\gamma^{2d}(G-uv) \leqslant |D| = \gamma^{2d}(G) \leqslant \gamma^{2d}(G)+2$. Assume now that D is not a double dominating set of $G-uv$. Then $D \cap \{u,v\} \neq \emptyset$. We consider two possible cases.

Case 1. $|D \cap \{u, v\}| = 1$, say $v \in D$ and $u \notin D$. Now it is easy to observe that $D \cup \{u\}$ is a double dominating set of $G - uv$ and so $\gamma^{2d}(G - uv) \leqslant |D| + 1 = \gamma^{2d}(G) + 1 \leqslant \gamma^{2d}(G) + 2$.

Case 2. $|D \cap \{u, v\}| = 2$. Since D is not a double dominating set of $G - uv$, then at least one vertex v or u does not have a neighbour in D. Assume that $N_{G-uv}(v) \cap D = \emptyset$ and $N_{G-uv}(u) \cap D \neq \emptyset$. Then $D \cup \{x\}$ is a double dominating set of $G - uv$ for a vertex $x \in N_{G-uv}(v)$ and so $\gamma^{2d}(G - uv) \leqslant |D| + 1 = \gamma^{2d}(G) + 1 \leqslant \gamma^{2d}(G) + 2$. Finally, if $N_{G-uv}(v) \cap D = \emptyset$ and $N_{G-uv}(u) \cap D = \emptyset$, then it is easy to observe that $D \cup \{x, y\}$ is a double domianting set of $G - uv$ for a vertex $x \in N_{G-uv}(v)$ and a vertex $y \in N_{G-uv}(u)$ and therefore $\gamma^{2d}(G - uv) \leqslant |D| + 2 = \gamma^{2d}(G) + 2$.

■

Corollary 10 *Let G be a unicyclic graph with $\gamma^{2d}(G) = a$, then $\gamma^{2d}(\mathcal{T}(G)) \subseteq \{a, a+1, a+2\}$.*

Although Lemma 9 is true for unicyclic graphs, γ^{2d} is not an interpolating function. It follows from the counterexample shown in Fig. 1, in which the unicyclic graph G has only three nonisomorphic spanning trees T_1, T_2 and T_3 with $\gamma^{2d}(T_1) = \gamma^{2d}(T_2) = 10$ and $\gamma^{2d}(T_3) = 12$. The marked vertices of each tree indicate a minimum double dominating set in this tree.

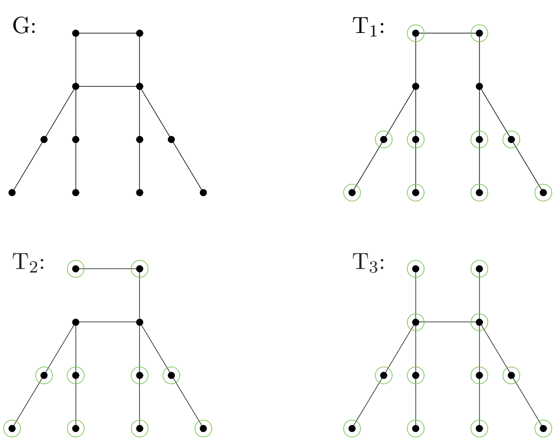

Fig. 1. A graph G with its spanning trees and $\gamma^{2d}(\mathcal{T}(G)) = \{10, 12\}$

We now study the interpolating character of two opposite variants of domination parameters—the weak and strong domination numbers of a graph introduced in [11].

We say that a vertex v *weakly dominates a vertex* u in a graph G if $u \in N_G[v]$ and $deg(v) \leqslant deg(u)$. Similarly, we say that a vertex u *strongly dominates a vertex* v in G if $v \in N_G[u]$ and $deg(u) \geqslant deg(v)$. A set D of vertices of a graph G is a *weak dominating set* of G if every vertex in $V(G) - D$ is weakly dominated by at least one vertex in D. Similarly, $D \subseteq V(G)$ is said to be a *strongly dominating set* of G if every vertex in $V(G) - D$ is strongly dominated by at least one vertex in D. The *weak (strong) domination number* of G, denoted by $\gamma_w(G)$ ($\gamma_s(G)$), is the minimum cardinality of a weak (strong) dominating set of G.

The weak domination number γ_w is not an interpolating function. This follows from the counterexample shown in Fig. 2, in which the unicyclic graph G has only three nonisomorphic spanning trees T_1, T_2 and T_3. The marked vertices of each tree indicate a minimum weak dominating set in this tree. Thus, $\gamma_w(\mathcal{T}(G)) = \{11, 12, 14\}$.

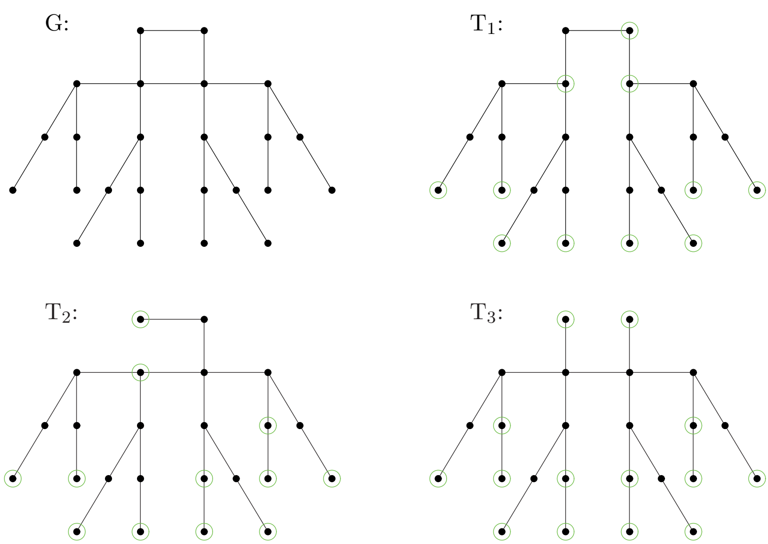

Fig. 2. A graph G with its spanning trees and $\gamma_w(\mathcal{T}(G)) = \{11, 12, 14\}$

For 2-connected graphs we have the following theorem.

Theorem 11 *The weak domination number* γ_w *interpolates over every* 2*-connected graph.*

Proof. Let G be a 2-connected graph, and let m and M be the smallest and the largest integer of $\gamma_w(\mathcal{T}(G))$, respectively. Let $T_\circ, T^*$ be spanning trees of G such that $\gamma_w(T_\circ) = m$ and $\gamma_w(T^*) = M$. Since G is a 2-connected graph then it follows from Theorem 1 that there exists a sequence of end edge-exchanges $T_\circ, T_1, \ldots, T_n = T^*$

transforming $T_\circ$ into T^*. We now need only to show that each step of the end edge-exchange may increase the value of γ_w by at most one. Then $\gamma_w(T_{l+1}) \leqslant \gamma_w(T_l) + 1$ for $l = 0, \ldots, n-1$ and it implies that the sequence $(\gamma_w(T_\circ), \gamma_w(T_1), \ldots, \gamma_w(T_n))$ contains $(m, m+1, \ldots, M)$ as a subsequence and consequently $\gamma_w(\mathcal{T}(G)) = \{m, m+1, \ldots, M\}$. Let D be a minimum weak dominating set of T_l and assume that $T_{l+1} = T_l - uv + uw$, where u is an end vertex of T_l and of T_{l+1}. If D is also a weak dominating set of T_{l+1}, then $\gamma_w(T_{l+1}) \leqslant \gamma_w(T_l) + 1$. Assume now that D is not a weak dominating set of T_{l+1}. It follows from the definition of the weak dominating set that $u \in D$. Thus, v is the unique vertex in T_{l+1}, which is not weak dominated by any vertex of T_{l+1}. Therefore $D \cup \{v\}$ is a weak dominating set of T_{l+1} and so $\gamma_w(T_{l+1}) \leqslant |D| + 1 \leqslant \gamma_w(T_l) + 1$. ■

We now prove that the strong domination number, like the weak domination number, interpolates over every 2-connected graph.

Theorem 12 *The strong domination number γ_s interpolates over any 2-connected graph.*

Proof. Let G be a 2-connected graph. As in the proof of Theorem 11, it is enough to show that $\gamma_s(T') \leqslant \gamma_s(T) + 1$ for every end edge-exchange of a spanning tree T into a spanning tree $T' = T - uv + uw$ of G, where u is an end vertex of T and of T'. Let D be a minimum strong dominating set of T. If D is also a strong dominating set of T', then $\gamma_s(T') \leqslant \gamma_s(T) \leqslant \gamma_s(T) + 1$. Assume now that D is not a strong dominating set of T'. We may assume that $u \notin D$. Otherwise it follows from the minimality of D that $u \in D$ and $v \notin D$ and then $(D - \{u\}) \cup \{v\}$ is also a minimum strong dominating set of T. It is now easy to observe that $D \cup \{w\}$ is a strong dominating set of T', so $\gamma_s(T') \leqslant |D| + 1 \leqslant \gamma_s(T) + 1$. ■

In the end we prove that a very interesting both mathematical and historical variant of domination parameter is an interpolating function. This variant of the domination number, which is suggested by the article in *Scientific American* (1999) by Ian Stewart, entitled "Defend the Roman Empire!", was first defined and characterized in [3].

For a graph $G = (V, E)$, let $f : V \to \{0, 1, 2\}$ be a function, and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V | f(v) = i\}$ and $|V_i| = n_i$ for $i = 0, 1, 2$. Between the functions f and the ordered partitions (V_0, V_1, V_2) of V exists $1-1$ correspondence and so we write $f = (V_0, V_1, V_2)$. A function $f : V \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) of a graph $G = (V, E)$ if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of f is $f(V) = \sum_{v \in V(G)} f(v) = 2n_2 + n_1$. The *Roman domination number*, denoted by $\gamma_R(G)$, equals the minimum weight of an RDF of G. A function $f = (V_0, V_1, V_2)$ is a *γ_R-function* if it is an RDF and $f(V) = \gamma_R(G)$.

Theorem 13 *The Roman domination number γ_R is an interpolating function.*

Proof. By Corollary 3(1), it suffices to show that the inequality $\gamma_R(G) \leqslant \gamma_R(G - uv) \leqslant \gamma_R(G) + 1$ is true for every unicyclic graph G and every edge of the unique cycle of G. If $f = (V_0, V_1, V_2)$ is a γ_R-function on a graph $G - uv$, then $f = (V_0, V_1, V_2)$ is also an RDF on a graph G and so $\gamma_R(G) \leqslant |V_1| + 2|V_2| = \gamma_R(G - uv)$. Let now $g = (V_0, V_1, V_2)$ be a γ_R-function on a graph G, and let uv be an edge of the unique cycle of G. If one of the vertices v or u belongs to V_0 and the other belongs to V_2, say $v \in V_0$ and $u \in V_2$, then function $g^{'} = (V_0 - \{v\}, V_1 \cup \{v\}, V_2)$ is an RDF on $G - uv$. Thus, $\gamma_R(G - uv) \leqslant g^{'}(V(G)) = |V_1 \cup \{v\}| + 2|V_2| = |V_1| + 2|V_2| + 1 = \gamma_R(G) + 1$. In other simpler cases $g = (V_0, V_1, V_2)$ is also an RDF on a graph $G - uv$ and therefore we have $\gamma_R(G - uv) \leqslant \gamma_R(G) \leqslant \gamma_R(G) + 1$. ■

References

[1] C. A. Barefoot, Interpolation theorem for the number of pendant vertices of connected spanning subgraphs of equal size, *Discrete Math.* 49 (1984), 109–112.

[2] C. A. Barefoot, Interpolation and related topics, *J. Combin. Math. Combin. Comput.* 12 (1992), 65–96.

[3] E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi and S. T. Hedetniemi, Roman domination in graphs, *Discrete Math.* 278 (2004), 11–22.

[4] J. E. Dunbar, J. W. Grossman, J. H. Hattingh, S. T. Hedetniemi and A. A. McRae, On weakly connected domination in graphs, *Discrete Math.* 167/168 (1997), 261–269.

[5] F. Harary and T. W. Haynes, Double domination in graphs, *Ars Combin.* 55 (2000), 201–213.

[6] F. Harary, R. J. Mokken and M. J. Plantholt, Interpolation theorem for diameters of spanning trees, *IEEE Trans. Circuits Systems* 30 (1983), 429–432.

[7] F. Harary and M. J. Plantholt, Classification of interpolation theorems for spanning trees and other families of spanning subgraphs, *J. Graph Theory* 13 (1989), 703–712.

[8] F. Harary and S. Schuster, Interpolation theorems for the independence and domination numbers of spanning trees, *Ann. Discrete Math.* 41 (1989), 221–228.

[9] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, New York, 1998.

[10] Y. Lin, A simpler proof of interpolation theorem for spanning trees, *Kexue Tongbao* 30 (1985), 134.

[11] E. Sampathkumar and L. Pushpa Latha, Strong weak domination and domination balance in a graph, *Discrete Math.* 161 (1996), 235–242.

[12] E. Sampathkumar and H. B. Walikar, The connected domination number of a graph, *J. Math. Phys. Sci.* 13 (1979), 607–613.

[13] S. Schuster, Interpolation theorem for the number of end-vertices of spanning trees, *J. Graph Theory* 7 (1983), 203–208.

[14] J. Topp and P. D. Vestergaard, Interpolation theorems for domination numbers of a graph, *Discrete Math.* 191 (1998), 207–221.

[15] J. Topp and P. D. Vestergaard, Interpolation theorems for the (r, s)-domination number of spanning trees, *Australas. J. Combin.* 17 (1998), 99–107.

(Received 15 Aug 2008; revised 8 July 2009)