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Local material symmetry group for first- and second-order strain gradient fluids

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Abstract
Using an unified approach based on the local material symmetry group introduced for general first- and second-order strain gradient elastic media, we analyze the constitutive equations of strain gradient fluids. For the strain gradient medium there exists a strain energy density dependent on first- and higher-order gradients of placement vector, whereas for fluids a strain energy depends on a current mass density and its gradients. Both models found applications to modeling of materials with complex inner structure such as beam-lattice metamaterials and fluids at small scales. The local material symmetry group is formed through such transformations of a reference placement which cannot be experimentally detected within the considered material model. We show that considering maximal symmetry group, i.e. material with strain energy that is independent of the choice of a reference placement, one comes to the constitutive equations of gradient fluids introduced independently on general strain gradient continua.

Keywords
Symmetry group, strain gradient elasticity, first-order strain gradient fluid, second-order strain gradient fluid, nonlinear elasticity

1. Introduction

Various concepts of symmetry are closely related to mechanics. For example, a symmetry may reflect a certain invariant property of a problem under consideration such an axial or spherical symmetry that is widely used for derivation of particular classes of solutions, see, e.g., [1, 2]. Symmetries are widely applied to modeling of anisotropic materials such as crystals [3, 4]. A variational symmetry is closely related to conservation laws [5–7]. Considering material symmetries Noll [8] introduced the local material group as a set of such transformations of a reference placement which cannot be detected experimentally. For solids it includes groups used in crystallography but it may also include non-orthogonal transformations related to description of fluids and subfluids [9–10]. Using the local material symmetry group one can properly describe fluids, solids, and intermediate classes called subfluids, that are neither solids nor fluids [9]. The notion of liquid crystals is also used instead of subfluids [10] but here we will use wording subfluids in order to avoid mixing with the Ericksen–Lesli liquid crystals [11, 12]. In particular, an elastic fluid is defined as an elastic medium for which a strain energy density does not depend on the choice of a reference placement. In other words, for a fluid any mass density preserving static deformations of a reference placement cannot be experimentally detected. The concept of material symmetry groups is also very useful for a certain simplification of a form of constitutive equations.

For example, in the case of simple (Cauchy) materials a strain energy density of an isotropic solid depends only on three principal invariants of the Cauchy–Green strain measure whereas for an elastic fluid we have a dependence on mass density [10]. Considering generalized models of continua we have rather complex constitutive equations, in general. Application of the material symmetry requirements to generalized models of continua is even more important than for simple materials as for the latter we have more arguments, in general. For example, for polar media a strain energy depends on few non-symmetric second-order tensors such as stretch and wryness tensor as well as a parametric tensor of microinertia [13, 14] or of microcurvature [15]. For these media, the local material symmetry group was discussed in [13, 16–18] where the corresponding system of strain invariants and the reduced constitutive equations were also presented. A similar definition of the material symmetry group for micromorphic media was given in [19]. For strain gradient media the material symmetry was discussed in [20–23]. In particular, the definition of the material symmetry group for third-order gradient elasticity was proposed in [23]. The material symmetry is also closely related to the analysis of elastic moduli tensors performed recently in [24–27] for strain gradient solids.

Recently the significant interest grows to a new class of materials called beam-lattice metamaterials, which are widely used in civil, mechanical, and aerospace engineering, see, e.g., [28, 29]. They consist of periodic or almost periodic networks of interconnected beams and in a certain sense mimic crystalline lattices and their properties. Among their properties such as a lightweight, relatively high stiffness, flexibility, it is worth also to mention many other useful characteristics such as acoustic response, thermal insulation, energy absorption, etc., that makes these materials very interesting for engineering. As an example of beam-lattice materials one can recall the open-cell foams [28, 30]. Straightforward calculations using real geometry with imperfections require expensive time-consuming calculations so one need to introduce robust models, which can be more easily optimized. Nowadays it is well-established that the homogenization of composite materials with high contrast in properties may lead to generalized continuum models, such as strain-gradient elasticity, see, e.g., [31–39], and references therein. These enriched models demonstrate the efficiency of homogenized models to modeling beam-lattice metamaterials. Within the strain-gradient elasticity there exists a strain energy density introduced as a functions of strains and the higher-order gradients of displacement vector [40–44].

In addition to modeling of composite materials, the strain-gradient elasticity found applications in description of solids and fluids at small scales, see, e.g., [45–47]. In the case of fluids, gradient models can be related to seminal works by van der Waals [48] and Korteweg [49], where the model was proposed with a strain energy density dependent on a current mass density and its spatial gradient. Nowadays this model constitutes a basis of co-called molecular theory of capillarity including wetting and dewetting phenomena [50, 51]. The foundations of the model were also given in the landscape making works by Cahn and Hilliard [52,53]. Thus, nowadays the model is known as the Korteweg fluid or the Cahn–Hilliard fluid or the first strain gradient fluid. In particular, within the model we can describe not only wetting and dewetting phenomena but a formation of an interfacial layer of finite thickness between uid and its vapour [54–58]. The natural generalization of the first strain gradient fluid is the models of higher order, i.e., second and *n*th strain gradient fluid models discussed in [45, 59–63].

It is interesting to note that considering various possible stiffness tensors, Milton and Cherkaev [64] proposed a model called nowadays a pentamode metamaterial with fluid-like properties, so it could be also called a metafluid [65]. Nowadays using additive technologies such structures can be relatively easily reproduced [66, 67].

As a result, one can see that:

1. beam-lattice metamaterials can be modeled as a strain-gradient continuum;
2. some beam-lattice metamaterials may exhibit fluid-like properties.

Thus, as in the case of simple (Cauchy) materials the symmetry analysis may enlighten the relation between the general form of constitutive relations of strain-gradient continua and the particular form of strain-gradient fluids. Note that the both models were introduced independently. In order to clarify these relations we use the analysis based on the local material symmetry group introduced for strain-gradient media.

Thus, the aim of this paper is to discuss the constitutive equations of a strain gradient fluids of first- and second-order from the point of view of the local material symmetry group introduced for general strain gradient continua undergoing large deformations.

This paper is organized as follows. In Section 2 we recall the constitutive equations of the strain gradient continua of first- and second-order under finite deformations. Following [23], we introduce the local material symmetry group. In Section 3 we briefly introduce first and second strain gradient elastic fluids using direct

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approach. Finally, in Section 4 we compare the constitutive relations derived with the help of the symmetry group and those introduced earlier in Section 3.

2. Strain energy density for strain gradient continua

Following [23, 47, 68] let us consider finite deformations of an elastic body $\mathfrak{B}$ modeled within strain gradient elasticity. For each material particle $x \in \mathfrak{B}$ we introduce its position vectors $\mathbf{x}$ and $\mathbf{X}$ in two states called reference (initial) $\varkappa$ and current (deformed) χ placement, respectively. Thus, a deformation of $\mathfrak{B}$ is defined as an invertible differentiable mapping from a reference placement into current one as

$$\mathbf{x} = \mathbf{x}(\mathbf{X}), \tag{1}$$

where the vector-valued function $\mathbf{x}(\mathbf{X})$ is assumed to be smooth enough.

Material modeling of hyperelastic media is based on introduction of a strain energy density function [10, 69]. In what follows, we introduce these functions for simple (Cauchy), first- and second-order strain gradient media by the relations

$$\mathcal{W} = \mathcal{W}_0(\mathbf{F}), \quad \mathbf{F} = \nabla_\varkappa \mathbf{x}, \tag{2}$$

$$\mathcal{W} = \mathcal{W}_1(\mathbf{F}, \nabla_\varkappa \mathbf{F}), \tag{3}$$

$$\mathcal{W} = \mathcal{W}_2(\mathbf{F}, \nabla_\varkappa \mathbf{F}, \nabla_\varkappa \nabla_\varkappa \mathbf{F}), \tag{4}$$

where $\mathbf{F}$ and $\nabla_\varkappa$ are the deformation gradient and referential nabla-operator defined as in [70–72]. Hereinafter the indices 0, 1, and 2 denotes simple, first-order, and second-order strain gradient models, respectively. Applying to (2)–(4) the principle of material frame indifference [10] we came to the following representations

$$\mathcal{W} = \mathcal{W}_0(\mathbf{C}), \quad \mathbf{C} = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}}, \tag{5}$$

$$\mathcal{W} = \mathcal{W}_1(\mathbf{C}, \mathbf{K}_1), \quad \mathbf{K}_1 = \nabla_\varkappa \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} \tag{6}$$

$$\mathcal{W} = \mathcal{W}_2(\mathbf{F}, \mathbf{K}_1, \mathbf{K}_2), \qquad \mathbf{K}_2 = \nabla_\varkappa \nabla_\varkappa \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}}, \tag{7}$$

where we keep the same notation for the constitutive functions, $\mathbf{C}$ is the Cauchy–Green strain tensor, T is the symbol of transposition operation, and the centered dot denotes the scalar product of two vectors [72, 73]. It can be also applied to tensors of any order. For example, for two dyads we have

$$(\mathbf{a} \otimes \mathbf{b}) \cdot (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \otimes \mathbf{d},$$

where $\otimes$ denotes the dyadic product and $\mathbf{a}, \mathbf{b}, \ldots$, denote arbitrary vectors. Note that $\mathbf{C}$, $\mathbf{K}_1$, and $\mathbf{K}_2$ are tensors of second-, third- and fourth-order, respectively. This makes the possible forms of constitutive equations rather complex as they includes a large number of material parameters [23, 68].

Note that in [23] another set of strain measures was used. Namely, the following constitutive relations were introduced

$$\mathcal{W} = \mathcal{W}_1(\mathbf{C}, \widetilde{\mathbf{K}}_1), \quad \widetilde{\mathbf{K}}_1 = \nabla_\varkappa \mathbf{F} \cdot \mathbf{F}^{-1} \tag{8}$$

$$\mathcal{W} = \mathcal{W}_2(\mathbf{F}, \widetilde{\mathbf{K}}_1, \widetilde{\mathbf{K}}_2), \quad \widetilde{\mathbf{K}}_2 = \nabla_\varkappa \nabla_\varkappa \mathbf{F} \cdot \mathbf{F}^{-1}. \tag{9}$$

In order to simplify constitutive relations using *a priori* known material symmetry, Noll's definition of the local material symmetry group (isotropy group) was extended for strain gradient materials in [23, 68]. Let us briefly recall these definitions including also the definition for simple materials. We consider a density-preserving transformation of $\varkappa$ into another reference placement $\varkappa^*$. The corresponding mapping $\varkappa \to \varkappa^*$ has the form

$$\mathbf{X}^* = \mathbf{X}^*(\mathbf{X}), \tag{10}$$

where $\mathbf{X}^*$ is the position vector of $x \in \mathfrak{B}$ related to $\varkappa^*$. Thus, we can describe deformations of $\mathfrak{B}$ as schematically shown in Figure 1. In what follows we denote all quantities calculated with respect to $\varkappa^*$ with the index "$*$".

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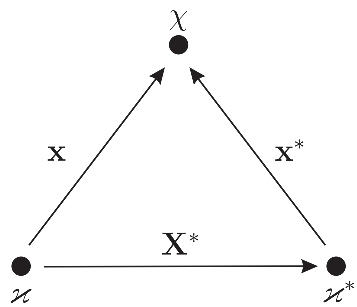

Figure 1. Mappings and corresponding placements $\varkappa$, $\varkappa^*$, and χ.

In order to introduce the definition of the local material symmetry group we have to find the correspondence relations between arguments of strain energy density functions, i.e., relations

$$\mathbf{C} \to \mathbf{C}^*, \quad \mathbf{K}_1 \to \mathbf{K}_1^*, \quad \mathbf{K}_2 \to \mathbf{K}_2^*.$$

As $\mathcal{W}_k$, $k = 0, 1, 2$, describes an energy stored in an infinitesimal neighborhood of a point $x \in \mathfrak{B}$ related to the current placement, we should have the relation $\mathcal{W}_k = \mathcal{W}_k^*$. Of course, the form of constitutive function could be dependent on the choice of the reference placement.

In the following, we are looking for such transformations of a reference placement which cannot be detected experimentally. From the mathematical point of view this requirement results in the determination of local transformations $\varkappa \to \varkappa^*$ such that for considered models of materials the following invariance properties are fulfilled

$$\begin{aligned} \mathcal{W}_0(\mathbf{C}) =& \mathcal{W}_0(\mathbf{C}^*), && (11)\\ \mathcal{W}_1(\mathbf{C}, \mathbf{K}_1) =& \mathcal{W}_1(\mathbf{C}^*, \mathbf{K}_1^*), && (12)\\ \mathcal{W}_2(\mathbf{C}, \mathbf{K}_1, \mathbf{K}_2) =& \mathcal{W}_2(\mathbf{C}^*, \mathbf{K}_1^*, \mathbf{K}_2^*). && (13) \end{aligned}$$

In order to complete (11)–(13) we have to find the exact correspondence between $\mathbf{C}$ and $\mathbf{C}^*$, $\mathbf{K}_1$ and $\mathbf{K}_1^*$, and $\mathbf{K}_2$ and $\mathbf{K}_2^*$.

Using identities for the first differentials

$$\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{X} \cdot \mathbf{F} = \mathrm{d}\mathbf{X}^* \cdot \mathbf{F}^* = \mathrm{d}\mathbf{X} \cdot \mathbf{P} \cdot \mathbf{F}^*,$$

we have the relation between deformation gradients and nabla-operators

$$\mathbf{F} = \mathbf{P} \cdot \mathbf{F}^*, \quad \nabla_\varkappa = \mathbf{P} \cdot \nabla^*, \tag{14}$$

where $\mathbf{P} = \nabla_\varkappa \mathbf{X}^*$ is the deformation gradient related to (10). As a result, we obtain the first correspondence relation

$$\mathbf{C} = \mathbf{P} \cdot \mathbf{C}^* \cdot \mathbf{P}^{\mathrm{T}}. \tag{15}$$

For the second differentials we have the formulas

$$\begin{aligned} \mathrm{d}^2\mathbf{x} \equiv& \mathrm{d}(\mathrm{d}\mathbf{x}) = \mathrm{d}(\mathrm{d}\mathbf{X} \cdot \mathbf{F}) = \mathrm{d}^2\mathbf{X} \cdot \mathbf{F} + \mathrm{d}\mathbf{X} \cdot \mathrm{d}\mathbf{F}\\ =& \mathrm{d}^2\mathbf{X} \cdot \mathbf{F} + \mathrm{d}\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot \nabla_\varkappa \mathbf{F}), && (16)\\ \mathrm{d}^2\mathbf{X}^* =& \mathrm{d}^2\mathbf{X} \cdot \mathbf{P} + \mathrm{d}\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot \nabla_\varkappa \mathbf{P}), && (17)\\ \mathrm{d}^2\mathbf{x} =& \mathrm{d}^2\mathbf{X}^* \cdot \mathbf{F}^* + \mathrm{d}\mathbf{X}^* \cdot (\mathrm{d}\mathbf{X}^* \cdot \nabla^* \mathbf{F}^*)\\ =& \mathrm{d}^2\mathbf{X} \cdot \mathbf{P} \cdot \mathbf{F}^* + \mathrm{d}\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot \nabla_\varkappa \mathbf{P}) \cdot \mathbf{F}^*\\ &+ \mathrm{d}\mathbf{X} \cdot \mathbf{P} \cdot (\mathrm{d}\mathbf{X} \cdot \mathbf{P} \cdot \nabla^* \mathbf{F}^*). && (18) \end{aligned}$$

Comparing (16) with (18) we see that

$$\mathrm{d}\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot \nabla_\varkappa \mathbf{F}) = \mathrm{d}\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot \nabla_\varkappa \mathbf{P}) \cdot \mathbf{F}^* + \mathrm{d}\mathbf{X} \cdot \mathbf{P} \cdot (\mathrm{d}\mathbf{X} \cdot \mathbf{P} \cdot \nabla^* \mathbf{F}^*)$$

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and obtain the formula for the second deformation gradient

$$\nabla_{\varkappa}\mathbf{F} = \nabla_{\varkappa}\mathbf{P} \cdot \mathbf{F}^* + (\mathbf{P} * \nabla^*\mathbf{F}^*) \cdot \mathbf{P}^{-\mathrm{T}}. \tag{19}$$

Here we introduce the Rayleigh product "$*$" of a second-order tensor and other tensors, see, e.g., [23] for more detail. In particular, for dyads, triads, tetrads, and other polyads, the Rayleigh product is defined as follows

$$\begin{aligned}
&\mathbf{P} * (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{P} \cdot \mathbf{a}) \otimes (\mathbf{P} \cdot \mathbf{b}) = \mathbf{P} \cdot (\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{P}^{\mathrm{T}},\\
&\mathbf{P} * (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) = (\mathbf{P} \cdot \mathbf{a}) \otimes (\mathbf{P} \cdot \mathbf{b}) \otimes (\mathbf{P} \cdot \mathbf{c}),\\
&\mathbf{P} * (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d}) = (\mathbf{P} \cdot \mathbf{a}) \otimes (\mathbf{P} \cdot \mathbf{b}) \otimes (\mathbf{P} \cdot \mathbf{c}) \otimes (\mathbf{P} \cdot \mathbf{d}), \quad \text{etc.},
\end{aligned}$$

where $\mathbf{P}$ is a second-order tensor. Obviously, as for dyads, for second-order tensors we have more simple formula for the Rayleigh product

$$\mathbf{P} * \mathbf{C} = \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^{\mathrm{T}}.$$

The Rayleigh product has properties

$$(\mathbf{P}_1 \cdot \mathbf{P}_2) * \mathbf{K} = \mathbf{P}_1 * (\mathbf{P}_2 * \mathbf{K}), \quad \mathbf{P}^{-1} * (\mathbf{P} * \mathbf{K}) = \mathbf{K}. \tag{20}$$

Multiplying (19) by $\mathbf{F}^{\mathrm{T}}$ from the right we obtain

$$\begin{aligned}
\mathbf{K}_1 \equiv & \nabla_{\varkappa}\mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} = \nabla_{\varkappa}\mathbf{P} \cdot \mathbf{F}^* \cdot \mathbf{F}^{\mathrm{T}} + (\mathbf{P} * \nabla^*\mathbf{F}^*) \cdot \mathbf{P}^{-\mathrm{T}} \cdot \mathbf{F}^{\mathrm{T}}\\
= & \nabla_{\varkappa}\mathbf{P} \cdot \mathbf{F}^* \cdot \mathbf{F}^{*\mathrm{T}} \cdot \mathbf{P}^{\mathrm{T}} + (\mathbf{P} * \nabla^*\mathbf{F}^*) \cdot \mathbf{P}^{-\mathrm{T}} \cdot \mathbf{F}^{*\mathrm{T}} \cdot \mathbf{P}^{\mathrm{T}}\\
= & \nabla_{\varkappa}\mathbf{P} \cdot \mathbf{C}^* \cdot \mathbf{P}^{\mathrm{T}} + \mathbf{P} * (\nabla^*\mathbf{F}^* \cdot \mathbf{F}^{*\mathrm{T}})\\
= & \nabla_{\varkappa}\mathbf{P} \cdot \mathbf{P}^{-1} \cdot \mathbf{P} \cdot \mathbf{C}^* \cdot \mathbf{P}^{\mathrm{T}} + \mathbf{P} * \mathbf{K}_1^*.
\end{aligned} \tag{21}$$

With (21) we have the second correspondence

$$\mathbf{K}_1 = \mathbf{P} * \mathbf{K}_1^* + \mathbf{B} \cdot \mathbf{P} \cdot \mathbf{C}^* \cdot \mathbf{P}^{\mathrm{T}}, \tag{22}$$

where $\mathbf{B} = \nabla_{\varkappa}\mathbf{P} \cdot \mathbf{P}^{-1}$.

In the following, we use the following transposition operation denoted by $T(m,n)$ applied for high-order tensors. For a polyad of Nth order it is defined as

$$\begin{aligned}
&(\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_m \otimes \cdots \otimes \mathbf{e}_n \otimes \cdots \otimes \mathbf{e}_N)^{\mathrm{T}(m,n)}\\
&= \mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_n \otimes \cdots \otimes \mathbf{e}_m \otimes \cdots \otimes \mathbf{e}_N.
\end{aligned} \tag{23}$$

For example, for triads we have

$$(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c})^{\mathrm{T}(1,2)} = \mathbf{b} \otimes \mathbf{a} \otimes \mathbf{c}, \quad (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c})^{\mathrm{T}(1,3)} = \mathbf{c} \otimes \mathbf{b} \otimes \mathbf{a}.$$

As $\mathbf{B}$ is a third-order tensor symmetric with respect to two first indices we have

$$\mathbf{B}^{\mathrm{T}(1,2)} = \mathbf{B}. \tag{24}$$

In order to find a relation between $\mathbf{K}_2$ and $\mathbf{K}_2^*$ we derive the third differentials of $\mathbf{x}$ and $\mathbf{X}^*$ as follows

$$\begin{aligned}
\mathrm{d}^3\mathbf{x} \equiv & \mathrm{d}(\mathrm{d}^2\mathbf{x}) = \mathrm{d}\left(\mathrm{d}^2\mathbf{X} \cdot \mathbf{F} + \mathrm{d}\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot \nabla_{\varkappa}\mathbf{F})\right)\\
= & \mathrm{d}^3\mathbf{X} \cdot \mathbf{F} + 2\mathrm{d}^2\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot \nabla_{\varkappa}\mathbf{F}) + \mathrm{d}\mathbf{X} \cdot (\mathrm{d}^2\mathbf{X} \cdot \nabla_{\varkappa}\mathbf{F})\\
& + \mathrm{d}\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot \nabla_{\varkappa}\nabla_{\varkappa}\mathbf{F})),
\end{aligned} \tag{25}$$

$$\begin{aligned}
\mathrm{d}^3\mathbf{X}^* = & \mathrm{d}^3\mathbf{X} \cdot \mathbf{P} + 2\mathrm{d}^2\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot \nabla_{\varkappa}\mathbf{P}) + \mathrm{d}\mathbf{X} \cdot (\mathrm{d}^2\mathbf{X} \cdot \nabla_{\varkappa}\mathbf{P})\\
& + \mathrm{d}\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot \nabla_{\varkappa}\nabla_{\varkappa}\mathbf{P})),
\end{aligned} \tag{26}$$

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$$
\begin{aligned}
\mathrm{d}^3\mathbf{x} =& \mathrm{d}\left(\mathrm{d}^2\mathbf{X}^* \cdot \mathbf{F}^* + \mathrm{d}\mathbf{X}^* \cdot (\mathrm{d}\mathbf{X}^* \cdot \nabla^*\mathbf{F}^*)\right) \\
=& \mathrm{d}^3\mathbf{X}^* \cdot \mathbf{F}^* + 2\mathrm{d}^2\mathbf{X}^* \cdot (\mathrm{d}\mathbf{X}^* \cdot \nabla^*\mathbf{F}^*) + \mathrm{d}\mathbf{X}^* \cdot (\mathrm{d}^2\mathbf{X}^* \cdot \nabla^*\mathbf{F}^*) \\
&+ \mathrm{d}\mathbf{X}^* \cdot (\mathrm{d}\mathbf{X}^* \cdot (\mathrm{d}\mathbf{X}^* \cdot \nabla^*\nabla^*\mathbf{F}^*)) .
\end{aligned} \tag{27}
$$

As $\nabla_\varkappa\mathbf{F} = \nabla_\varkappa\nabla_\varkappa\mathbf{x}$, $\nabla_\varkappa\mathbf{P} = \nabla_\varkappa\nabla_\varkappa\mathbf{X}^*$ and $\nabla^*\mathbf{F}^* = \nabla^*\nabla^*\mathbf{x}$, these third-order tensors are symmetric with respect to first two indices that results in the identities

$$
\begin{aligned}
&\mathrm{d}^2\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot \nabla_\varkappa\mathbf{F}) = \mathrm{d}\mathbf{X} \cdot (\mathrm{d}^2\mathbf{X} \cdot \nabla_\varkappa\mathbf{F}), \\
&\mathrm{d}^2\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot \nabla_\varkappa\mathbf{P}) = \mathrm{d}\mathbf{X} \cdot (\mathrm{d}^2\mathbf{X} \cdot \nabla_\varkappa\mathbf{P}), \\
&\mathrm{d}^2\mathbf{X}^* \cdot (\mathrm{d}\mathbf{X}^* \cdot \nabla^*\mathbf{F}^*) = \mathrm{d}\mathbf{X}^* \cdot (\mathrm{d}^2\mathbf{X}^* \cdot \nabla^*\mathbf{F}^*).
\end{aligned}
$$

Substituting (17) and (26) into (27) and comparing the result with (25) we get

$$
\begin{aligned}
\mathrm{d}\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot \nabla_\varkappa\nabla_\varkappa\mathbf{F})) =& \mathrm{d}\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot \nabla_\varkappa\mathbf{P})) \cdot \mathbf{F}^* \\
&+ 3\mathrm{d}\mathbf{X} \cdot \mathbf{P} \cdot (\mathrm{d}\mathbf{X} \cdot (\mathrm{d}\mathbf{X} \cdot \nabla_\varkappa\mathbf{P}) \cdot \nabla^*\mathbf{F}^*) \\
&+ \mathrm{d}\mathbf{X} \cdot \mathbf{P} \cdot (\mathrm{d}\mathbf{X} \cdot \mathbf{P} \cdot (\mathrm{d}\mathbf{X} \cdot \mathbf{P} \cdot \nabla^*\nabla^*\mathbf{F}^*)) .
\end{aligned} \tag{28}
$$

From (28) it follows that

$$
\begin{aligned}
\nabla_\varkappa\nabla_\varkappa\mathbf{F} =& \nabla_\varkappa\nabla_\varkappa\mathbf{P} \cdot \mathbf{F}^* + (\mathbf{P} * \nabla^*\nabla^*\mathbf{F}^*) \cdot \mathbf{P}^{-\mathrm{T}} \\
&+ 3 \operatorname*{sym}_{(1,2,3)} \left[\mathbf{B} \cdot \mathbf{P} \cdot \left((\nabla^*\mathbf{F}^*)^{\mathrm{T}(2,3)} \cdot \mathbf{P}^{\mathrm{T}} \right)^{\mathrm{T}(2,3)} \right] .
\end{aligned} \tag{29}
$$

Here we have introduced the symmetrization with respect the first three indices. For a triad and tetrad it is given by

$$
\begin{aligned}
\operatorname*{sym}_{(1,2,3)}(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) =& \frac{1}{6}(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} + \mathbf{b} \otimes \mathbf{a} \otimes \mathbf{c} + \mathbf{a} \otimes \mathbf{c} \otimes \mathbf{b} \\
&+ \mathbf{c} \otimes \mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{a} + \mathbf{c} \otimes \mathbf{b} \otimes \mathbf{a}), \\
\operatorname*{sym}_{(1,2,3)}(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d}) =& \operatorname*{sym}_{(1,2,3)}(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) \otimes \mathbf{d}.
\end{aligned}
$$

The symmetrization is commutative with the Rayleigh product

$$
\operatorname*{sym}_{(1,2,3)}(\mathbf{P} * \mathbf{T}) = \mathbf{P} * \operatorname*{sym}_{(1,2,3)}(\mathbf{T}) \tag{30}
$$

for any tensor $\mathbf{T}$.

Multiplying (29) by $\mathbf{F}^{\mathrm{T}}$ from the right we obtain

$$
\begin{aligned}
\mathbf{K}_2 \equiv& \nabla_\varkappa\nabla_\varkappa\mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} = \\
=& \nabla_\varkappa\nabla_\varkappa\mathbf{P} \cdot \mathbf{F}^* \cdot \mathbf{F}^{*\mathrm{T}} \cdot \mathbf{P}^{\mathrm{T}} + (\mathbf{P} * \nabla^*\nabla^*\mathbf{F}^*) \cdot \mathbf{P}^{-\mathrm{T}} \cdot \mathbf{F}^{*\mathrm{T}} \cdot \mathbf{P}^{\mathrm{T}} \\
&+ 3 \operatorname*{sym}_{(1,2,3)} \left[\mathbf{B} \cdot \mathbf{P} \cdot \left((\nabla^*\mathbf{F}^*)^{\mathrm{T}(2,3)} \cdot \mathbf{P}^{\mathrm{T}} \right)^{\mathrm{T}(2,3)} \right] \cdot \mathbf{F}^{*\mathrm{T}} \cdot \mathbf{P}^{\mathrm{T}} \\
=& \nabla_\varkappa\nabla_\varkappa\mathbf{P} \cdot \mathbf{P}^{-1} \cdot \mathbf{P} \cdot \mathbf{C}^* \cdot \mathbf{P}^{\mathrm{T}} + \mathbf{P} * (\nabla^*\nabla^*\mathbf{F}^* \cdot \mathbf{F}^{*T}) \\
&+ 3 \operatorname*{sym}_{(1,2,3)} \left[\mathbf{B} \cdot \left(\mathbf{P} * (\nabla^*\mathbf{F}^* \cdot \mathbf{F}^{*T}) \right) \right] \\
=& \mathbf{P} * \mathbf{K}_2^* + \mathbf{W} \cdot \mathbf{P} \cdot \mathbf{C}^* \cdot \mathbf{P}^{\mathrm{T}} + 3 \operatorname*{sym}_{(1,2,3)} \left[\mathbf{B} \cdot (\mathbf{P} * \mathbf{K}_1^*) \right] .
\end{aligned} \tag{31}
$$

In (31) we have introduced a fourth-order tensor W as follows

$$
\mathbf{W} = \nabla_\varkappa\nabla_\varkappa\mathbf{P} \cdot \mathbf{P}^{-1},
$$

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which is symmetric with respect to the first three indices

$$\mathbf{W}^{T(1,2)} = \mathbf{W}, \quad \mathbf{W}^{T(1,3)} = \mathbf{W}, \quad \mathbf{W}^{T(2,3)} = \mathbf{W}. \tag{32}$$

As we restrict to mass-density-preserving transformations, $\mathbf{P}$ is an unimodular tensor, $|\det \mathbf{P}| = 1$. Thus, the gradient of $\det \mathbf{P}$ is zero. Using the identities

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\mathbf{P}} \det \mathbf{P} = (\det \mathbf{P})\mathbf{P}^{-T}, \\
&\mathrm{d}(\det \mathbf{P}) = \mathrm{d}\mathbf{X} \cdot \nabla_{\varkappa}(\det \mathbf{P}), \\
&\mathrm{d}(\det \mathbf{P}) = \mathrm{d}\mathbf{P} : \frac{\mathrm{d}}{\mathrm{d}\mathbf{P}} \det \mathbf{P} = (\mathrm{d}\mathbf{X} \cdot \nabla_{\varkappa}\mathbf{P}) : \frac{\mathrm{d}}{\mathrm{d}\mathbf{P}} \det \mathbf{P},
\end{aligned}$$

we obtain

$$\nabla_{\varkappa}(\det \mathbf{P}) = (\det \mathbf{P})\nabla_{\varkappa}\mathbf{P} : \mathbf{P}^{-T} = \pm(\nabla_{\varkappa}\mathbf{P} \cdot \mathbf{P}^{-1}) : \mathbf{1} = \pm\mathbf{B} : \mathbf{1} = 0. \tag{33}$$

Here $\mathbf{1}$ is the unit tensor and ":" denotes the double dot product defined for dyads, triads, and other polyads by relations

$$(\mathbf{a} \otimes \mathbf{b}) : (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}),$$

$$(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) : (\mathbf{d} \otimes \mathbf{e}) = (\mathbf{b} \cdot \mathbf{d})(\mathbf{c} \cdot \mathbf{e})\mathbf{a},$$

$$(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) : (\mathbf{d} \otimes \mathbf{e} \otimes \mathbf{f}) = (\mathbf{b} \cdot \mathbf{d})(\mathbf{c} \cdot \mathbf{e})\mathbf{a} \otimes \mathbf{f}, \quad \text{etc.}$$

Thus, we have that

$$(\mathbf{a} \otimes \mathbf{b}) : \mathbf{1} = \mathbf{a} \cdot \mathbf{b}, \quad (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) : \mathbf{1} = (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a}, \quad \text{etc.}$$

In fact, in addition to (24), Equation (33) constitutes an additional constraint for $\mathbf{B}$, which plays the same role as the constraint $|\det \mathbf{P}| = 1$.

As $\det \mathbf{P} = \pm 1$, its second gradient is also zero. Using (33) we obtain

$$\nabla_{\varkappa}\nabla_{\varkappa} \det \mathbf{P} = \det \mathbf{P}[\mathbf{W} : \mathbf{1} - \mathbf{B} : \mathbf{B}^{T(1,3)}]. \tag{34}$$

Thus, we have the additional constraint for $\mathbf{W}$ and $\mathbf{B}$

$$\mathbf{W} : \mathbf{1} = \mathbf{B} : \mathbf{B}^{T(1,3)}. \tag{35}$$

Note that $\mathbf{W} : \mathbf{1}$ and $\mathbf{B} : \mathbf{B}^{T(1,3)}$ are both symmetric second-order tensors.

As we consider $\mathbf{P}$, $\nabla_{\varkappa}\mathbf{P}$, and $\nabla_{\varkappa}\nabla_{\varkappa}\mathbf{P}$ (or $\mathbf{P}$, $\mathbf{B}$ and $\mathbf{W}$) to be determined in a point, these tensors can be regarded as mutually independent, which means that they do not have to fulfill any additional integrability conditions.

In the following, we use the following groups: the group of unimodular tensors Unim, i.e., the group with respect to multiplication which contains second-order tensors $\mathbf{P}$ such $\det \mathbf{P} = \pm 1$, and the groups Lin_3 and Lin_4 with respect of addition which consist of third- and fourth-order tensors, respectively.

Using (15), (22), and (31), we transform (11)–(13) into the following definition of the local material symmetry groups for considered classes of materials.

Definition 2.1 (Local material symmetry group). *We call the local material symmetry group at x the following sets of tensors:*

- *for simple materials a set of second-order tensors*

$$\mathcal{G}^{(0)}_{\varkappa} = \{\mathbf{P} : \mathbf{P} \in \mathit{Unim}\}$$

such that

$$\mathcal{W}_0(\mathbf{C}) = \mathcal{W}_0(\mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^{T});$$

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- *for first-order strain gradient materials a set of ordered couples*

$$\mathcal{G}_\varkappa^{(1)} = \{\mathbb{X} = (\mathbf{P}, \mathbf{B}) : \mathbf{P} \in \mathit{Unim}, \mathbf{B} \in \mathit{Lin}_3, \mathbf{B}^{\mathrm{T}(1,2)} = \mathbf{B}, \mathbf{B} : \mathbf{1} = \mathbf{0}\}$$

such that

$$\mathcal{W}_1(\mathbf{C}, \mathbf{K}_1) = \mathcal{W}_1(\mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^{\mathrm{T}}, \mathbf{P} * \mathbf{K}_1 + \mathbf{B} \cdot \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^{\mathrm{T}});$$

- *for second-order strain gradient materials a set of ordered triples*

$$\begin{aligned}\mathcal{G}_\varkappa^{(2)} =\{\mathbb{Z} = (\mathbf{P}, \mathbf{B}, \mathbf{W}) : \mathbf{P} \in \mathit{Unim},& \\ \mathbf{B} \in \mathit{Lin}_3, \quad \mathbf{B}^{\mathrm{T}(1,2)} = \mathbf{B}, \quad \mathbf{B} : \mathbf{1} = \mathbf{0},& \\ \mathbf{W} \in \mathit{Lin}_4, \quad \mathbf{W}^{\mathrm{T}(1,2)} = \mathbf{W}, \quad \mathbf{W}^{\mathrm{T}(1,3)} = \mathbf{W},& \\ \mathbf{W}^{\mathrm{T}(2,3)} = \mathbf{W}, \quad \mathbf{W} : \mathbf{1} = \mathbf{B} : \mathbf{B}^{\mathrm{T}(1,3)}\}&\end{aligned}$$

such that

$$\begin{aligned}\mathcal{W}_2(\mathbf{C}, \mathbf{K}_1, \mathbf{K}_2) =&\mathcal{W}_2(\mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^{\mathrm{T}}, \mathbf{P} * \mathbf{K}_1 + \mathbf{B} \cdot \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^{\mathrm{T}}, \mathbf{P} * \mathbf{K}_2 + \mathbf{W} \cdot \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^{\mathrm{T}} \\ &+ 3 \operatorname*{sym}_{(1,2,3)} [\mathbf{B} \cdot (\mathbf{P} * \mathbf{K}_1)])\end{aligned}$$

for all $\mathbf{C}$, $\mathbf{K}_1$, $\mathbf{K}_2$ *defined at x in domains of corresponding energy density functions.*

Sets $\mathcal{G}_\varkappa^{(0)}$, $\mathcal{G}_\varkappa^{(1)}$, *and* $\mathcal{G}_\varkappa^{(2)}$ *constitute groups with respect to the following group operations:*

- *the multiplication as the group operation for* $\mathcal{G}_\varkappa^{(0)}$, *i.e., if* $\mathbf{P}_1, \mathbf{P}_2 \in \mathcal{G}_\varkappa^{(0)}$, *then* $\mathbf{P} = \mathbf{P}_1 \cdot \mathbf{P}_2$ *belongs to* $\mathcal{G}_\varkappa^{(0)}$*;*
- *for two elements* $\mathbb{X}_1 = (\mathbf{P}_1, \mathbf{B}_1)$ *and* $\mathbb{X}_2 = (\mathbf{P}_2, \mathbf{B}_2)$ *in* $\mathcal{G}_\varkappa^{(1)}$ *the group operation results in*

$$\mathbb{X}_1 \circ \mathbb{X}_2 = \left(\mathbf{P}_1 \cdot \mathbf{P}_2, \mathbf{B}_1 + \mathbf{P}_1 * (\mathbf{B}_2 \cdot \mathbf{P}_1^{-1} \cdot \mathbf{P}_1^{-\mathrm{T}})\right);$$

- *for two elements* $\mathbb{Z}_1 = (\mathbf{P}_1, \mathbf{B}_1, \mathbf{W}_1)$ *and* $\mathbb{Z}_2 = (\mathbf{P}_2, \mathbf{B}_2, \mathbf{W}_2)$ *in* $\mathcal{G}_\varkappa^{(2)}$ *the group operation results in*

$$\begin{aligned}\mathbb{Z}_1 \circ \mathbb{Z}_2 = \quad & \mathbf{P}_1 \cdot \mathbf{P}_2, \mathbf{B}_1 + \mathbf{P}_1 * (\mathbf{B}_2 \cdot \mathbf{P}_1^{-1} \cdot \mathbf{P}_1^{-\mathrm{T}}), \\ & \mathbf{W}_1 + \mathbf{P}_1 * (\mathbf{W}_2 \cdot \mathbf{P}_1^{-1} \cdot \mathbf{P}_1^{-\mathrm{T}}) \\ & +3 \operatorname*{sym}_{(1,2,3)} \left[\mathbf{B}_1 \cdot \left(\mathbf{P}_1 * (\mathbf{B}_2 \cdot \mathbf{P}_1^{-1} \cdot \mathbf{P}_1^{-\mathrm{T}})\right)\right]\Bigg).\end{aligned}$$

Thus, $\mathcal{G}_\varkappa^{(0)}$ is a subgroup of the full unimodular group Unim.

Let us prove the property of new group operations. Let $\mathbb{X}_1 = (\mathbf{P}_1, \mathbf{B}_1)$ and $\mathbb{X}_2 = (\mathbf{P}_2, \mathbf{B}_2)$ be two arbitrary elements of $\mathcal{G}_\varkappa^{(1)}$. So the following relations are fulfilled

$$\begin{aligned}\mathcal{W}_1(\mathbf{C}, \mathbf{K}_1) =&\mathcal{W}_1(\mathbf{P}_1 \cdot \mathbf{C} \cdot \mathbf{P}_1^{\mathrm{T}}, \mathbf{P}_1 * \mathbf{K}_1 + \mathbf{B}_1 \cdot \mathbf{P}_1 \cdot \mathbf{C} \cdot \mathbf{P}_1^{\mathrm{T}}) \\ =&\mathcal{W}_1(\mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}}, \mathbf{P}_2 * \mathbf{K}_1 + \mathbf{B}_2 \cdot \mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}})\end{aligned}$$

for all $\mathbf{C}$ and $\mathbf{K}_1$ in the domain of $\mathcal{W}_1$. By definition $\mathbb{X} = \mathbb{X}_1 \circ \mathbb{X}_2$ is given by

$$\mathbb{X} \equiv (\mathbf{P}, \mathbf{B}) = (\mathbf{P}_1 \cdot \mathbf{P}_2, \mathbf{B}_1 + \mathbf{P}_1 * (\mathbf{B}_2 \cdot \mathbf{P}_1^{-1} \cdot \mathbf{P}_1^{-\mathrm{T}})).$$

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Let us check that $\mathbb{X}$ belongs to $\mathcal{G}_{\varkappa}^{(1)}$. We have the identities

$$
\begin{aligned}
&\mathcal{W}_1(\mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}^{\mathrm{T}}, \mathbf{P}*\mathbf{K}_1+\mathbf{B}\cdot\mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}^{\mathrm{T}})\\
&=\mathcal{W}_1\big[\mathbf{P}_1\cdot\mathbf{P}_2\cdot\mathbf{C}\cdot\mathbf{P}_2^{\mathrm{T}}\cdot\mathbf{P}_1^{\mathrm{T}},\ (\mathbf{P}_1\cdot\mathbf{P}_2)*\mathbf{K}_1\\
&\qquad+(\mathbf{B}_1+\mathbf{P}_1*(\mathbf{B}_2\cdot\mathbf{P}_1^{-1}\cdot\mathbf{P}_1^{-\mathrm{T}}))\cdot\mathbf{P}_1\cdot\mathbf{P}_2\cdot\mathbf{C}\cdot\mathbf{P}_2^{\mathrm{T}}\cdot\mathbf{P}_1^{\mathrm{T}}\big]\\
&=\mathcal{W}_1\big[\mathbf{P}_1\cdot(\mathbf{P}_2\cdot\mathbf{C}\cdot\mathbf{P}_2^{\mathrm{T}})\cdot\mathbf{P}_1^{\mathrm{T}},\mathbf{P}_1*(\mathbf{P}_2*\mathbf{K}_1)\\
&\qquad+\mathbf{B}_1\cdot\mathbf{P}_1\cdot(\mathbf{P}_2\cdot\mathbf{C}\cdot\mathbf{P}_2^{\mathrm{T}})\cdot\mathbf{P}_1^{\mathrm{T}}\\
&\qquad+(\mathbf{P}_1*(\mathbf{B}_2\cdot\mathbf{P}_1^{-1}\cdot\mathbf{P}_1^{-\mathrm{T}}))\cdot\mathbf{P}_1\cdot\mathbf{P}_2\cdot\mathbf{C}\cdot\mathbf{P}_2^{\mathrm{T}}\cdot\mathbf{P}_1^{\mathrm{T}}\big]\\
&=\mathcal{W}_1\big[\mathbf{P}_1\cdot(\mathbf{P}_2\cdot\mathbf{C}\cdot\mathbf{P}_2^{\mathrm{T}})\cdot\mathbf{P}_1^{\mathrm{T}},\\
&\qquad\mathbf{P}_1*[\mathbf{P}_2*\mathbf{K}_1+\mathbf{B}_2\cdot\mathbf{P}_2\cdot\mathbf{C}\cdot\mathbf{P}_2^{\mathrm{T}}]+\mathbf{B}_1\cdot\mathbf{P}_1\cdot(\mathbf{P}_2\cdot\mathbf{C}\cdot\mathbf{P}_2^{\mathrm{T}})\cdot\mathbf{P}_1^{\mathrm{T}}\big]\\
&=\mathcal{W}_1(\mathbf{P}_2\cdot\mathbf{C}\cdot\mathbf{P}_2^{\mathrm{T}},\mathbf{P}_2*\mathbf{K}_1+\mathbf{B}_2\cdot\mathbf{P}_2\cdot\mathbf{C}\cdot\mathbf{P}_2^{\mathrm{T}})\\
&=\mathcal{W}_1(\mathbf{C},\mathbf{K}_1).
\end{aligned}
\tag{36}
$$

Moreover, $\mathbf{B}$ has the same symmetry as $\mathbf{B}_1$ and $\mathbf{B}_2$, and $\mathbf{B}:\mathbf{1}=\mathbf{0}$. Thus, $\mathbb{X}=\mathbb{X}_1\circ\mathbb{X}_2\in\mathcal{G}_{\varkappa}^{(1)}$. Note, that here we used the identities (20) and

$$
(\mathbf{P}*\mathbf{B})\cdot\mathbf{P}^{-\mathrm{T}}\cdot\mathbf{C}\cdot\mathbf{P}^{\mathrm{T}}=\mathbf{P}*(\mathbf{B}\cdot\mathbf{C}).
\tag{37}
$$

Similarly, we prove that if $\mathbb{Z}_1$ and $\mathbb{Z}_2\in\mathcal{G}_{\varkappa}^{(2)}$, then $\mathbb{Z}=\mathbb{Z}_1\circ\mathbb{Z}_2\in\mathcal{G}_{\varkappa}^{(2)}$. First, we have

$$
\begin{aligned}
\mathcal{W}_2(\mathbf{C},\mathbf{K}_1,\mathbf{K}_2)=&\mathcal{W}_2(\mathbf{P}_1\cdot\mathbf{C}\cdot\mathbf{P}_1^{\mathrm{T}},\mathbf{P}_1*\mathbf{K}_1+\mathbf{B}_1\cdot\mathbf{P}_1\cdot\mathbf{C}\cdot\mathbf{P}_1^{\mathrm{T}},\\
&\mathbf{P}_1*\mathbf{K}_2+\mathbf{W}_1\cdot\mathbf{P}_1\cdot\mathbf{C}\cdot\mathbf{P}_1^{\mathrm{T}}+3\operatorname*{sym}_{(1,2,3)}[\mathbf{B}_1\cdot(\mathbf{P}_1*\mathbf{K}_1)])\\
=&\mathcal{W}_2(\mathbf{P}_2\cdot\mathbf{C}\cdot\mathbf{P}_2^{\mathrm{T}},\mathbf{P}_2*\mathbf{K}_1+\mathbf{B}_2\cdot\mathbf{P}_2\cdot\mathbf{C}\cdot\mathbf{P}_2^{\mathrm{T}},\\
&\mathbf{P}_2*\mathbf{K}_2+\mathbf{W}_2\cdot\mathbf{P}_2\cdot\mathbf{C}\cdot\mathbf{P}_2^{\mathrm{T}}+3\operatorname*{sym}_{(1,2,3)}[\mathbf{B}_2\cdot(\mathbf{P}_2*\mathbf{K}_1)]).
\end{aligned}
$$

For

$$
\begin{aligned}
\mathbb{Z}\equiv\mathbb{Z}_1\circ\mathbb{Z}_2=(\mathbf{P},\mathbf{B},\mathbf{W})=\quad&\mathbf{P}_1\cdot\mathbf{P}_2,\mathbf{B}_1+\mathbf{P}_1*(\mathbf{B}_2\cdot\mathbf{P}^{-1}\cdot\mathbf{P}_1^{-\mathrm{T}}),\\
&\mathbf{W}_1+\mathbf{P}_1*(\mathbf{W}_2\cdot\mathbf{P}_1^{-1}\cdot\mathbf{P}_1^{-\mathrm{T}})+3\operatorname*{sym}_{(1,2,3)}\big[\mathbf{B}_1\cdot\big(\mathbf{P}_1*(\mathbf{B}_2\cdot\mathbf{P}_1^{-1}\cdot\mathbf{P}_1^{-\mathrm{T}})\big)\big]\Big)
\end{aligned}
$$

we obtain the identities

$$
\begin{aligned}
&\mathcal{W}_2\Big(\mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}^{\mathrm{T}},\mathbf{P}*\mathbf{K}_1+\mathbf{B}\cdot\mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}^{\mathrm{T}},\\
&\quad\mathbf{P}*\mathbf{K}_2+\mathbf{W}\cdot\mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}^{\mathrm{T}}+3\operatorname*{sym}_{(1,2,3)}[\mathbf{B}\cdot(\mathbf{P}*\mathbf{K}_1)]\Big)\\
&=\mathcal{W}_2\Big[\mathbf{P}_1\cdot\mathbf{P}_2\cdot\mathbf{C}\cdot\mathbf{P}_2^{\mathrm{T}}\cdot\mathbf{P}_1^{\mathrm{T}},(\mathbf{P}_1\cdot\mathbf{P}_2)*\mathbf{K}_1\\
&\quad+(\mathbf{B}_1+\mathbf{P}_1*(\mathbf{B}_2\cdot\mathbf{P}_1^{-1}\cdot\mathbf{P}_1^{-\mathrm{T}}))\cdot\mathbf{P}_1\cdot\mathbf{P}_2\cdot\mathbf{C}\cdot\mathbf{P}_2^{\mathrm{T}}\cdot\mathbf{P}_1^{\mathrm{T}},\\
&\quad(\mathbf{P}_1\cdot\mathbf{P}_2)*\mathbf{K}_1\\
&\quad+\big[\mathbf{W}_1+\mathbf{P}_1*(\mathbf{W}_2\cdot\mathbf{P}_1^{-1}\cdot\mathbf{P}_1^{-\mathrm{T}})\\
&\quad+3\operatorname*{sym}_{(1,2,3)}\big[\mathbf{B}_1\cdot\big(\mathbf{P}_1*(\mathbf{B}_2\cdot\mathbf{P}_1^{-1}\cdot\mathbf{P}_1^{-\mathrm{T}})\big)\big]\big]\cdot\mathbf{P}_1\cdot\mathbf{P}_2\cdot\mathbf{C}\cdot\mathbf{P}_2^{\mathrm{T}}\cdot\mathbf{P}_1^{\mathrm{T}}\\
&\quad+3\operatorname*{sym}_{(1,2,3)}\big[(\mathbf{B}_1+\mathbf{P}_1*(\mathbf{B}_2\cdot\mathbf{P}_1^{-1}\cdot\mathbf{P}_1^{-\mathrm{T}}))\cdot((\mathbf{P}_1\cdot\mathbf{P}_2)*\mathbf{K}_1)\big]\Big].
\end{aligned}
$$

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Using the same transformations of $\mathbf{C}$ and $\mathbf{K}_1$ as for $\mathcal{W}_1$, (30), and the identities

$$\mathbf{P}_1 * (\mathbf{W}_2 \cdot \mathbf{P}_1^{-1} \cdot \mathbf{P}_1^{-\mathrm{T}}) \cdot \mathbf{P}_1 \cdot \mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}} \cdot \mathbf{P}_1^{\mathrm{T}} = \mathbf{P}_1 * (\mathbf{W}_2 \cdot \mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}}),$$
$$\mathbf{P}_1 * (\mathbf{B}_2 \cdot \mathbf{P}_1^{-1} \cdot \mathbf{P}_1^{-\mathrm{T}}) \cdot ((\mathbf{P}_1 \cdot \mathbf{P}_2) * \mathbf{K}_1) = \mathbf{P}_1 * (\mathbf{B}_2 \cdot (\mathbf{P}_2 * \mathbf{K}_1))$$

we come to

$$\begin{aligned}
&\mathcal{W}_2\Big[\mathbf{P}_1 \cdot (\mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}}) \cdot \mathbf{P}_1^{\mathrm{T}},\\
&\quad \mathbf{P}_1 * [\mathbf{P}_2 * \mathbf{K}_1 + \mathbf{B}_2 \cdot \mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}}] + \mathbf{B}_1 \cdot \mathbf{P}_1 \cdot (\mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}}) \cdot \mathbf{P}_1^{\mathrm{T}},\\
&\quad \mathbf{P}_1 * \big[\mathbf{P}_2 * \mathbf{K}_2 + \mathbf{W}_2 \cdot \mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}} + 3 \operatorname*{sym}_{(1,2,3)} [\mathbf{B}_2 \cdot (\mathbf{P}_2 * \mathbf{K}_1)]\big]\\
&\quad + \mathbf{W}_1 \cdot \mathbf{P}_1 \cdot (\mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}}) \cdot \mathbf{P}_1^{\mathrm{T}} + 3 \operatorname*{sym}_{(1,2,3)} \big[\mathbf{B}_1 \cdot \big(\mathbf{P}_1 * (\mathbf{P}_2 * \mathbf{K}_1)\big)\big]\\
&\quad + 3 \operatorname*{sym}_{(1,2,3)} \big[\mathbf{B}_1 \cdot \big(\mathbf{P}_1 * (\mathbf{B}_2 \cdot \mathbf{P}_1^{-1} \cdot \mathbf{P}_1^{-\mathrm{T}})\big)\big]\big] \cdot \mathbf{P}_1 \cdot (\mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}}) \cdot \mathbf{P}_1^{\mathrm{T}}\Big]\\[1em]
&= \mathcal{W}_2\Big[\mathbf{P}_1 \cdot (\mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}}) \cdot \mathbf{P}_1^{\mathrm{T}},\\
&\quad \mathbf{P}_1 * [\mathbf{P}_2 * \mathbf{K}_1 + \mathbf{B}_2 \cdot \mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}}] + \mathbf{B}_1 \cdot \mathbf{P}_1 \cdot (\mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}}) \cdot \mathbf{P}_1^{\mathrm{T}},\\
&\quad \mathbf{P}_1 * \big[\mathbf{P}_2 * \mathbf{K}_2 + \mathbf{W}_2 \cdot \mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}} + 3 \operatorname*{sym}_{(1,2,3)} [\mathbf{B}_2 \cdot (\mathbf{P}_2 * \mathbf{K}_1)]\big]\\
&\quad + \mathbf{W}_1 \cdot \mathbf{P}_1 \cdot (\mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}}) \cdot \mathbf{P}_1^{\mathrm{T}}\\
&\quad + 3 \operatorname*{sym}_{(1,2,3)} \big[\mathbf{B}_1 \cdot \big(\mathbf{P}_1 * (\mathbf{P}_2 * \mathbf{K}_1 + \mathbf{B}_2 \cdot (\mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}}))\big)\big]\Big]\\
&= \mathcal{W}_2\big(\mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}}, \mathbf{P}_2 * \mathbf{K}_1 + \mathbf{B}_2 \cdot \mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}},\\
&\quad \mathbf{P}_2 * \mathbf{K}_2 + \mathbf{W}_2 \cdot \mathbf{P}_2 \cdot \mathbf{C} \cdot \mathbf{P}_2^{\mathrm{T}} + 3 \operatorname*{sym}_{(1,2,3)} [\mathbf{B}_2 \cdot (\mathbf{P}_2 * \mathbf{K}_1)]\big)\\
&= \mathcal{W}_2(\mathbf{C}, \mathbf{K}_1, \mathbf{K}_2).
\end{aligned}$$

Finally, let us note that $\mathbf{W}$ has the same symmetry as $\mathbf{W}_1$ and $\mathbf{W}_2$. Moreover, we can prove that $\mathbf{W} : \mathbf{1} = \mathbf{B} : \mathbf{B}^{\mathrm{T}(1,3}$. To this end, we have used the following formulas

$$3 \operatorname*{sym}_{(1,2,3)} \left(\mathbf{B}^2\right) : \mathbf{1} = 2\mathbf{B} : \mathbf{B}^{\mathrm{T}(1,3}, \tag{38}$$

$$3 \operatorname*{sym}_{(1,2,3)} (\mathbf{B}_1 \cdot \mathbf{B}_2) : \mathbf{1} = \mathbf{B}_1 : \mathbf{B}_2^{\mathrm{T}(1,3} + \mathbf{B}_2 : \mathbf{B}_1^{\mathrm{T}(1,3} \tag{39}$$

for third-order tensors $\mathbf{B}$, $\mathbf{B}_1$, and $\mathbf{B}_2$ such that $\mathbf{B} : \mathbf{1} = \mathbf{B}_1 : \mathbf{1} = \mathbf{B}_2 : \mathbf{1} = \mathbf{0}$, which can be proved through straightforward calculations. Thus, $\mathbb{Z} = \mathbb{Z}_1 \circ \mathbb{Z}_2 \in \mathcal{G}_\varkappa^{(2)}$.

The unit elements of $\mathcal{G}_\varkappa^{(0)}$, $\mathcal{G}_\varkappa^{(1)}$ and $\mathcal{G}_\varkappa^{(2)}$ are $\mathbf{1}$, $\mathbb{I} \equiv (\mathbf{1}, \mathbf{0})$ and $\mathbb{I}^{(2)} \equiv (\mathbf{1}, \mathbf{0}, \mathbf{0})$, respectively. Inverse elements of these groups are

$$\begin{aligned}
\mathbf{P} \in \mathcal{G}_\varkappa^{(0)} :&\quad \mathbf{P}^{-1};\\
\mathbb{X} \equiv (\mathbf{P}, \mathbf{B}) \in \mathcal{G}_\varkappa^{(1)} :&\quad \mathbb{X}^{-1} = (\mathbf{P}^{-1}, -\mathbf{P}^{-1} * (\mathbf{B} \cdot \mathbf{P} \cdot \mathbf{P}^{\mathrm{T}}));\\
\mathbb{Z} \equiv (\mathbf{P}, \mathbf{B}, \mathbf{W}) \in \mathcal{G}_\varkappa^{(2)} :&\\
&\mathbb{Z}^{-1} = \big(\mathbf{P}^{-1}, -\mathbf{P}^{-1} * (\mathbf{B} \cdot \mathbf{P} \cdot \mathbf{P}^{\mathrm{T}}), -\mathbf{P}^{-1} * (\mathbf{W} \cdot \mathbf{P} \cdot \mathbf{P}^{\mathrm{T}})\\
&\quad -3 \operatorname*{sym}_{(1,2,3)} \big(\mathbf{P}^{-1} * (\mathbf{B} \cdot \mathbf{P} \cdot \mathbf{P}^{\mathrm{T}}) \cdot \mathbf{P}^{-1} * (\mathbf{B} \cdot \mathbf{P} \cdot \mathbf{P}^{\mathrm{T}})\big)\big).
\end{aligned}$$

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For example, $\mathbb{X}^{-1}$ satisfies to (24) and (33), and

$$\begin{aligned}\mathbb{X}^{-1}\circ\mathbb{X} =&(\mathbf{P}^{-1}\cdot\mathbf{P}, -\mathbf{P}^{-1}*(\mathbf{B}\cdot\mathbf{P}\cdot\mathbf{P}^{\mathrm{T}})+\mathbf{P}^{-1}*(\mathbf{B}\cdot\mathbf{P}\cdot\mathbf{P}^{\mathrm{T}}))\\ =&(\mathbf{1},\mathbf{0}),\\ \mathbb{X}\circ\mathbb{X}^{-1} =&\left(\mathbf{P}\cdot\mathbf{P}^{-1},\mathbf{B}-\mathbf{P}*\left(\mathbf{P}^{-1}*(\mathbf{B}\cdot\mathbf{P}\cdot\mathbf{P}^{\mathrm{T}})\cdot\mathbf{P}^{-1}\cdot\mathbf{P}^{-\mathrm{T}}\right)\right)\\ =&(\mathbf{1},\mathbf{0}).\end{aligned}$$

Similarly, an element $\mathbb{Z}^{-1}$ satisfies to $\mathbb{Z}^{-1}\circ\mathbb{Z}=\mathbb{Z}\circ\mathbb{Z}^{-1}=(\mathbf{1},\mathbf{0},\mathbf{0})$ and to (32) and (35).

Let us note that this definition introduces the symmetry group locally, i.e., for a considered point x. Thus for non-homogeneous materials the material symmetry maybe different in different points. Moreover, the symmetry groups depend also on the choice of a reference placement, in general.

Our definition of the local material symmetry group is slightly different from that given in [23] as here we used another strain measures. Moreover, in order to keep constraints (24) and (33), (32) and (35) we change the group operation, cf. [23, equation (103)]. Note if we consider instead of Unim a subgroup of the full orthogonal group Orth, the group operation transforms into more simple form

$$\mathbb{X}_1\circ\mathbb{X}_2=(\mathbf{P}_1\cdot\mathbf{P}_2,\mathbf{B}_1+\mathbf{P}_1*\mathbf{B}_2),$$

which is similar to that used in [23].

In the following, we consider a particular class of materials called elastic fluids. We define an elastic fluid as a hyperelastic medium whose symmetry group contains all admissible elements, i.e., the corresponding symmetry group is a maximal. As a result, it does not depend on the choice of reference placement. For example, for a simple fluid the symmetry group coincides with the full unimodular group: $\mathcal{G}^{(0)}_{\varkappa}=\mathsf{Unim}$ [10]. Thus, we formulate

Definition 2.2 (Strain gradient fluids). *An elastic strain gradient material is called a* strain gradient fluid *at x if there exists a reference placement $\varkappa$, called undistorted, such that the material symmetry group is maximal.*

In other words, here $\mathcal{G}^{(0)}_{\varkappa}=\mathsf{Unim}$, $\mathcal{G}^{(1)}_{\varkappa}=\mathcal{U}^{(1)}$, and $\mathcal{G}^{(2)}_{\varkappa}=\mathcal{U}^{(2)}$, where $\mathcal{U}^{(1)}$ and $\mathcal{U}^{(2)}$ contains *all* elements of Unim, Lin_3, and Lin_4 with required symmetries and constraints.

In order to clarify the connections between symmetry groups and complex fluids let us first recall the constitutive relations of elastic first- and second-order strain gradient fluids which were introduced independently within so-called direct approach.

3. First- and second-order strain gradient fluids

Let us briefly recall the constitutive equations of strain gradient fluids. Let $\mathcal{U}$ be a strain energy density function defined per unit mass in a current placement χ and ρ be a mass density in χ. For the classic Euler fluid the constitutive equation has the form [62, 70]

$$\mathcal{U}=\mathcal{U}_0(\rho), \tag{40}$$

whereas for first- and second-order strain gradient fluids we have the following constitutive dependencies [45, 62, 72]

$$\mathcal{U}=\mathcal{U}_1(\rho,\nabla\rho), \tag{41}$$
$$\mathcal{U}=\mathcal{U}_2(\rho,\nabla\rho,\nabla\nabla\rho), \tag{42}$$

respectively. Applying the principle of material frame indifference to (41) and (42) we obtain the following invariance properties

$$\mathcal{U}_1(\rho,\nabla\rho)=\mathcal{U}_1(\rho,\mathbf{Q}\cdot\nabla\rho), \tag{43}$$
$$\mathcal{U}_2(\rho,\nabla\rho,\nabla\nabla\rho)=\mathcal{U}_2(\rho,\mathbf{Q}\cdot\nabla\rho,\mathbf{Q}\cdot\nabla\nabla\rho\cdot\mathbf{Q}^{\mathrm{T}})\quad\forall\ \mathbf{Q}:\quad\mathbf{Q}^{\mathrm{T}}=\mathbf{Q}^{-1}. \tag{44}$$

Equations (43) and (44) state that $\mathcal{U}_1$ and $\mathcal{U}_2$ are isotropic functions of their arguments arguments [72, 74, 75]. Using the theory of invariants [74, 75] we obtain the following representations

$$\mathcal{U}_1=\mathcal{U}_1(\rho,j_0), \tag{45}$$
$$\mathcal{U}_2=\mathcal{U}_2(\rho,j_0,j_1,j_2,j_3,j_4,j_5), \tag{46}$$

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where we introduce the following invariants

$$
\begin{aligned}
j_0 =& \nabla\rho\cdot\nabla\rho, \quad j_1 = \operatorname{tr}\mathbf{D} = \nabla\cdot\nabla\rho, \quad j_2 = \operatorname{tr}\mathbf{D}^2, \quad j_3 = \operatorname{tr}\mathbf{D}^3,\\
j_4 =& \nabla\rho\cdot\mathbf{D}\cdot\nabla\rho, \quad j_5 = \nabla\rho\cdot\mathbf{D}^2\cdot\nabla\rho,
\end{aligned}
\tag{47}
$$

and $\mathbf{D} = \nabla\nabla\rho$ is a symmetric second-order tensor.

Obviously, as $\mathcal{U}_0$, $\mathcal{U}_1$, and $\mathcal{U}_2$ do not depend on any reference placement, these constitutive relations give examples of fluids defined from the point of view of symmetry group. Let us discuss this matter in more detail.

4. Strain gradient fluids from the point of view of the symmetry group

For the simple materials, the correspondence between the elastic fluid and the symmetry group is straightforward and could be found in many textbooks on continuum mechanics, see, e.g., [10, 70, 72]. Indeed, as $\mathcal{G}^{(0)}_{\varkappa} = \mathsf{Unim}$, from Definition 2.1 we obtain

$$
\mathcal{W}_0(\mathbf{C}) = \mathcal{W}_0(\mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}^{\mathrm{T}}) \quad \forall\mathbf{P}\in\mathsf{Unim}. \tag{48}
$$

Substituting into (48) $\mathbf{P} = (\det\mathbf{F})\mathbf{F}^{-1}$ we obtain

$$
\mathcal{W}_0(\mathbf{C}) = \mathcal{W}_0((\det\mathbf{F})^2\mathbf{F}^{-1}\cdot\mathbf{C}\cdot\mathbf{F}^{-\mathrm{T}}) = \mathcal{W}_0((\det\mathbf{F})^2\mathbf{1}) = \mathcal{W}_0(\det\mathbf{F}).
$$

Thus, $\mathcal{W}_0$ depends only on $\det\mathbf{F}$ and here we again keep the same notation for the constitutive function. As $\det\mathbf{F}$ relates mass densities in $\varkappa$ and χ as

$$
|\det\mathbf{F}|\rho = \rho_{\varkappa},
$$

we can replace it through ρ. Thus, we came to an elastic simple fluid with the constitutive relation in the form

$$
\mathcal{W}_0 = \mathcal{W}_0(\rho).
$$

As $\mathcal{W}_0$ was defined per unit volume in the reference placement whereas $\mathcal{U}_0$ was defined per unit mass in the current placement, these functions can be related as follows

$$
\mathcal{U}_0 = \frac{\rho_{\varkappa}}{\rho^2}\mathcal{W}_0.
$$

Let us apply a similar approach to non-trivial cases of strain gradient fluids.

4.1. First-order strain gradient fluid

According to Definitions 2.1 and 2.2 we have the following invariance property

$$
\mathcal{W}_1(\mathbf{C},\mathbf{K}_1) = \mathcal{W}_1(\mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}^{\mathrm{T}}, \mathbf{P}*\mathbf{K}_1 + \mathbf{B}\cdot\mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}^{\mathrm{T}}) \tag{49}
$$

for *all* $\mathbf{P}\in\mathsf{Unim}$ and for *all* $\mathbf{B}\in\mathsf{Lin}_3$ such that $\mathbf{B} = \mathbf{B}^{\mathrm{T}(1,2)}$ and $\mathbf{B}:\mathbf{1} = \mathbf{0}$. As in the case of simple fluids, substituting $\mathbf{P} = (\det\mathbf{F})\mathbf{F}^{-1}$ into (49) we obtain

$$
\begin{gathered}
\mathcal{W}_1(\mathbf{C},\mathbf{K}_1) = \mathcal{W}_1\left((\det\mathbf{F})^2\mathbf{1}, (\det\mathbf{F})^3\mathbf{F}^{-1}*\mathbf{K}_1 + (\det\mathbf{F})^2\mathbf{B}\right)\\
\forall\mathbf{B}: \quad \mathbf{B} = \mathbf{B}^{\mathrm{T}(1,2)}, \quad \mathbf{B}:\mathbf{1} = \mathbf{0}.
\end{gathered}
\tag{50}
$$

Using Lemma 5.2 we can reduce (51) as follows

$$
\mathcal{W}_1(\mathbf{C},\mathbf{K}_1) = \mathcal{W}_1\left((\det\mathbf{F})^2\mathbf{1}, (\det\mathbf{F})^3(\mathbf{F}^{-1}*\mathbf{K}_1):\mathbf{1}\right). \tag{51}
$$

Keeping, for simplicity, the same notation for the strain energy density, we came to the dependence

$$
\mathcal{W}_1(\mathbf{C},\mathbf{K}_1) = \mathcal{W}_1\left(\det\mathbf{F}, (\mathbf{F}^{-1}*\mathbf{K}_1):\mathbf{1}\right). \tag{52}
$$

Similarly (33) we derive the formula for $\nabla\det\mathbf{F}$:

$$
\nabla\det\mathbf{F} = (\nabla\mathbf{F}\cdot\mathbf{F}^{-1}):\mathbf{1}. \tag{53}
$$

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Using (53) we can prove that

$$(\mathbf{F}^{-1} * \mathbf{K}_1) : \mathbf{1} = (\nabla \mathbf{F} \cdot \mathbf{F}^{-1}) : \mathbf{1} \equiv \nabla \det \mathbf{F}.$$

As a result, we see that

$$\mathcal{W}_1(\mathbf{C}, \mathbf{K}_1) = \mathcal{W}_1(\det \mathbf{F}, \nabla \det \mathbf{F}), \tag{54}$$

which can be transformed into dependence on ρ and $\nabla \rho$.

Thus, we have proven that the constitutive relation of the first-order strain gradient fluid (41) follows from Definition 2.2.

Let us note that the constraint $\mathbf{B} : \mathbf{1} = \mathbf{0}$ plays here a crucial role. Indeed, if we restrict ourselves to arbitrary symmetric tensors $\mathbf{B}$ we come to an elastic simple fluid with the constitutive relation $\mathcal{W}_1 = \mathcal{W}_1(\det \mathbf{F})$. In other words, a less-restrictive symmetry group definition results in the trivial result.

4.2. Second-order strain gradient fluid

Here we have the invariance

$$\begin{aligned} \mathcal{W}_2(\mathbf{C}, \mathbf{K}_1, \mathbf{K}_2) =& \mathcal{W}_2(\mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^{\mathrm{T}}, \mathbf{P} * \mathbf{K}_1 + \mathbf{B} \cdot \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^{\mathrm{T}}, \mathbf{P} * \mathbf{K}_2 \\ & + \mathbf{W} \cdot \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^{\mathrm{T}} + 3 \operatorname*{sym}_{(1,2,3)} [\mathbf{B} \cdot (\mathbf{P} * \mathbf{K}_1)]) \end{aligned} \tag{55}$$

for all admissible $\mathbf{P}$, $\mathbf{B}$ and $\mathbf{W}$. Substituting again $\mathbf{P} = (\det \mathbf{F})\mathbf{F}^{-1}$ into (55) we obtain

$$\begin{aligned} \mathcal{W}_2(\mathbf{C}, \mathbf{K}_1, \mathbf{K}_2) =& \mathcal{W}_2\Big((\det \mathbf{F})^2 \mathbf{1}, (\det \mathbf{F})^3 \mathbf{F}^{-1} * \mathbf{K}_1 + (\det \mathbf{F})^2 \mathbf{B}, \\ & (\det \mathbf{F})^4 \mathbf{F}^{-1} * \mathbf{K}_2 + (\det \mathbf{F})^2 \mathbf{W} \\ & + 3(\det \mathbf{F})^3 \operatorname*{sym}_{(1,2,3)} \left[\mathbf{B} \cdot (\mathbf{F}^{-1} * \mathbf{K}_1)\right]\Big) \\ \forall \mathbf{B} : \quad & \mathbf{B} = \mathbf{B}^{\mathrm{T}(1,2)}, \quad \mathbf{B} : \mathbf{1} = \mathbf{0}, \\ \forall \mathbf{W} : \quad & \mathbf{W} = \mathbf{W}^{\mathrm{T}(1,2)} = \mathbf{W}^{\mathrm{T}(1,3)} = \mathbf{W}^{\mathrm{T}(2,3)}, \\ & \mathbf{W} : \mathbf{1} = \mathbf{B} : \mathbf{B}^{\mathrm{T}(1,3)}. \end{aligned} \tag{56}$$

Now let us consider $\mathbf{B} = \mathbf{0}$. In this case $\mathbf{W}$ has the property $\mathbf{W} : \mathbf{1} = \mathbf{0}$, so (56) transforms into

$$\begin{aligned} \mathcal{W}_2(\mathbf{C}, \mathbf{K}_1, \mathbf{K}_2) =& \mathcal{W}_2\big((\det \mathbf{F})^2 \mathbf{1}, (\det \mathbf{F})^3 \mathbf{F}^{-1} * \mathbf{K}_1, \\ & (\det \mathbf{F})^4 \mathbf{F}^{-1} * \mathbf{K}_2 + (\det \mathbf{F})^2 \mathbf{W}\big) \\ \forall \mathbf{W} : \quad & \mathbf{W} = \mathbf{W}^{\mathrm{T}(1,2)} = \mathbf{W}^{\mathrm{T}(1,3)} = \mathbf{W}^{\mathrm{T}(2,3)}, \\ & \mathbf{W} : \mathbf{1} = \mathbf{B} : \mathbf{B}^{\mathrm{T}(1,3)}. \end{aligned} \tag{57}$$

This means that we can apply Lemma 5.3. Thus, $\mathcal{W}_2$ depends on $\mathbf{K}_2$ through $\mathbf{K}_2 : \mathbf{1}$ only. Now with (35) we can exclude $\mathbf{W}$ from (56) and obtain the following invariance expressed in terms of $\mathbf{B}$:

$$\begin{aligned} \mathcal{W}_2(\mathbf{C}, \mathbf{K}_1, \mathbf{K}_2) =& \mathcal{W}_2\Big((\det \mathbf{F})^2 \mathbf{1}, (\det \mathbf{F})^3 \mathbf{F}^{-1} * \mathbf{K}_1 + (\det \mathbf{F})^2 \mathbf{B}, \\ & (\det \mathbf{F})^4 (\mathbf{F}^{-1} * \mathbf{K}_2) : \mathbf{1} + (\det \mathbf{F})^2 \mathbf{B} : \mathbf{B}^{\mathrm{T}(1,3)} \\ & + 3(\det \mathbf{F})^3 \operatorname*{sym}_{(1,2,3)} \left[\mathbf{B} \cdot (\mathbf{F}^{-1} * \mathbf{K}_1)\right] : \mathbf{1}\Big) \\ \forall \mathbf{B} : \quad & \mathbf{B} = \mathbf{B}^{\mathrm{T}(1,2)}, \quad \mathbf{B} : \mathbf{1} = \mathbf{0}. \end{aligned} \tag{58}$$

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Using (39) we transform (58) into

$$\mathcal{W}_2 = \mathcal{W}_2\big((\det\mathbf{F})^2\mathbf{1}, (\det\mathbf{F})^3\mathbf{F}^{-1} * \mathbf{K}_1 + (\det\mathbf{F})^2\mathbf{B},$$
$$(\det\mathbf{F})^4(\mathbf{F}^{-1} * \mathbf{K}_2) : \mathbf{1} + (\det\mathbf{F})^2\mathbf{B} : \mathbf{B}^{\mathrm{T}(1,3)}$$
$$+ (\det\mathbf{F})^3\left[\mathbf{B} : (\mathbf{F}^{-1} * \mathbf{K}_1)^{\mathrm{T}(1,3)} + (\mathbf{F}^{-1} * \mathbf{K}_1) : \mathbf{B}^{\mathrm{T}(1,3)}\right]\big). \tag{59}$$

We have the identity

$$(\mathbf{F}^{-1} * \mathbf{K}_2) : \mathbf{1} = (\nabla\nabla\mathbf{F}\cdot\mathbf{F}^{-1}) : \mathbf{1}. \tag{60}$$

Moreover, similarly to (34) we have another identity

$$\nabla\nabla\det\mathbf{F} = \det\mathbf{F}\left[(\nabla\nabla\mathbf{F}\cdot\mathbf{F}^{-1}) : \mathbf{1} - (\nabla\mathbf{F}\cdot\mathbf{F}^{-1}) : (\nabla\mathbf{F}\cdot\mathbf{F}^{-1})^{\mathrm{T}(1,3)}\right].$$

As

$$(\mathbf{F}^{-1} * \mathbf{K}_1) : (\mathbf{F}^{-1} * \mathbf{K}_1)^{\mathrm{T}(1,3)} = (\nabla\mathbf{F}\cdot\mathbf{F}^{-1}) : (\nabla\mathbf{F}\cdot\mathbf{F}^{-1})^{\mathrm{T}(1,3)}$$

we see that

$$\det\mathbf{F}(\mathbf{F}^{-1} * \mathbf{K}_2) : \mathbf{1} = \nabla\nabla\det\mathbf{F} + \det\mathbf{F}(\mathbf{F}^{-1} * \mathbf{K}_1) : (\mathbf{F}^{-1} * \mathbf{K}_1)^{\mathrm{T}(1,3)}.$$

As a result, we transform (59) into

$$\mathcal{W}_2 = \mathcal{W}_2\big((\det\mathbf{F})^2\mathbf{1}, (\det\mathbf{F})^3\mathbf{F}^{-1} * \mathbf{K}_1 + (\det\mathbf{F})^2\mathbf{B}, (\det\mathbf{F})^3\nabla\nabla\det\mathbf{F}$$
$$+ (\det\mathbf{F})^2\left[(\det\mathbf{F})\mathbf{F}^{-1} * \mathbf{K}_1 + \mathbf{B}\right] : \left[(\det\mathbf{F})\mathbf{F}^{-1} * \mathbf{K}_1 + \mathbf{B}\right]^{\mathrm{T}(1,3)}\big). \tag{61}$$

In other words, $\mathcal{W}_2$ has the form

$$\mathcal{W}_2 = \mathcal{W}_2\big(\det\mathbf{F}, (\det\mathbf{F})\mathbf{F}^{-1} * \mathbf{K}_1 + \mathbf{B}, \nabla\nabla\det\mathbf{F}\big), \tag{62}$$

so it can be transformed into a dependence on $\det\mathbf{F}$, $\nabla\det\mathbf{F}$, and $\nabla\nabla\det\mathbf{F}$. This can be transformed into dependence on ρ, $\nabla\rho$, and $\nabla\nabla\rho$. Note that here we used constraint (56). Thus, one can see that constraints followed from the incompressibility condition $\det\mathbf{P} = \pm 1$ play a crucial role for transformation of a general gradient medium into gradient-type fluid.

5. Conclusions

Considering relations between capillary fluids and general strain gradient elasticity models, we have demonstrated that the local material symmetry group constitutes a unified approach to material modeling, i.e., to classification and further of constitutive equations of gradient elastic media. Indeed, defining a gradient fluid as a material insensitive to any mass-density-preserving transformations of a reference placement, we came to the constitutive relations of capillary fluids. In a similar way more complex cases related to strain gradient subfluids can be analyzed. The latter could be useful for the group description of some beam-lattice metamaterials [32, 35, 38]. Let us note that a certain invariance of a strain energy density may also result in non-trivial conservation laws using the technique [5–7]. Such conservation laws may serve as additional identities and as benchmark solutions within the statics of strain gradient elasticity.

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Appendix. Some representations of invariant scalar functions

Here we present few propositions related to a representation of scalar functions of tensorial arguments. We begin from a preliminary proposition.

Lemma 5.1. *Let f be a scalar function of a tensor $\mathbf{X}$ of any order such that*

$$f(\mathbf{X}+\mathbf{B}) = f(\mathbf{X}) \quad \forall \mathbf{B}: \quad \mathbf{B}:\mathbf{1} = \mathbf{0}. \tag{63}$$

Then f has the form

$$f(\mathbf{X}) = f(\mathbf{X}:\mathbf{1}). \tag{64}$$

In other words, Lemma 5.1 states that under condition (63) f depends on $\mathbf{X}$ through $\mathbf{X}:\mathbf{1}$.

Proof. Obviously, if f is given by (64) it satisfies (63). Indeed, in this case we have

$$f(\mathbf{X}) = f(\mathbf{X}+\mathbf{B}) = f((\mathbf{X}+\mathbf{B}):\mathbf{1}) = f(\mathbf{X}:\mathbf{1}).$$

In order to prove the converse we decompose $\mathbf{X}$ as follows

$$\mathbf{X} = \mathbf{X}_{tr} + \mathbf{X}_{dev}, \quad \mathbf{X}_{tr} = \frac{1}{3}(\mathbf{X}:\mathbf{1})\otimes\mathbf{1}, \quad \mathbf{X}_{dev} = \mathbf{X} - \frac{1}{3}(\mathbf{X}:\mathbf{1})\otimes\mathbf{1}. \tag{65}$$

This decomposition is similar to the decomposition of a second-order tensor into a sum of the spherical and deviatoric parts [72]. We have

$$\mathbf{X}_{tr}:\mathbf{1} = \mathbf{X}:\mathbf{1}, \quad \mathbf{X}_{dev}:\mathbf{1} = \mathbf{0}.$$

Using (65) we transform (63) into

$$f(\mathbf{X}) = f(\mathbf{X}+\mathbf{B}) = f(\mathbf{X}_{tr}+\mathbf{X}_{dev}+\mathbf{B}) \quad \forall \mathbf{B}: \quad \mathbf{B}:\mathbf{1} = \mathbf{0}.$$

Taking here $\mathbf{B} = -\mathbf{X}_{dev}$ we come to

$$f(\mathbf{X}) = f(\mathbf{X}_{tr}) = f\left(\frac{1}{3}(\mathbf{X}:\mathbf{1})\right)$$

from that it follows (64). □

Let us now consider a more specific case. Let $\mathbf{X}$ be a third-order tensor symmetric with respect to the first two indices, $\mathbf{X} = \mathbf{X}^{\mathrm{T}(1,2)}$. Unfortunately, the introduced decomposition (65) does not keep the symmetry. Indeed, one can see that

$$\mathbf{X}_{tr} \neq \mathbf{X}_{tr}^{\mathrm{T}(1,2)} \quad \text{and} \quad \mathbf{X}_{dev} \neq \mathbf{X}_{dev}^{\mathrm{T}(1,2)}.$$

In order to keep this symmetry, we introduce the symmetrized decomposition

$$\mathbf{X} = \mathbf{X}^s_{tr} + \mathbf{X}^s_{dev}, \tag{66}$$

$$\mathbf{X}^s_{tr} = \frac{1}{4}\mathbf{X}:\mathbf{1}\otimes\mathbf{1} + \frac{1}{4}\left(\mathbf{X}:\mathbf{1}\otimes\mathbf{1}\right)^{\mathrm{T}(1,2)}, \tag{67}$$

$$\mathbf{X}^s_{dev} = \mathbf{X} - \frac{1}{4}\mathbf{X}:\mathbf{1}\otimes\mathbf{1} - \frac{1}{4}\left(\mathbf{X}:\mathbf{1}\otimes\mathbf{1}\right)^{\mathrm{T}(1,2)}. \tag{68}$$

One can see that we still have identities

$$\mathbf{X}^s_{tr}:\mathbf{1} = \mathbf{X}:\mathbf{1}, \quad \mathbf{X}^s_{dev}:\mathbf{1} = \mathbf{0}.$$

With this decomposition we prove the following result.

Lemma 5.2. *Let f be a scalar function of a third-order tensor $\mathbf{X}$, $\mathbf{X} = \mathbf{X}^{\mathrm{T}(1,2)}$, such that*

$$f(\mathbf{X}+\mathbf{B}) = f(\mathbf{X}) \quad \forall \mathbf{B}: \quad \mathbf{B} = \mathbf{B}^{\mathrm{T}(1,2)}, \quad \mathbf{B}:\mathbf{1} = \mathbf{0}. \tag{69}$$

Then f has the form (64).

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Proof. Again, we see that if f is given by (64) it satisfies to (69). As $\mathbf{B} = \mathbf{B}^s_{dev}$ considering decomposition (69)

$$f(\mathbf{X}) = f(\mathbf{X} + \mathbf{B}) = f(\mathbf{X}^s_{tr} + \mathbf{X}^s_{dev} + \mathbf{B})$$

we can take $\mathbf{B} = -\mathbf{X}^s_{dev}$, so we obtain

$$\begin{aligned} f(\mathbf{X}) =& f(\mathbf{X}^s_{tr}) = f\left(\frac{1}{4}\mathbf{X} : \mathbf{1} \otimes \mathbf{1} + \frac{1}{4}(\mathbf{X} : \mathbf{1} \otimes \mathbf{1})^{\mathrm{T}(1,2)}\right) \\ =& f(\mathbf{X} : \mathbf{1}) \end{aligned}$$

and we again came to (64). □

Now let us consider fourth-order tensors. Let $\mathbf{Y}$ be a fourth-order tensor symmetric with respect to the first three indices. We again introduce its "spherical" and "deviatoric" parts as follows

$$\mathbf{Y} = \mathbf{Y}^s_{tr} + \mathbf{Y}^s_{dev}, \tag{70}$$

where

$$\mathbf{Y}^s_{tr} = \frac{3}{5} \operatorname*{sym}_{(1,2,3)} [\mathbf{Y} : \mathbf{1} \otimes \mathbf{1}], \quad \mathbf{Y}^s_{dev} = \mathbf{Y} - \mathbf{Y}^s_{tr}. \tag{71}$$

Thus, we have

$$\mathbf{Y}^s_{tr} : \mathbf{1} = \mathbf{Y} : \mathbf{1}, \quad \mathbf{Y}^s_{dev} : \mathbf{1} = \mathbf{0}.$$

Using this decomposition, we formulate the following proposition:

Lemma 5.3. *Let f be a scalar function of a fourth-order tensor $\mathbf{Y}$ symmetric with respect to the first three indices, $\mathbf{Y} : \mathbf{1} = \mathbf{0}$, such that*

$$f(\mathbf{Y} + \mathbf{W}) = f(\mathbf{Y}) \tag{72}$$

$$\forall \mathbf{W} : \quad \mathbf{W} = \mathbf{W}^{\mathrm{T}(1,2)} = \mathbf{W}^{\mathrm{T}(1,3)} = \mathbf{W}^{\mathrm{T}(2,3)}, \quad \mathbf{W} : \mathbf{1} = \mathbf{0}.$$

Then f has the form (64).

The proof mimics previous proofs and we omit it.

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