Central European Science Journals
www.cesj.com

Central European Journal of Mathematics
Central European Science Journals
CEJM 2(4) 2004 561–572

Local properties of the solution set of the operator equation in Banach spaces in a neighbourhood of a bifurcation point

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Received 13 January 2004; accepted 7 September 2004

Abstract: In this work we study the problem of the existence of bifurcation in the solution set of the equation $F(x,\lambda)=0$, where $F\colon X\times R^k\to Y$ is a C^2-smooth operator, X and Y are Banach spaces such that $X\subset Y$. Moreover, there is given a scalar product $\langle\cdot,\cdot\rangle\colon Y\times Y\to R^1$ that is continuous with respect to the norms in X and Y. We show that under some conditions there is bifurcation at a point $(0,\lambda_0)\in X\times R^k$ and we describe the solution set of the studied equation in a small neighbourhood of this point.

Keywords: bifurcation, finite-dimensional reduction, Fredholm operator, implicit function, variational gradient
MSC (2000): 34K18

1 Introduction

Let X and Y be real Banach spaces and $F\colon X\times R^k\to Y$ be a continuous map. Suppose that the equation

$$F(x,\lambda)=0, \tag{1}$$

where $x\in X$ and $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_k)\in R^k$, possesses the trivial family of solutions

$$\Lambda=\{(0,\lambda)\in X\times R^k\colon \lambda\in R^k\}.$$

A point (x,λ) such that $F(x,\lambda)=0$ and $x\neq 0$ is called a nontrivial solution of (1). Bifurcation theory is concerned in part with the existence of nontrivial solutions of (1)

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in a small neighbourhood of Λ. A point $(0, \lambda_0) \in \Lambda$ is called a bifurcation point of (1) if every neighbourhood of $(0, \lambda_0)$ contains a nontrivial solution of (1).

Methods of bifurcation theory are often applied in mathematical physics. Let us mention some applications to mechanics of elastic constructions and hydromechanics. In [3] the buckling of a thin elastic plate subject to arbitrary forces and stresses along its boundary is studied by the use of a perturbation theory and a variational method. In [6] to describe a deformation of the minimal interface of two fluids in a vertical tube in a gravitational field one applies a method based on the Crandall-Rabinowitz bifurcation theorem and representation theory. In [9] the buckling of a thin elastic rectangular plate simply supported on sides is studied numerically. In [14] the forms of equilibrium of a thin elastic circular plate lying on an elastic foundation and simply supported along its boundary are investigated via a finite-dimensional reduction and the Krasnosielski bifurcation theorem. Finally, in [16] the buckling of a homogeneous finite beam clamped at the edges to an elastic foundation is studied by a method of a key function due to Sapronov.

Assume that F is C^1-smooth. For every $\lambda \in R^k$, let $F'_x(0, \lambda)\colon X \to Y$ denote the Fréchet derivative of F with respect to x at $(0, \lambda)$. Let $N(\lambda) = \ker F'_x(0, \lambda)$ and $R(\lambda) = \operatorname{im} F'_x(0, \lambda)$. It is easily seen that if $F'_x(0, \lambda_0)\colon X \to Y$ is a Fredholm operator of index zero then a necessary condition for $(0, \lambda_0)$ to be a bifurcation point of (1) is

$$\dim N(\lambda_0) > 0.$$

In this paper we investigate bifurcation at $(0, \lambda_0)$ when X is a linear subspace of Y, there is given a scalar product $\langle \cdot, \cdot \rangle\colon Y \times Y \to R^1$ that is continuous with respect to the norms in X and Y, and F is a C^p-smooth map ($p \geq 2$) that satisfies the following conditions:

(I_1) $F(0, \lambda) = 0$ for every $\lambda \in R^k$,

(I_2) $\dim N(\lambda_0) = 1$,

(I_3) $N(\lambda_0) \perp R(\lambda_0)$,

(I_4) $F'_x(0, \lambda_0)\colon X \to Y$ is a Fredholm operator of index 0.

Our aim is to prove a theorem on bifurcation at $(0, \lambda_0)$ and a local structure of a solution set of equation (1) in a neighbourhood of a bifurcation point. We apply a kind of finite-dimensional reduction of Liapunov-Schmidt type and the implicit function theorem. We are motivated by applications in mathematical physics [6], [14], [16] in which the problems under considerations (see above) are described by (1) with F that satisfies (I_1)–(I_4) and is a variational gradient. The main results of this work are Theorem 3.7 and its variational version: Conclusion 3.10. Theorem 3.7 is an analogue of the Crandall-Rabinowitz bifurcation theorem (see [17], [21]). However, our theorem is formulated in terms of a finite-dimensional reduction and in a variational case it seems to be easier to apply. Conclusion 3.10 is well adapted to a class of nonlinear problems of elasticity described by the von Kármán equations with one or a few parameters (see [4], [15], [16]) in the case when the linearization space is one-dimensional. An example is given in Section 4.

The paper is divided into four sections. In Section 2 we introduce some notions and we briefly sketch a scheme of finite-dimensional reduction. Section 3 is devoted to the study of bifurcation and local properties of the solution set of (1) near a bifurcation point. In Section 4 some applications of our results are indicated.

In practice it suffices to suppose that F is defined in a neighbourhood of $(0, \lambda_0)$ in $X \times R^k$, but we want to omit inessential details.

2 Finite-dimensional reduction

In this section we describe a kind of a finite-dimensional reduction of the Liapunov-Schmidt type. The scheme we present is adapted from [21] (see also [10], [11], [17], [20]).

From now on we assume that $X \subset Y$ are real Banach spaces with a scalar product $\langle \cdot, \cdot \rangle \colon Y \times Y \to R^1$ that is continuous with respect to the norms in X and Y. The norms in X and Y can be defined independently of the scalar product $\langle \cdot, \cdot \rangle$, and the norm in X does not have to be induced by the norm in Y. In particular, X and Y with $\langle \cdot, \cdot \rangle$ may be Hilbert spaces. Let $F \colon X \times R^k \to Y$ be a C^p-smooth map, where $p \geq 1$, satisfying conditions: (I_1), (I_3), (I_4) and
(I_2') $\dim N(\lambda_0) = n \neq 0$.

The aim is to show that under the above assumptions the problem of bifurcation for equation (1) at the point $(0, \lambda_0) \in X \times R^k$ is reducible to the problem of bifurcation for the equation $\varphi(\xi, \lambda) = 0$ with a certain map $\varphi \colon S \subset R^n \times R^k \to R^n$ at the point $(0, \lambda_0) \in R^n \times R^k$. The reader may find the proofs of the propositions given below in [13] and [15].

Proposition 2.1. For every $\lambda \in R^k$ the following equality holds:

$$Y = R(\lambda) \oplus N(\lambda). \tag{2}$$

Let $G \colon X \times R^n \times R^k \to Y$ be a map defined by

$$G(x, \xi, \lambda) = F(x, \lambda) + \sum_{i=1}^{n} (\xi_i - \langle x, e_i \rangle) e_i, \tag{3}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $\{e_1, e_2, \ldots, e_n\}$ is a fixed orthonormal base of $N(\lambda_0)$.

Proposition 2.2. The operator $G_x'(0, 0, \lambda_0) \colon X \to Y$ is an isomorphism.

It is easily seen that G is C^p-smooth. From the implicit function theorem it follows that there exist two open sets $U \subset X$ and $S \subset R^n \times R^k$ such that $0 \in U$, $(0, \lambda_0) \in S$ and the solution set of the equation

$$G(x, \xi, \lambda) = 0 \tag{4}$$

in $U \times S$ is a graph of a certain C^p-smooth function $x \colon S \to U$ such that $x(0, \lambda_0) = 0$. Moreover, it is obvious that $x(0, \lambda) = 0$ for all $(0, \lambda) \in S$, because $G(0, 0, \lambda) = 0$. Let

$\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)\colon S \to R^n$ be defined by coordinates as follows:

$$\varphi_i(\xi, \lambda) = \xi_i - \langle x(\xi, \lambda), e_i \rangle, \ i = 1, \ldots, n. \tag{5}$$

Proposition 2.3. $(0, \lambda_0) \in \Lambda$ is a bifurcation point of equation (1) if and only if $(0, \lambda_0) \in S$ is a bifurcation point of equation

$$\varphi(\xi, \lambda) = 0. \tag{6}$$

3 Theorem on bifurcation

In this section our main results are stated and proved.

Let $F\colon X \times R^k \to Y$ be a C^p-smooth map, $p \geq 2$, satisfying conditions (I_1)-(I_4) (see p. 562). Fix $e \in N(\lambda_0)$ such that $\langle e, e \rangle = 1$ and denote $\lambda_0 = (\lambda_{01}, \lambda_{02}, \ldots, \lambda_{0k})$. We will describe the solution set of (1) in terms of the finite-dimensional reduction. Notice that now in the formulas of maps G and φ there are $n = 1$ and $e_1 = e$. Differentiating the equality $G(x(\xi, \lambda), \xi, \lambda) = 0$ with respect to ξ at $(0, \lambda_0)$ we obtain

$$F'_x(0, \lambda_0)x'_\xi(0, \lambda_0) + (1 - \langle x'_\xi(0, \lambda_0), e \rangle)e = 0.$$

From this and (I_3) it follows that $x'_\xi(0, \lambda_0) = e$.

Theorem 3.1. There exist open sets $V_0 \subset X$ and $V \subset R^k$ such that $(0, \lambda_0) \in V_0 \times V$ and for every $(x, \lambda) \in V_0 \times V$ we have $F(x, \lambda) = 0$ if and only if $(\langle x, e \rangle, \lambda) \in S$ and $x = x(\langle x, e \rangle, \lambda)$.

Proof 3.2. Suppose contrary to our claim, that there are no open sets $V_0 \subset X$ and $V \subset R^k$ with the above properties. Then for every $n \in N$ there exists $(x_n, \lambda_n) \in X \times R^k$ such that $||x_n||_X \leq \frac{1}{n}$, $|\lambda_n - \lambda_0| \leq \frac{1}{n}$ and one of the following conditions is satisfied:
1. $F(x_n, \lambda_n) = 0$ and $(\langle x_n, e \rangle, \lambda_n) \notin S$,
2. $F(x_n, \lambda_n) = 0$, $(\langle x_n, e \rangle, \lambda_n) \in S$ and $x_n \neq x(\langle x_n, e \rangle, \lambda_n)$,
3. $F(x_n, \lambda_n) \neq 0$, $(\langle x_n, e \rangle, \lambda_n) \in S$ and $x_n = x(\langle x_n, e \rangle, \lambda_n)$.

If $(\langle x_n, e \rangle, \lambda_n) \in S$ and $x_n = x(\langle x_n, e \rangle, \lambda_n)$ then $F(x_n, \lambda_n) = F(x(\langle x_n, e \rangle, \lambda_n), \lambda_n) + (\langle x_n, e \rangle - \langle x(\langle x_n, e \rangle, \lambda_n), e \rangle)e = G(x(\langle x_n, e \rangle, \lambda_n), \langle x_n, e \rangle, \lambda_n) = 0$.
Since $x_n \to 0$ in X, there exists $n_0 \in N$ such that $x_n \in U$ for every $n \geq n_0$. If for some $n \geq n_0$ we have $F(x_n, \lambda_n) = 0$ and $(\langle x_n, e \rangle, \lambda_n) \in S$ then $0 = F(x_n, \lambda_n) + (\langle x_n, e \rangle - \langle x_n, e \rangle)e = G(x_n, \langle x_n, e \rangle, \lambda_n)$, and so $x_n = x(\langle x_n, e \rangle, \lambda_n)$.
Since $(\langle x_n, e \rangle, \lambda_n) \to (0, \lambda_0) \in S$ there exists $n_1 \in N$ such that $(\langle x_n, e \rangle, \lambda_n) \in S$ for every $n \geq n_1$ — a contradiction.

The equality $\langle G(x(\xi, \lambda), \xi, \lambda), e \rangle = 0$ implies

$$\varphi(\xi, \lambda) = -\langle F(x(\xi, \lambda), \lambda), e \rangle. \tag{7}$$

From (7) we obtain

$$\varphi'_\xi(\xi, \lambda) = -\langle F'_x(x(\xi, \lambda), \lambda)x'_\xi(\xi, \lambda), e \rangle,$$

and hence $\varphi'_\xi(0,\lambda_0) = 0$. Moreover, since $\varphi(0,\lambda) = 0$ for every $(0,\lambda) \in S$ we have $\varphi^{(m)}_{\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_m}}(0,\lambda_0) = 0$ for all $i_1, i_2, \ldots, i_m \in \{1,2,\ldots,k\}$ and $m \in N$. In order to get our main result we have to assume that there is $i \in \{1,2,\ldots,k\}$ such that $\varphi''_{\xi\lambda_i}(0,\lambda_0) \neq 0$. There is no loss of generality if we assume

(I_5) $\varphi''_{\xi\lambda_k}(0,\lambda_0) \neq 0$.

From now on, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k) \in R^k$, $\lambda' = (\lambda_1, \lambda_2, \ldots, \lambda_{k-1}) \in R^{k-1}$ we will write $\lambda = (\lambda', \lambda_k)$.

Proposition 3.3. There exist open sets $\Omega_0 \subset R^1 \times R^{k-1}$ and $\Omega \subset R^1$ such that $(0,\lambda'_0) \in \Omega_0$, $\lambda_{0k} \in \Omega$ and there exists a C^p-smooth map $f\colon \Omega_0 \to \Omega$ that satisfies the following conditions:

(1) $f(0,\lambda'_0) = \lambda_{0k}$,

(2) for every $(\xi,\lambda') \in \Omega_0$ and $\lambda_k \in \Omega$ we have $\varphi(\xi,\lambda',\lambda_k) = 0$ if and only if $\xi = 0$ or $\lambda_k = f(\xi,\lambda')$.

Proof 3.4. Let $\psi\colon S \to R^1$ be a function defined by

$$\psi(\xi,\lambda) = \int_0^1 \varphi'_\xi(t\xi,\lambda)dt. \tag{8}$$

Observe that we have

$$\varphi(\xi,\lambda) = \xi\psi(\xi,\lambda). \tag{9}$$

Hence $\varphi(\xi,\lambda) = 0$ only if $\xi = 0$ or $\psi(\xi,\lambda) = 0$. From (8) and (I_5) it follows that $\psi(0,\lambda_0) = \varphi'_\xi(0,\lambda_0) = 0$ and $\psi'_{\lambda_k}(0,\lambda_0) = \varphi''_{\xi\lambda_k}(0,\lambda_0) \neq 0$. Applying the implicit function theorem we get the desired claim.

Let $B_r(\lambda'_0)$ denote a ball in R^{k-1} of radius r centered at λ'_0, and $B_\delta(0)$ a ball in X of radius δ centered at 0.

Theorem 3.5. Let $f\colon \Omega_0 \to \Omega$ be a function of Proposition 3.3 and $r > 0$ be a number such that $(-r,r) \times B_r(\lambda'_0) \subset \Omega_0$. There exist open sets $\tilde{V}_0 \subset X$ and $\tilde{V} \subset B_r(\lambda'_0) \times \Omega$ such that $(0,\lambda_0) \in \tilde{V}_0 \times \tilde{V}$ and for every $(x,\lambda) \in \tilde{V}_0 \times \tilde{V}$ we have $F(x,\lambda) = 0$ if and only if $x = 0$ or there exists $\xi \in (-r,r)$ such that $\lambda_k = f(\xi,\lambda')$ and $x = x(\xi,\lambda',f(\xi,\lambda'))$.

Proof 3.6. There exists $\delta \in (0,r)$ such that for every $x \in X$ if $||x||_X < \delta$ then $|\langle x,e\rangle| < r$. Let $\tilde{V}_0 = V_0 \cap B_\delta(0)$ and $\tilde{V} = V \cap (B_r(\lambda'_0) \times \Omega)$, where $V_0 \subset X$ and $V \subset R^k$ are open sets of Theorem 3.1. Take $(x,\lambda) \in \tilde{V}_0 \times \tilde{V}$.

($\Rightarrow$) By Theorem 3.1, if $F(x,\lambda) = 0$ then $(\langle x,e\rangle,\lambda) \in S$ and $x = x(\langle x,e\rangle,\lambda)$, which gives $\varphi(\langle x,e\rangle,\lambda) = 0$. From Proposition 3.3 it follows that $\langle x,e\rangle = 0$ or $\lambda_k = f(\langle x,e\rangle,\lambda')$. If $\langle x,e\rangle = 0$ then $x = x(0,\lambda) = 0$. If $\lambda_k = f(\langle x,e\rangle,\lambda')$ then $x = x(\langle x,e\rangle,\lambda',f(\langle x,e\rangle,\lambda'))$.

($\Leftarrow$) Assume now that $x = 0$ or there exists $\xi \in (-r,r)$ such that $\lambda_k = f(\xi,\lambda')$ and $x = x(\xi,\lambda',f(\xi,\lambda'))$. In the first case, $F(x,\lambda) = F(0,\lambda) = 0$. In the second case, by Proposition 3.3, we have $\varphi(\xi,\lambda) = 0$, and hence $F(x,\lambda) = F(x,\lambda) + \varphi(\xi,\lambda)e = F(x,\lambda) + (\xi - \langle x(\xi,\lambda),e\rangle)e = F(x,\lambda) + (\xi - \langle x,e\rangle)e = G(x,\xi,\lambda) = 0$.

We are now in a position to prove our main result.

Theorem 3.7. Under assumptions (I_1)-(I_5), the solution set of equation (1) in a certain neighbourhood of $(0, \lambda_0) \in \Lambda$ is the union of two sets: Λ and Ξ. The set Ξ is given by

$$\Xi = \{(\hat{x}(\xi, \lambda'), \lambda', f(\xi, \lambda')) \colon |\xi| < r, |\lambda' - \lambda'_0| < r\},$$

where $\hat{x}$ and f are C^p-smooth functions such that $\hat{x}(0, \lambda'_0) = 0$, $f(0, \lambda'_0) = \lambda_{0k}$, $\hat{x}'_\xi(0, \lambda'_0) = e$, $f'_\xi(0, \lambda'_0) = -\frac{1}{2}\frac{\varphi''_{\xi\xi}(0,\lambda_0)}{\varphi''_{\xi\lambda_k}(0,\lambda_0)}$, $\hat{x}'_{\lambda_s}(0, \lambda'_0) = 0$ and $f'_{\lambda_s}(0, \lambda'_0) = -\frac{\varphi''_{\xi\lambda_s}(0,\lambda_0)}{\varphi''_{\xi\lambda_k}(0,\lambda_0)}$ for every $s \in \{1, 2, \ldots, k-1\}$.
Moreover, the intersection of Λ and Ξ in a sufficiently small neighbourhood of $(0, \lambda_0)$ can be parametrized as follows

$$I_{\Lambda,\Xi} = \{(0, \lambda', f(\hat{\xi}(\lambda'), \lambda')) \colon |\lambda' - \lambda'_0| < \varrho\}$$

where $0 < \varrho \leq r$ and $\hat{\xi}$ is a C^p-smooth function such that $\hat{\xi}(\lambda'_0) = 0$ and $\hat{\xi}'_{\lambda_s}(\lambda'_0) = 0$ for every $s \in \{1, 2, \ldots, k-1\}$, which gives that $(0, \lambda_0)$ is a bifurcation point of (1).

Proof 3.8. Let $f \colon \Omega_0 \to \Omega$ be a function of Proposition 3.3. Fix $r > 0$ such that $(-r, r) \times B_r(\lambda'_0) \subset \Omega_0$. Let $\hat{x} \colon (-r, r) \times B_r(\lambda'_0) \to X$ be given by $\hat{x}(\xi, \lambda') = x(\xi, \lambda', f(\xi, \lambda'))$. Then $f(0, \lambda'_0) = \lambda_{0k}$ and $\hat{x}(0, \lambda'_0) = x(0, \lambda_0) = 0$. Differentiating $\hat{x}$ we get $\hat{x}'_\xi(0, \lambda'_0) = e$ and $\hat{x}'_{\lambda_s}(0, \lambda'_0) = 0$ for every $s \in \{1, 2, \ldots, k-1\}$. Moreover, differentiating the equality $\psi(\xi, \lambda', f(\xi, \lambda')) = 0$ we obtain $f'_\xi(0, \lambda'_0) = -\frac{\psi'_\xi(0,\lambda_0)}{\psi'_{\lambda_k}(0,\lambda_0)} = -\frac{1}{2}\frac{\varphi''_{\xi\xi}(0,\lambda_0)}{\varphi''_{\xi\lambda_k}(0,\lambda_0)}$ and $f'_{\lambda_s}(0, \lambda'_0) = -\frac{\psi'_{\lambda_s}(0,\lambda_0)}{\psi'_{\lambda_k}(0,\lambda_0)} = -\frac{\varphi''_{\xi\lambda_s}(0,\lambda_0)}{\varphi''_{\xi\lambda_k}(0,\lambda_0)}$ for every $s \in \{1, 2, \ldots, k-1\}$. From Theorem 3.5 it follows that there exist open sets $\tilde{V}_0 \subset X$ and $\tilde{V} \subset B_r(\lambda'_0) \times \Omega$ such that $(0, \lambda_0) \in \tilde{V}_0 \times \tilde{V}$ and $\{(x, \lambda) \in \tilde{V}_0 \times \tilde{V} \colon F(x, \lambda) = 0\} = \{(x, \lambda) \in \tilde{V}_0 \times \tilde{V} : x = 0\} \cup \{(x, \lambda) \in \tilde{V}_0 \times \tilde{V} \colon \exists_{\xi \in (-r,r)}\, x = x(\xi, \lambda', f(\xi, \lambda')) \wedge \lambda_k = f(\xi, \lambda')\} = (\Lambda \cup \Xi) \cap \tilde{V}_0 \times \tilde{V}$. A point $(x, \lambda) \in \Lambda \cap \Xi$ only if it satisfies the following system

$$\begin{cases} x = \hat{x}(\xi, \lambda'), \\ \lambda_k = f(\xi, \lambda'),\ \xi \in (-r, r),\ \lambda' \in B_r(\lambda'_0), \\ x = 0. \end{cases}$$

Since $\hat{x}(0, \lambda'_0) = 0$ and $\hat{x}'_\xi(0, \lambda'_0) = e \neq 0$, there exist: $0 < \varrho \leq r$, an open set $B \subset (-r, r)$ such that $0 \in B$ and a C^p-smooth function $\hat{\xi} \colon B_\varrho(\lambda'_0) \to B$ such that $\hat{\xi}(\lambda'_0) = 0$ and for all $(\xi, \lambda') \in B \times B_\varrho(\lambda'_0)$ we have $\hat{x}(\xi, \lambda') = 0$ only if $\xi = \hat{\xi}(\lambda')$. Differentiating the equality $\hat{x}(\hat{\xi}(\lambda'), \lambda') = 0$ we receive $\hat{x}'_\xi(\hat{\xi}(\lambda'), \lambda')\hat{\xi}'_{\lambda_s}(\lambda') + \hat{x}'_{\lambda_s}(\hat{\xi}(\lambda'), \lambda') = 0$ for every $s \in \{1, 2, \ldots, k-1\}$, and hence $\hat{\xi}'_{\lambda_s}(\lambda'_0) = 0$. Summarizing $I_{\Lambda,\Xi} \subset \Lambda \cap \Xi$ and in a sufficiently small neighbourhood of $(0, \lambda_0)$ the intersection $\Lambda \cap \Xi$ is equal to $I_{\Lambda,\Xi}$.

Conclusion 3.9. Assume that (I_1)-(I_5) hold and $k = 2$. Then the solution set of (1) in a small neighbourhood of $(0, \lambda_0) \in \Lambda$ is the union of two surfaces: Λ and Ξ. The surface Ξ can be parametrized as follows

$$\Xi = \{(\hat{x}(\xi, \lambda_1), \lambda_1, f(\xi, \lambda_1)) \colon (\xi, \lambda_1) \in (-r, r) \times (\lambda_{01} - r, \lambda_{01} + r)\},$$

where $\hat{x}\colon (-r,r)\times(\lambda_{01}-r,\lambda_{01}+r)\to X$ and $f\colon (-r,r)\times(\lambda_{01}-r,\lambda_{01}+r)\to R^1$ are C^p-smooth functions such that $\hat{x}(0,\lambda_{01})=0$, $f(0,\lambda_{01})=\lambda_{02}$, $\hat{x}'_\xi(0,\lambda_{01})=e$, $\hat{x}'_{\lambda_1}(0,\lambda_{01})=0$, $f'_\xi(0,\lambda_{01})=-\frac{1}{2}\frac{\varphi''_{\xi\xi}(0,\lambda_{01})}{\varphi''_{\xi\lambda_2}(0,\lambda_{01})}$ and $f'_{\lambda_1}(0,\lambda_{01})=-\frac{\varphi''_{\xi\lambda_1}(0,\lambda_{01})}{\varphi''_{\xi\lambda_2}(0,\lambda_{01})}$. In a sufficiently small neighbourhood of $(0,\lambda_0)$ the surfaces Λ and Ξ intersect only along the curve

$$I_{\Lambda,\Xi}=\{(0,\lambda_1,f(\hat{\xi}(\lambda_1),\lambda_1))\colon \lambda_1\in(\lambda_{01}-\varrho,\lambda_{01}+\varrho)\},$$

where $0<\varrho\le r$ and $\hat{\xi}\colon(\lambda_{01}-\varrho,\lambda_{01}+\varrho)\to(-r,r)$ is a C^p-smooth function such that $\hat{\xi}(\lambda_{01})=\hat{\xi}'(\lambda_{01})=0$, and hence $(0,\lambda_0)$ is a bifurcation point of (1).

Let us consider the following condition:
(I'_3) $F\colon X\times R^k\to Y$ is a variational gradient of a certain functional $E\colon X\times R^k\to R^1$ with respect to the scalar product $\langle\cdot,\cdot\rangle$, i.e. for all $x,y\in X$ and $\lambda\in R^k$

$$E'_x(x,\lambda)y=\langle F(x,\lambda),y\rangle.$$

It is evident that (I'_3) implies (I_3). Furthermore, by formula (7) we obtain

$$\varphi''_{\xi\lambda_s}(0,\lambda_0)=-E^{(3)}_{xx\lambda_s}(0,\lambda_0)(e,e,1) \tag{10}$$

for $s\in\{1,2,\dots,k\}$. From this it follows that if F satisfies (I'_3) then (I_5) can be replaced by the equivalent condition:
(I'_5) $E^{(3)}_{xx\lambda_k}(0,\lambda_0)(e,e,1)\neq 0$.
By (7) we also obtain

$$\varphi''_{\xi\xi}(0,\lambda_0)=-E^{(3)}_{xxx}(0,\lambda_0)(e,e,e). \tag{11}$$

Summarizing, in a variational case we have the following result.

Conclusion 3.10. Under assumptions: (I_1), (I_2), (I'_3), (I_4) and (I'_5), the solution set of equation (1) in a certain neighbourhood of $(0,\lambda_0)\in\Lambda$ is the union of two sets: Λ and Ξ. The set Ξ is given by

$$\Xi=\{(\hat{x}(\xi,\lambda'),\lambda',f(\xi,\lambda'))\colon |\xi|<r,|\lambda'-\lambda'_0|<r\},$$

where $\hat{x}$ and f are C^p-smooth functions such that $\hat{x}(0,\lambda'_0)=0$, $f(0,\lambda'_0)=\lambda_{0k}$, $\hat{x}'_\xi(0,\lambda'_0)=e$, $f'_\xi(0,\lambda'_0)=-\frac{1}{2}\frac{E^{(3)}_{xxx}(0,\lambda_0)(e,e,e)}{E^{(3)}_{xx\lambda_k}(0,\lambda_0)(e,e,1)}$, $\hat{x}'_{\lambda_s}(0,\lambda'_0)=0$ and $f'_{\lambda_s}(0,\lambda'_0)=-\frac{E^{(3)}_{xx\lambda_s}(0,\lambda_0)(e,e,1)}{E^{(3)}_{xx\lambda_k}(0,\lambda_0)(e,e,1)}$ for every $s\in\{1,2,\dots,k-1\}$.
Moreover, the intersection of Λ and Ξ in a sufficiently small neighbourhood of $(0,\lambda_0)$ can be parametrized as follows

$$I_{\Lambda,\Xi}=\{(0,\lambda',f(\hat{\xi}(\lambda'),\lambda'))\colon |\lambda'-\lambda'_0|<\varrho\}$$

where $0<\varrho\le r$ and $\hat{\xi}$ is a C^p-smooth function such that $\hat{\xi}(\lambda'_0)=0$ and $\hat{\xi}'_{\lambda_s}(\lambda'_0)=0$ for every $s\in\{1,2,\dots,k-1\}$, which gives that $(0,\lambda_0)$ is a bifurcation point of (1).

4 Applications

It is obvious that if we assume that F is a map from a small neighbourhood of the point $(0,\lambda_0)$ in $X \times R^k$ to Y, our results remain true. After this remark we are ready to give an example of application of Conclusion 3.10 to mathematical physics. All the results of Section 4 were proved either in [12] or [15]. However, to make this exposition self-sufficient we give the main ideas of the proofs.

For every $m \in N$ and $\mu \in (0,1)$, let $C^{m,\mu}(\bar{D})$ denote the real Hölder space of functions defined on $D = \{(u,v) \in R^2 \colon u^2+v^2<1\}$ with the standard norm

$$||x; C^{m,\mu}(\bar{D})|| = \max_{|\alpha|\leq m} \sup\{|D^\alpha x(u,v)| \colon (u,v) \in D\} + \max_{|\alpha|\leq m} \sup\left\{\frac{|D^\alpha x(u,v) - D^\alpha x(\bar{u},\bar{v})|}{|(u-\bar{u}, v-\bar{v})|^\mu} \colon (u,v),(\bar{u},\bar{v}) \in D, (u,v) \neq (\bar{u},\bar{v})\right\},$$

where $D^\alpha x = \frac{\partial^{|\alpha|}x}{\partial^{\alpha_1}u\partial^{\alpha_2}v}$, $\alpha = (\alpha_1,\alpha_2) \in N_0 \times N_0$, $N_0 = N \cup \{0\}$ and $|\alpha| = \alpha_1+\alpha_2$. It is well-known that $C^{m,\mu}(\bar{D})$ is a Banach space (see [1]). Let

- $C^{4,\mu}_{0,0}(\bar{D}) = \{f \in C^{4,\mu}(\bar{D}) \colon \Delta f|_{\partial D} = f|_{\partial D} = 0\}$,
- $C^{2,\mu}_{0}(\bar{D}) = \{f \in C^{2,\mu}(\bar{D}) \colon f|_{\partial D} = 0\}$,
- $X = C^{4,\mu}_{0,0}(\bar{D}) \times C^{4,\mu}_{0,0}(\bar{D})$,
- $Y = C^{0,\mu}(\bar{D}) \times C^{0,\mu}(\bar{D})$.

The norms in X and Y are defined by coordinates. That is as the maximum (or the sum) of norms of both coordinates of a given element. The function given by

$$\langle(x_1,x_2),(y_1,y_2)\rangle = \frac{1}{\pi}\iint_D (x_1y_1+x_2y_2)dudv$$

is a scalar product in Y, which is continuous with respect to the norms in X and Y. We define $F \colon X \times R^2_+ \to Y$ as follows

$$F(x,\lambda) = (\Delta^2 x_1 - [x_1,x_2] + 2\lambda_1\Delta x_1 + \lambda_2 x_1 - \gamma x_1^3, -\Delta^2 x_2 - \frac{1}{2}[x_1,x_1]), \tag{12}$$

where $R_+ = (0,+\infty)$, $x = (x_1,x_2)$, $\lambda = (\lambda_1,\lambda_2)$, γ is a positive constant and $[\cdot,\cdot] \colon X \to Y$ is given by

$$[x_1,x_2] = \frac{\partial^2 x_1}{\partial u^2}\frac{\partial^2 x_2}{\partial v^2} - 2\frac{\partial^2 x_1}{\partial u\partial v}\frac{\partial^2 x_2}{\partial u\partial v} + \frac{\partial^2 x_1}{\partial v^2}\frac{\partial^2 x_2}{\partial u^2}.$$

The equation

$$F(x,\lambda) = 0 \tag{13}$$

with F given by (12) is called the von Kárman equation for a thin circular elastic plate which lies on an elastic base and is uniformly radially compressed along its boundary. In mechanics x_1 is a deflection function, x_2 is a stress function, λ_1 is a value of a compressing force, λ_2 and γ are parameters of an elastic foundation. The solutions of (13) lying in a sufficiently small neighbourhood of the set of trivial solutions of (13) are called the forms of equilibrium of a plate. The map F is C^∞-smooth and an easy computation shows that for all $y = (y_1, y_2) \in X$

$$F'_x(x,\lambda)y = (\Delta^2 y_1 - [y_1, x_2] - [x_1, y_2] + 2\lambda_1 \Delta y_1 + \lambda_2 y_1 - 3\gamma x_1^2 y_1, -\Delta^2 y_2 - [x_1, y_1]). \quad (14)$$

Let $E\colon X \times R^2_+ \to R^1$ be given by

$$\begin{aligned} E(x,\lambda) = {} & \frac{1}{2\pi} \iint_D \left((\Delta x_1)^2 - (\Delta x_2)^2 - [x_1, x_1]x_2\right) dudv + \\ & \frac{1}{2\pi} \iint_D \left(-2\lambda_1 \left(\left(\frac{\partial x_1}{\partial u}\right)^2 + \left(\frac{\partial x_1}{\partial v}\right)^2\right) + \lambda_2 x_1^2 - \frac{1}{2}\gamma x_1^4\right) dudv. \end{aligned} \quad (15)$$

E is easily seen to be C^∞-smooth.

Theorem 4.1 (see Th. 2.4 of [12]). The map F is a variational gradient of the functional E with respect to the scalar product $\langle \cdot, \cdot \rangle$.

Sketch of the proof 4.2. For all $x, y \in X$ and $\lambda \in R^2_+$, we have

$$\begin{aligned} E'_x(x,\lambda)y = \frac{d}{dt}E(x+ty,\lambda)|_{t=0} = {} & \frac{1}{\pi}\iint_D \Delta x_1 \Delta y_1 dudv - \frac{1}{\pi}\iint_D \Delta x_2 \Delta y_2 dudv \\ & -\frac{1}{\pi}\iint_D [x_1, y_1]x_2 dudv - \frac{1}{2\pi}\iint_D [x_1, x_1]y_2 dudv \\ & -\frac{1}{\pi}\iint_D 2\lambda_1 \left(\frac{\partial x_1}{\partial u}\frac{\partial y_1}{\partial u} + \frac{\partial x_1}{\partial v}\frac{\partial y_1}{\partial v}\right) dudv \\ & +\frac{1}{\pi}\iint_D (\lambda_2 x_1 y_1 - \gamma x_1^3 y_1) dudv. \end{aligned}$$

Integrating by part we receive

$$\begin{aligned} \iint_D \Delta x_1 \Delta y_1 dudv &= \iint_D (\Delta^2 x_1) y_1 dudv, \\ \iint_D \Delta x_2 \Delta y_2 dudv &= \iint_D (\Delta^2 x_2) y_2 dudv, \\ \iint_D [x_1, y_1]x_2 dudv &= \iint_D [x_1, x_2]y_1 dudv \end{aligned}$$

and

$$\iint_D \left(\frac{\partial x_1}{\partial u}\frac{\partial y_1}{\partial u} + \frac{\partial x_1}{\partial v}\frac{\partial y_1}{\partial v}\right) dudv = -\iint_D (\Delta x_1) y_1 dudv.$$

Hence $E'_x(x,\lambda)y = \langle F(x,\lambda), y\rangle$, which completes the proof.

Theorem 4.3 (see Th. 2.2 of [12]). For every $\lambda \in R^2_+$, $F'_x(0,\lambda)\colon X \to Y$ is a Fredholm map of index 0.

Sketch of the proof 4.4. Fix $\lambda \in R^2_+$. By (14) we get

$$F'_x(0,\lambda)y = (\Delta^2 y_1 + 2\lambda_1 \Delta y_1 + \lambda_2 y_1, -\Delta^2 y_2). \quad (16)$$

We can write (16) as

$$F'_x(0,\lambda)y = A(y) + B(y),$$

where $A, B\colon X \to Y$ are given as follows:

$$A(y) = (\Delta^2 y_1, -\Delta^2 y_2), \quad B(y) = (2\lambda_1 \Delta y_1 + \lambda_2 y_1, 0).$$

It is known that $\Delta\colon C_0^{2,\mu}(\overline{D}) \to C^{0,\mu}(\overline{D})$ is an isomorphism. Moreover, it is a simple matter to check that B is compact, which finishes the proof.

Let $J_k\colon R \to R$, $k \in N_0$, denote the k-th Bessel function. It is well-known (see [8], [18]) that $\alpha \in R$ is an eigenvalue of $\Delta\colon C_0^{2,\mu}(\overline{D}) \to C^{0,\mu}(\overline{D})$ if and only if $\alpha < 0$ and there is $k \in N_0$ such that $J_k(\sqrt{-\alpha}) = 0$. Furthermore, if $J_0(\sqrt{-\alpha}) = 0$ then the eigenspace corresponding to α is one-dimensional. If $J_k(\sqrt{-\alpha}) = 0$ for a certain $k \in N$ then the corresponding eigenspace is two-dimensional.

For $\lambda = (\lambda_1, \lambda_2) \in R_+^2$, let $\delta = (\lambda_1)^2 - \lambda_2$, $a = -\lambda_1 - \sqrt{\delta}$ and $b = -\lambda_1 + \sqrt{\delta}$. Of course, a and b are determined on condition $\delta \geq 0$. Let $\Delta^2 + 2\lambda_1\Delta + \lambda_2 \mathrm{I}\colon C_{0,0}^{4,\mu}(\overline{D}) \to C^{0,\mu}(\overline{D})$ and $\Delta - a\mathrm{I}, \Delta - b\mathrm{I}\colon C_0^{2,\mu}(\overline{D}) \to C^{0,\mu}(\overline{D})$, where $\mathrm{I}(h) = h$ are natural embeddings of the appropriate Hölder spaces.

Lemma 4.5 (see Lemmas 4.1-4.3 of [12]). Under the above assumptions:
(i) If $\delta < 0$ then $\ker(\Delta^2 + 2\lambda_1\Delta + \lambda_2\mathrm{I}) = \{0\}$.
(ii) If $\delta = 0$ then $\ker(\Delta^2 + 2\lambda_1\Delta + \lambda_2\mathrm{I}) = \ker(\Delta + \lambda_1\mathrm{I})$.
(iii) If $\delta > 0$ then $\ker(\Delta^2 + 2\lambda_1\Delta + \lambda_2\mathrm{I}) = \ker(\Delta - a\mathrm{I}) \oplus \ker(\Delta - b\mathrm{I})$.

By (16), $N(\lambda) = \ker(\Delta^2 + 2\lambda_1\Delta + \lambda_2\mathrm{I}) \times \{0\}$. From this and Lemma 4.5 we obtain what follows.

Theorem 4.6. $\dim N(\lambda) = 1$ if and only if one of the below conditions is satisfied:
(I) $\delta = 0$ and $J_0(\sqrt{\lambda_1}) = 0$,
(II) $\delta > 0$, $J_0(\sqrt{-a}) = 0$ and $J_k(\sqrt{-b}) \neq 0$ for every $k \in N_0$,
(III) $\delta > 0$, $J_0(\sqrt{-b}) = 0$ and $J_k(\sqrt{-a}) \neq 0$ for every $k \in N_0$.

Suppose that $\lambda_0 = (\lambda_{01}, \lambda_{02})$ and $\dim N(\lambda_0) = 1$. Fix $e = (e_1, 0) \in N(\lambda_0)$ such that $\langle e, e\rangle = 1$. Set

$$c_0 = \begin{cases} a_0 & \text{if (I) or (II)}, \\ b_0 & \text{if (III)}, \end{cases}$$

where $a_0 = -\lambda_{01} - \sqrt{\delta_0}$, $b_0 = -\lambda_{01} + \sqrt{\delta_0}$ and $\delta_0 = (\lambda_{01})^2 - \lambda_{02}$. A trivial verification combining Theorem 4.1 with (14) shows that

$$E'''_{xx\lambda_1}(x,\lambda)(y,z,1) = \frac{2}{\pi}\iint_D (\Delta y_1) z_1 \, du dv,$$
$$E'''_{xx\lambda_2}(x,\lambda)(y,z,1) = \frac{1}{\pi}\iint_D y_1 z_1 \, du dv,$$

and

$$E'''_{xxx}(x,\lambda)(y,z,w) = -\frac{1}{\pi}\iint_D([y_1,z_2]+[y_2,z_1]+6\gamma x_1y_1z_1)w_1dudv$$
$$-\frac{1}{\pi}\iint_D[y_1,z_1]w_2dudv,$$

where $x=(x_1,x_2)$, $y=(y_1,y_2)$, $z=(z_1,z_2)$, $w=(w_1,w_2)$. From this and Lemma 4.5 we receive

$$E'''_{xx\lambda_1}(0,\lambda_0)(e,e,1) = \frac{2}{\pi}\iint_D(\Delta e_1)e_1dudv = 2c_0\langle e,e\rangle = 2c_0,$$
$$E'''_{xx\lambda_2}(0,\lambda_0)(e,e,1) = \frac{1}{\pi}\iint_D e_1^2dudv = \langle e,e\rangle = 1,$$
$$E'''_{xxx}(0,\lambda_0)(e,e,e) = 0.$$

Applying Conclusion 3.10 we get the following theorem.

Theorem 4.7. Let $\lambda_0 \in R^2_+$ satisfy the above assumptions. Then the solution set of equation (13) in a certain neighbourhood of $(0,\lambda_0) \in X \times R^2_+$ is the union of two sets: Λ and Ξ. The set Ξ is given by

$$\Xi = \{(\hat{x}(\xi,\lambda_1),\lambda_1,f(\xi,\lambda_1))\colon |\xi|<r, |\lambda_1-\lambda_{01}|<r\},$$

where $\hat{x}$ and f are C^∞-smooth functions such that $\hat{x}(0,\lambda_{01}) = 0$, $f(0,\lambda_{01}) = \lambda_{02}$, $\hat{x}'_\xi(0,\lambda_{01}) = e$, $f'_\xi(0,\lambda_{01}) = 0$, $\hat{x}'_{\lambda_1}(0,\lambda_{01}) = 0$ and $f'_{\lambda_1}(0,\lambda_{01}) = -2c_0$.
Moreover, the intersection of Λ and Ξ in a sufficiently small neighbourhood of $(0,\lambda_0)$ can be parametrized as follows

$$I_{\Lambda,\Xi} = \{(0,\lambda_1,f(\hat{\xi}(\lambda_1),\lambda_1))\colon |\lambda_1-\lambda_{01}|<\varrho\},$$

where $0<\varrho\le r$ and $\hat{\xi}$ is a C^∞-smooth function such that $\hat{\xi}(\lambda_{01})=0$ and $\hat{\xi}'(\lambda_{01})=0$, which gives that $(0,\lambda_0)$ is a bifurcation point of (13).

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