

**Method for universal detection of two-photon polarization entanglement**Karol Bartkiewicz,<sup>1,2,\*</sup> Paweł Horodecki,<sup>3,4</sup> Karel Lemr,<sup>2</sup> Adam Miranowicz,<sup>1</sup> and Karol Życzkowski<sup>5,6</sup><sup>1</sup>*Faculty of Physics, Adam Mickiewicz University, 61-614 Poznań, Poland*<sup>2</sup>*RCPTM, Joint Laboratory of Optics of Palacký University and Institute of Physics of Academy of Sciences of the Czech Republic, 17 Listopadu 12, 772 07 Olomouc, Czech Republic*<sup>3</sup>*Faculty of Applied Physics and Mathematics, Gdańsk University of Technology, 80-233 Gdańsk, Poland*<sup>4</sup>*National Quantum Information Centre in Gdańsk, 81-824 Sopot, Poland*<sup>5</sup>*Institute of Physics, Jagiellonian University, ul Łojasiewicza 11, 30-348 Krakow, Poland*<sup>6</sup>*Center for Theoretical Physics, Polish Academy of Sciences, Aleja Lotników 32/46, 02-668 Warsaw, Poland*

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Detecting and quantifying quantum entanglement of a given unknown state poses problems that are fundamentally important for quantum information processing. Surprisingly, no direct (i.e., without quantum tomography) universal experimental implementation of a necessary and sufficient test of entanglement has been designed even for a general two-qubit state. Here we propose an experimental method for detecting a collective universal witness, which is a necessary and sufficient test of two-photon polarization entanglement. It allows us to detect entanglement for any two-qubit mixed state and to establish tight upper and lower bounds on its amount. A different element of this method is the sequential character of its main components, which allows us to obtain relatively complicated information about quantum correlations with the help of simple linear-optical elements. As such, this proposal realizes a universal two-qubit entanglement test within the present state of the art of quantum optics. We show the optimality of our setup with respect to the minimal number of measured quantities.

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**I. INTRODUCTION**

Quantum entanglement [1,2] is a fascinating phenomenon considered to be one of the main resources in quantum information and quantum engineering (for reviews, see Refs. [3–5]). In general, detecting entanglement in various physical scenarios poses a significant problem. Most widely used methods are based on measuring entanglement witnesses (see Ref. [4]), which are efficient but, typically, not universal and require some information about the state prior to its measurement. On the other hand, by performing standard methods of quantum tomography of a given state, one obtains complete information about that state. Thus, information concerning its entanglement can be extracted by an explicit calculation, through the postprocessing of the complete experimental data. However, full tomography requires measuring a large number of parameters; this number scales with the square of the total dimension of a measured state. Moreover, there remains one conceptually fundamental question, namely, what is the minimal number of parameters that are experimentally feasible (in the sense of, e.g., linear optics) that will nevertheless provide complete information about quantum entanglement independently of a general input state. This can be viewed as a question about a quantum processor with a quantum input (state) and a classical output (giving a yes or no answer or some quantitative information about entanglement) with minimal processing of classical (incoherent) information inside.

Early proposals regarding the detection and quantification of quantum entanglement without state reconstruction were based on the identification of polynomial moments. These methods made it possible to retrieve information on entanglement from the data spectrum of the partial transpose

of the two-qubit Wootters concurrence (see Refs. [6–8] for a significant quantum-noise reduction). They enabled sharp two-qubit entanglement tests, but required nonlinear postprocessing of the data to retrieve the original information about entanglement.

Independently of the above-mentioned line of research, the concept of collective entanglement witnesses [9] made it possible to construct collective observables for describing entanglement quantitatively in experiments [10]. Moreover, the analysis of the concurrence of Ref. [11] (see also Ref. [12] for recent developments) eventually led to a quantitative experimental estimation of entanglement in terms of specific two-copy collective witnesses [13,14].

Another interesting example of collective entanglement witnesses is a two-copy witness based on the geometric intuition of the concept of metric [15]. A number of multipartite tests based on the nonlinear functions of simple multicopy observables were developed [16] and several other quantitative [17–20] and qualitative [15,16,21–24] methods of detecting entanglement without quantum tomography were proposed. Nevertheless, these techniques, although quite powerful, are not universal and the quality of their results depends on a given state.

Experimental adaptive approaches [25,26] were also proposed for the case of two qubits, which we shall focus on in this paper. Although these methods are an elegant improvement, they do not satisfy the universality requirement.

Let us stress, however, that there exists a universal witness of entanglement (UWE) for a two-qubit state, as introduced in Ref. [27] and defined here in Sec. II. This UWE can be measured by performing the joint measurements on the four copies of a given state [27]. However, so far no experimental implementation for such a measurement has been proposed. The aim of this paper is to propose a constructive measurement procedure that outputs the mean value  $\langle W_{\text{univ}}^{(4)} \rangle$

\*bark@amu.edu.pl

of the above witness for any two-qubit polarization state of a pair of photons, thereby allowing us to detect the arbitrary quantum entanglement of such systems. This is an experimental proposal of a universal (sharp) entanglement test with (i) elementary (linear) optics and (ii) practically trivial (direct substitution for polynomial) postprocessing of experimental data. To be more specific, the procedure has the unique advantage that it can be (probabilistically) utilized with the help of just linear-optical methods involving only a sequence of beam splitters and the Hong-Ou-Mandel (HOM) interference. Quite remarkably, no polarizing beam splitters or phase rotators are needed. This is especially important here because we consider only the polarization-encoded qubits.

We start the presentation of our results with the analysis of the properties of the UWE symmetries of the observables needed for reproducing the three moments  $\Pi_i$  ( $i = 2, 3, 4$ ). Then we shall provide the optical HOM interference methods for reproducing the values of the moments. Having found these values, one just needs to substitute them into the polynomial (1) and to check the sign of the final value.

This paper is organized as follows. In Sec. II we recall the definition of the UWE and show how the negativity and concurrence are bounded by some functions of the UWE. In Sec. III we present the main idea of how to measure the UWE. A detailed derivation of one of our important formulas is given in the Appendix. In Sec. IV we relate our results to Makhlin's invariants and find the minimum number of independent measurements required for detecting entanglement. In Sec. V we describe our proposal of an experimental photonic implementation for the UWE detection. In Sec. VI we present two alternative implementations. We discuss our results and summarize in Sec. VII.

## II. UNIVERSAL ENTANGLEMENT WITNESS AND BOUNDS ON NEGATIVITY AND CONCURRENCE

The UWE, provided in Ref. [27], is an operator  $W_{\text{univ}}^{(4)}$  such that its expected value corresponds to the determinant of the partially transposed (marked by  $\Gamma$ ) two-qubit matrix  $\rho$ , i.e.,

$$\begin{aligned} \det \rho^\Gamma &= \langle W_{\text{univ}}^{(4)} \rangle := \text{tr}(W_{\text{univ}}^{(4)} \rho^{\otimes 4}) \\ &= \frac{1}{24}(1 - 6\Pi_4 + 8\Pi_3 + 3\Pi_2^2 - 6\Pi_2), \end{aligned} \quad (1)$$

which is given in terms of the moments  $\Pi_n = \text{tr}[(\rho^\Gamma)^n]$ . For convenience, we refer to the observable  $W_{\text{univ}}^{(4)}$ , but also to its expectation value  $\langle W_{\text{univ}}^{(4)} \rangle$  as the UWE. It follows from the positive partial transpose criterion that a two-qubit state is entangled if and only if  $\langle W_{\text{univ}}^{(4)} \rangle < 0$ . The explicit form of this witness, which is the mean value of the Hermitian observable  $W_{\text{univ}}^{(4)}$  on the four copies  $\rho^{\otimes 4}$  of qubit pairs in a given state  $\rho$ , is explicitly provided in Ref. [27] and constructed from permutation matrices. The main advantage of this UWE compared to other universal methods of two-qubit entanglement detection is that this is a linear observable that does not require solving (unfolding) nonlinear polynomial equations, which are more sensitive to errors, to obtain the information about a given state (see Refs. [6–8]). Another advantage of this witness is that its rescaled value  $w := \max[0, -16\langle W_{\text{univ}}^{(4)} \rangle]$  provides tight upper and lower bounds [27] on the negativity  $N(\rho)$  and concurrence

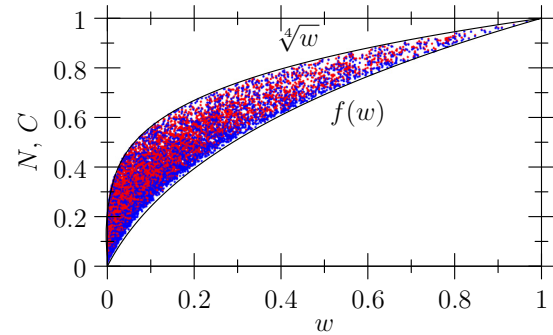


FIG. 1. (Color online) Tight lower and upper bounds for the two-qubit negativity  $N$  and concurrence  $C$  in terms of the universal witness value  $w = \max[0, -16\langle W_{\text{univ}}^{(4)} \rangle]$ . Red (blue) dots correspond to the concurrence (negativity) for  $10^4$  density matrices  $\rho$  generated by a Monte Carlo simulation (see also [27]).

$C(\rho)$  of a two-qubit state  $\rho$ ,

$$f(w) \leq N \leq C \leq \sqrt[4]{w}, \quad (2)$$

where  $f(w)$  is the inverse of the polynomial  $w(C) = C(C + 2)^3/27$  on the interval  $C \in [0, 1]$ . We recall that the negativity  $N$  of a two-qubit state  $\rho$  can be defined as [4]

$$N(\rho) = 2 \max\{0, -\min[\text{eig}(\rho^\Gamma)]\}, \quad (3)$$

i.e., via the minimum (negative) eigenvalue of the partially transposed density matrix  $\rho^\Gamma$ , while the concurrence  $C$  of a two-qubit state  $\rho$  is given by [28]

$$C(\rho) = \max\left(0, -\sum_j \lambda_j + 2 \max_j \lambda_j\right), \quad (4)$$

where  $\{\lambda_j^2\} = \text{eig}[\rho(\sigma_2 \otimes \sigma_2)\rho^*(\rho_2 \otimes \sigma_2)]$  and  $\sigma_2$  is the Pauli operator. The lower bound  $f(w)$  can be given explicitly in terms of the universal witness value  $w$  as

$$f(w) = \frac{1}{2}(-3 + \sqrt{z} + \sqrt{3 - z + 2/\sqrt{z}}), \quad (5)$$

where  $z = 1 + x - 36w/x$ , and

$$x = 3\sqrt[3]{2\sqrt{w^2(16w + 1)} - 2w}. \quad (6)$$

This lower bound and the upper bound are shown in Fig. 1.

## III. HOW TO MEASURE UNIVERSAL ENTANGLEMENT WITNESS: PRINCIPLE IDEA

In order to directly determine the witness  $\langle W_{\text{univ}}^{(4)} \rangle$ , we can measure all the moments  $\Pi_n = \text{tr}(\rho^\Gamma)^n$  separately. We know how to measure  $\Pi_2$  (see Ref. [29]), since it is equivalent to the purity of  $\rho = \rho_{a_1, b_1}$ . The remaining problem is how to measure  $\Pi_{3,4}$ . As already mentioned, these moments were originally reproduced as the mean values of the observables constructed from permutation operators. However, the direct measurement of the observables seems to be difficult due to their relatively complicated structure. Fortunately, we can express the moments  $\Pi_n$  (for  $n = 3, 4$ ) differently, i.e., by decomposing the  $n$ th cycle into the products of inversions (the SWAP operations  $S$ ) as

$$\Pi_n = \text{tr}(A_n B_n \rho^{\otimes n}), \quad (7)$$

where  $n = 2, 3, 4$  and

$$\begin{aligned}
 A_2 &= S_{a_1, a_2} \otimes I_{b_1} \otimes I_{b_2}, \\
 B_2 &= I_{a_1} \otimes I_{a_2} \otimes S_{b_1, b_2}, \\
 A_3 &= S_{a_1, a_2} \otimes I_{a_3} \otimes I_{b_1} \otimes S_{b_2, b_3}, \\
 B_3 &= I_{a_1} \otimes S_{a_2, a_3} \otimes S_{b_1, b_2} \otimes I_{b_3}, \\
 A_4 &= S_{a_1, a_2} \otimes S_{a_3, a_4} \otimes I_{b_1} \otimes S_{b_2, b_3} \otimes I_{b_4}, \\
 B_4 &= I_{a_1} \otimes S_{a_2, a_3} \otimes I_{a_4} \otimes S_{b_1, b_2} \otimes S_{b_3, b_4},
 \end{aligned} \tag{8}$$

together with the SWAP operator  $S = |HH\rangle\langle HH| + |HV\rangle\langle VH| + |VH\rangle\langle HV| + |VV\rangle\langle VV|$  and the single-qubit identity operator  $I$ , where  $|H\rangle$  and  $|V\rangle$  are the states of the horizontally and vertically polarized photons, respectively. The products  $A_n B_n$  are not Hermitian for  $n = 3, 4$ , so they cannot be measured directly. However, the operators  $A_n$  and  $B_n$  taken separately are Hermitian and have other useful properties, i.e.,  $\text{tr}(A_n B_n \rho^{\otimes n}) = \text{tr}(B_n A_n \rho^{\otimes n})$  and  $A_n^2 = B_n^2 = I^{\otimes n}$ . By applying these properties we can express higher-order moments of the partially transposed matrix as

$$\Pi_n = \frac{1}{2} \text{tr}[(A_n + B_n)^2 \rho^{\otimes n}] - 1, \quad n = 3, 4. \tag{9}$$

Note that this method displays some analogy to the method of calculating the collective spin of two parties. Let us define  $X_n = (A_n + B_n)^2$ . Then, in order to measure these two moments, we have to perform projections on the eigenspaces of  $X_3$  and  $X_4$ . The implementation of these operations might be difficult for two reasons: (i) the large number of different eigenvalues of the operators  $X_n$  (positive-operator-valued measures) and (ii) the complicated structure of the corresponding eigenspaces with the eigenvectors corresponding to entangled multiqubit states. Fortunately, the operator  $X_3$  has only two different eigenvalues (1, 4), resulting in two eigenspaces, and the operator  $X_4$  has only three different eigenvalues (0, 2, 4), hence it has three eigenspaces. Therefore, one has to perform only the measurement of five projections on some of the eigenspaces to measure both  $\Pi_3$  and  $\Pi_4$ . The remaining problem is to find the eigenspaces and associate them with the specific settings of a multiphoton interferometer. Measuring the second moment  $\Pi_2$  requires two projections. Thus, the complete measurement of  $W_{\text{univ}}^{(4)}$  can be decomposed into seven projections (this value may be even lower if some optimization is applied) onto subspaces spanned by highly entangled multiqubit states, which is twice as efficient as a full two-qubit tomography. There is, however, another method of measuring the products of  $A_n$  and  $B_n$  for  $n = 3, 4$  that is better regarding the complexity of these projections. We can express  $A_n = P_n^+ - P_n^-$  in terms of the projectors onto the symmetric ( $P_n^+$ ) and antisymmetric ( $P_n^-$ ) subspaces. Then, as shown in the Appendix, we have

$$\Pi_n = \text{tr}[B_n P_n^+ \rho^{(n)} P_n^+] - \text{tr}[B_n P_n^- \rho^{(n)} P_n^-] \tag{10}$$

for an arbitrary  $\rho$ . It is convenient to define  $P_{m,n}^\pm = \frac{1}{2}(I_{a_m} \otimes I_{a_n} \pm S_{a_m, a_n})$  and  $\bar{P}_{m,n}^\pm = \frac{1}{2}(I_{b_m} \otimes I_{b_n} \pm S_{b_m, b_n})$ . Then the symmetric ( $P_n^+$ ) and antisymmetric ( $P_n^-$ ) projectors for  $n = 3, 4$  read

$$P_3^\pm = P_{1,2}^\mp \otimes \bar{P}_{2,3}^\pm + P_{1,2}^\pm \otimes \bar{P}_{2,3}^\mp, \tag{11}$$

$$P_4^\pm = P_3^\mp \otimes P_{3,4}^\pm + P_3^\pm \otimes P_{3,4}^\mp. \tag{12}$$

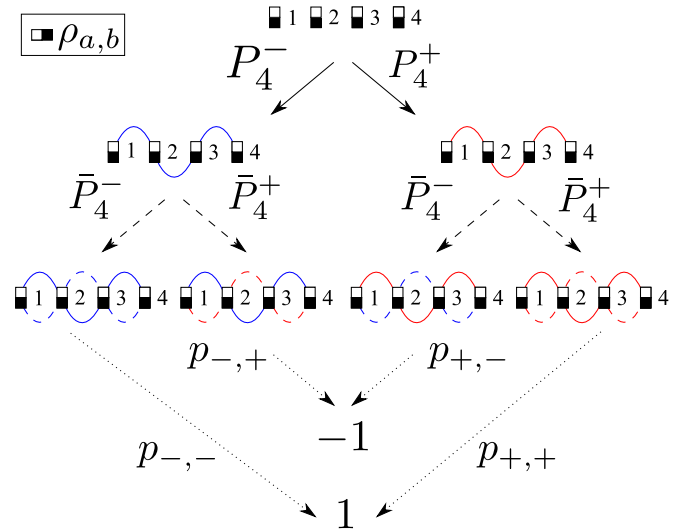


FIG. 2. (Color online) Conceptual diagram for measuring the moment  $\Pi_4$ . The four copies of a given two-photon state  $\rho_{a,b}$  are marked by small rectangles, where the white (black) part corresponds to the  $a$  ( $b$ ) photon. Solid (dashed) curves connect photons that are measured simultaneously at the first (second) stage. The state  $\rho_{a,b}$  is split into the symmetric and antisymmetric parts by the projectors  $P_4^+$  and  $P_4^-$  (solid arrows), respectively. This can be considered as a transformation that is deterministic in principle. Next, the two branches are split again into the symmetric (by the projector  $\bar{P}_4^+$ ) and antisymmetric ( $\bar{P}_4^-$ ) parts (dashed arrows). These projectors are applied to the two subgroups of qubits, as indicated by the red (blue) curves for the symmetric (antisymmetric) subspace projections. There are four possible outcomes of this procedure. The events, indicated by dotted arrows, correspond to measuring the value  $+1$  ( $-1$ ) with the probability  $p_{\pm,\pm}$  ( $p_{\mp,\pm}$ ), where  $p_{+,+} + p_{+,-} + p_{-,-} + p_{-,+} = 1$  and  $\langle \Pi_n \rangle = p_{+,+} - p_{+,-} - p_{-,-} + p_{-,+}$ . The same procedure can be used for measuring the lower-order moments  $\Pi_n$  for  $n = 2, 3$  if the last  $(4 - n)$  copies of  $\rho_{a,b}$  are removed.

For the operator  $B_n = \bar{P}_n^+ - \bar{P}_n^-$ , we can apply the same procedure but with the subsystems of the multiqubit density matrix swapped as  $a \leftrightarrow b$ . Then, we have

$$\Pi_n = \text{tr}(\bar{P}_n^+ Q \bar{P}_n^+) - \text{tr}(\bar{P}_n^- Q \bar{P}_n^-), \tag{13}$$

where  $Q = P_n^+ \rho^{(n)} P_n^+ - P_n^- \rho^{(n)} P_n^-$ , which means that

$$\Pi_n = \sum_{x,y=0}^1 (-1)^{x+y} \text{tr}[\bar{P}_n^x P_n^y \rho^{(n)} P_n^y \bar{P}_n^x], \tag{14}$$

where  $\bar{P}_n^0 = \bar{P}_n^+$  and  $\bar{P}_n^1 = \bar{P}_n^-$ . Thus, it appears to be more convenient to project  $\rho^{(n)}$  onto the symmetric or antisymmetric subspace of  $A_n$  first and then to measure  $B_n = \bar{P}_n^+ - \bar{P}_n^-$ , as shown in Fig. 2.

#### IV. MAKHLIN'S INVARIANTS AND MINIMAL NUMBER OF INDEPENDENT MEASUREMENTS

The moments  $\Pi_{\{1,2,3,4\}}$  are interrelated. In order to demonstrate this property, let us express the two-qubit density matrix  $\rho$  in terms of the Pauli matrices  $\sigma_i$  for  $i = 1, 2, 3$  and the single-qubit identity matrix  $\sigma_0$ . The resulting matrix

reads

$$\rho = \frac{1}{4}\sigma_0 \otimes \sigma_0 + \frac{1}{2}s_i\sigma_i \otimes \sigma_0 + \frac{1}{2}p_j\sigma_0 \otimes \sigma_j + \beta_{ij}\sigma_i \otimes \sigma_j, \quad (15)$$

where the elements of the correlation matrix  $\hat{\beta}$  are  $\beta_{ij} = \text{tr}[(\sigma_i \otimes \sigma_j)\rho]$  and the Bloch vectors  $\mathbf{s}$  and  $\mathbf{p}$  have the following elements of  $s_i = \text{tr}[(\sigma_i \otimes \sigma_0)\rho]$  and  $p_j = \text{tr}[(\sigma_0 \otimes \sigma_j)\rho]$ , respectively. It can be directly shown, after tedious calculations, that

$$\begin{aligned} \Pi_1 &= 1, \\ 4\Pi_2 &= 1 + x_1, \\ 16\Pi_3 &= 1 + 3x_1 + 6x_2, \\ 64\Pi_4 &= 1 + 6x_1 + 24x_2 + x_1^2 + 2x_3 + 4x_4, \end{aligned} \quad (16)$$

where

$$\begin{aligned} x_1 &= I_2 + I_4 + I_7, & x_2 &= I_1 + I_{12}, \\ x_3 &= I_2^2 - I_3, & x_4 &= I_5 + I_8 + I_{14} + I_4 I_7 \end{aligned} \quad (17)$$

are functions of nine local invariants of the two-qubit matrix  $\rho$  as defined by Makhlin in Ref. [30], i.e.,  $I_1 = \det \hat{\beta}$ ,  $I_2 = \text{tr}(\hat{\beta}^T \hat{\beta})$ ,  $I_3 = \text{tr}(\hat{\beta}^T \hat{\beta})^2$ ,  $I_4 = \mathbf{s}^2$ ,  $I_5 = [\mathbf{s}\hat{\beta}]^2$ ,  $I_7 = \mathbf{p}^2$ ,  $I_8 = [\hat{\beta}\mathbf{p}]^2$ ,  $I_{12} = \mathbf{s}\hat{\beta}\mathbf{p}$ , and  $I_{14} = e_{ijk}e_{lmn}s_i p_l \beta_{jm} \beta_{kn}$ , where  $e_{ijk}$  is the Levi-Civita symbol. It is apparent that only six (instead of nine) linear combinations of Makhlin's invariants need to be measured to estimate the values of  $x_n$  for  $n = 1, \dots, 4$ . These invariants read

$$\begin{aligned} y_1 &= I_2, & y_2 &= I_3, & y_3 &= I_4, & y_4 &= I_7, \\ y_5 &= I_1 + I_{12}, & y_6 &= I_5 + I_8 + I_{14}. \end{aligned} \quad (18)$$

Thus, in order to detect the entanglement via  $\det \rho^\Gamma$ , one needs to measure exactly six instead of nine independent linear combinations of fundamental invariants. It also happens that this is also the minimal number of independent fundamental quantities describing the negativity of an arbitrary two-qubit state [31].

### V. PROPOSAL OF EXPERIMENTAL IMPLEMENTATION

The analyzed projections  $P_n^\pm$  are not the products of projections (except  $n = 2$ ), thus they cannot be implemented by local (two-qubit) operations. Let us note that the  $P_n^\pm$  projectors split a collective multiqubit state into the states of positive and negative parities. This technique was also applied to, e.g., the cluster-state preparation [32].

We can, however, measure  $B_3 P_3^\pm$  directly as shown in Fig. 3. Note that  $B_3$  can be measured using only beam splitters and photon detectors analogously to the methods applied in Refs. [29,33,34]. In Figs. 3 and 4 we show a simple implementation of a  $B_n$  block (for  $n = 3, 4$ ). To measure the three parameters from Fig. 2 (four parameters without normalization) instead of using the  $B_n$  block we can reuse the  $P_n^\pm$  part of the relevant setup to perform a  $\bar{P}_n^\pm$  projection (see Sec. VI).

For any  $P_n^\pm$ , two qubits  $a_1$  and  $a_n$  ( $b_n$ ) for even (odd)  $n$  can be destroyed in this process. In the most complex case of  $P_4^\pm$ , we have to perform a nondestructive parity test on six qubits, where two of them can be destroyed before measuring  $B_4$ . The

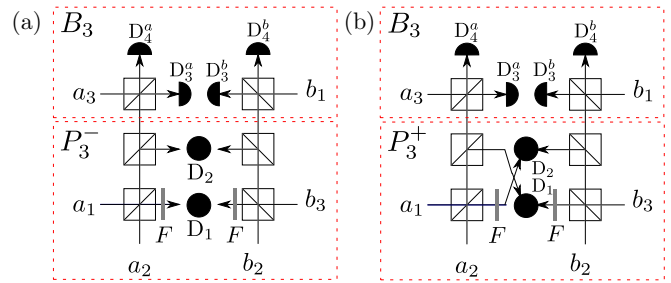


FIG. 3. (Color online) Linear-optical setup for measuring  $B_3 P_3^\pm$  directly. The principle of its operation is based on the fact that a beam splitter (BS) performs the projection  $P^+$  ( $P^-$ ) if the incoming photons in  $b_1$  and  $b_2$  are bunched (antibunched). If they are bunched, at least one photon is passed to the detection mode of the first BS. Next, if no photon is passed to the detection mode of the second BS, the output state is  $\text{tr}_{b_1}(P^- \rho_{b_1, b_2} P^-)$ . However, if there is no photon in the first detection mode and there is a photon in the second detection mode, the output state becomes  $\text{tr}_{b_1}(P^+ \rho_{b_1, b_2} P^+)$ . The setup works if both detectors  $D_1$  and  $D_2$  register a photon, and it is unknown from where the photons have arrived. The  $B_3$  part is implemented by distinguishing between  $P^+$  and  $P^-$  by means of detecting bunching and antibunching, respectively. To guarantee the optimality of this setup with respect to the minimal number of measured quantities, the information about parities of individual photon pairs should be erased as described in Sec. VI B. Here we assume that this is done by using photon-number-resolving detectors  $D_4^{a(b)}$ , but this can also be done probabilistically using bucket detectors [29,33]. Thus, assuming perfect detectors, one needs two measurements to determine  $\Pi_3$ . Finally, neutral density filters  $F$  of the transmittance  $1/2$  ensure that the setup works with probability  $1/16$ .

measurement of  $B_4 P_4^\pm$  is more challenging than that of  $\Pi_3$ . In the comparison to the  $\Pi_3$  setup, the main difficulty here is the necessity to condition the outcome of nondestructive measurements on  $b_2$  and  $b_3$ . This is because both  $b_2$  and  $b_3$

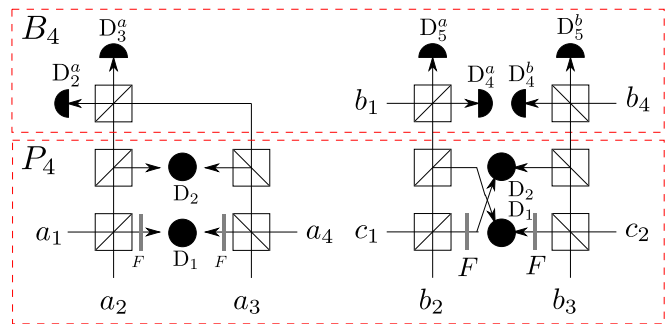


FIG. 4. (Color online) Linear-optical setup for direct measuring  $B_4 P_4^\pm$ . The principle of its operation is similar to that in Fig. 3. Here, for simplicity, we assume that the detectors  $D_1$  and  $D_2$  can distinguish between the even and odd numbers of photons. Note that this assumption is irrelevant in the setups discussed in Sec. VI. If the even (odd) number of photons is passed to detector  $D_1$ , the measurement result is that of the  $B_4$  measurement multiplied by  $1$  ( $-1$ ). The ancillary modes  $c_1$  and  $c_2$  are prepared in the polarization singlet state. Note that the right-hand-side module corresponds to that shown in Fig. 3(b) but with the replaced notation for the input modes and detectors:  $(a_1, a_2, a_3, b_1, b_2, b_3) \rightarrow (c_1, b_2, b_1, b_4, b_3, c_2)$  and  $(D_3^a, D_3^b, D_4^a, D_4^b) \rightarrow (D_4^a, D_4^b, D_5^a, D_5^b)$ , respectively.

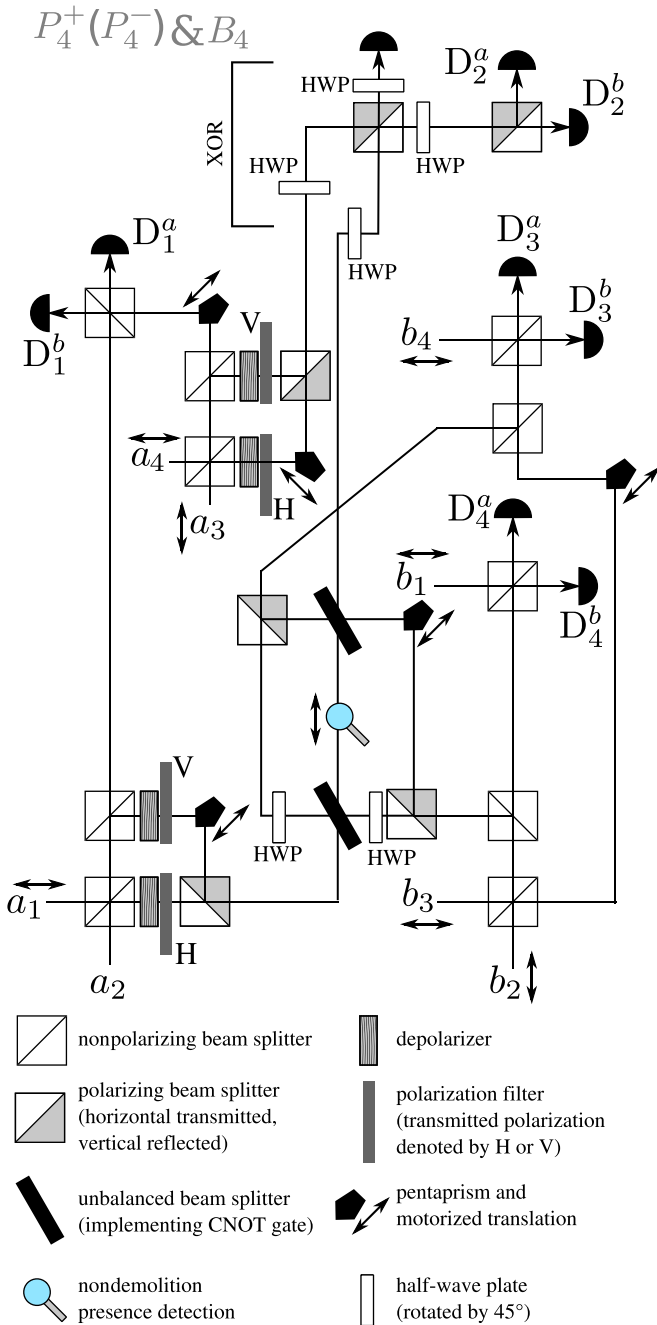


FIG. 5. (Color online) Alternative setup for the direct measurement of  $B_4P_4^\pm$ . The method uses three known quantum gates: the XOR, the CNOT (i.e., the reversible XOR), and the nondemolition presence detection gate. All these gates can be implemented using linear optics only. The XOR gate can easily be constructed using a polarizing beam splitter and a set of half-wave plates (HWP) (see, e.g., Ref. [40]). The CNOT gate can be built using a special partially polarizing beam splitter (see, e.g., Ref. [36]). Finally, the nondemolition presence detection can be achieved with the assistance of two ancillas [41] in the polarization singlet state. The method is successful if there are two photons registered by each pair of the photon-number-resolving detectors  $D_1$ ,  $D_3$ , and  $D_4$ , while one photon is detected by either  $D_2^a$  or  $D_2^b$ . It is easy to show that the photon detection by  $D_2^a$  corresponds to the  $B_4P_4^-$  measurement, while the detector  $D_2^b$  heralds the  $B_4P_4^+$  measurement.

are required in the latter part of the  $\Pi_4$  measurement. We can solve this problem by using ancillary photons prepared in the polarization singlet state in the modes  $c_1$  and  $c_2$ . The corresponding setup is shown in Fig. 4. In some experimental approaches, this setup can be further simplified by applying, e.g., the time-bin methods [33,35]. Our alternative proposal is discussed in the next section and shown in Fig. 5. Finally, note that the relevant moments  $\Pi_n$  can be measured with the subblocks of the setup in Fig. 4 if some information is ignored.

VI. ALTERNATIVE IMPLEMENTATIONS

A. Alternative setup for measuring  $\Pi_4$

The setup depicted in Fig. 4 is fairly efficient and experimentally not overly demanding. Unfortunately, it requires the photon-number parity measurement, which (i) is experimentally challenging and (ii) has photon losses in the setup that can result in incorrect measurement outcome. Especially the second limitation hinders the implementation of the method since photon losses and imperfect detection efficiencies are unavoidable in real experiments.

To overcome this problem, in addition to the time-bin approach as already mentioned, we have devised another setup, as shown in Fig. 5, for the direct measurement of the  $B_4P_4^\pm$  term. In contrast to the previous setup, shown in Fig. 4, no parity measurement is required. On the other hand, the new setup is much more complex and requires interferometric stability (at some places). The idea behind the method is to use two quantum gates: the controlled-NOT (CNOT) gate [36–39] and the exclusive OR (XOR) gate [40]. In order to join two CNOT gates, it is also required to introduce the nondemolition photon presence detection gate, which uses two additional ancillas in a Bell state [41,42]. By heralding the presence of a qubit, this gate informs that the preceding CNOT operation was successful. The entire measurement method is successful if two photons are detected by each detector pair among  $D_1$ ,  $D_3$ , and  $D_4$ , while one photon is detected by either of the  $D_2$  detectors and the presence detection gate also heralds a photon. Further, if the detector  $D_2^a$  fires, the method performs the  $B_4P_4^-$  measurement, while if the photon impinges on the  $D_2^b$  detector, the setup implements the  $B_4P_4^+$  measurement.

B. Alternative implementation of the  $B_{3(4)}$  block

If we take a look at the  $B_n$  blocks with  $n = 3, 4$  in Figs. 3 and 4, respectively, we will discover that we check for bunching and antibunching separately for each photon pair. In addition to the information about the outcome of the  $\bar{P}_n^\pm$  projections, we obtain the information about which of the photon pairs is bunched or antibunched. We do not use the which-pair information in any way (we just need to know how many pairs bunched or antibunched) and this measurement is not difficult to implement. However, one may argue that we gain more knowledge from our measurement that is necessary to measure  $\Pi_n$ .

To perform the  $B_n$  measurement and not to distinguish between the pairs of photons one would have to use the same block as for the  $P_n^\pm$  measurement and to swap the modes  $a \leftrightarrow b$

to perform the  $\bar{P}_n^\pm$  projections. This procedure would result in the four separate detection events  $P_n^+ \bar{P}_n^+$ ,  $P_n^+ \bar{P}_n^-$ ,  $P_n^- \bar{P}_n^+$ , and  $P_n^- \bar{P}_n^-$  [see Eq. (14)] associated with the single observable  $\Pi_n$ . This number of detection events is now smaller than in the case of analyzing bunching and antibunching for each pair at the original  $B_n$  blocks. The drawback of this method is that it is more experimentally challenging. However, by using the time-bin approach, analogously to that applied in Ref. [33], we could reuse the same physical  $P_n^\pm$  setup to measure  $\bar{P}_n^\pm$  at a later moment of time. To summarize, we may iterate the block measuring  $P_n^\pm$  to get exactly the statistics corresponding *only* to four exclusive events.

## VII. DISCUSSION AND CONCLUSION

We showed how to directly measure the universal entanglement witness  $W_{\text{univ}}^{(4)}$  by using four copies of an arbitrary two-qubit state  $\rho_{a,b}$  of a two-photon polarization state. Our approach consists of three measurements associated with three moments of the partially transposed matrix  $\rho_{a,b}^\Gamma$ , i.e.,  $\Pi_n = \text{tr}[(\rho^\Gamma)^n]$  for  $n = 2, 3, 4$ . The key issue is to calculate the number of parameters that were estimated in the process of measurement. Figure 2 shows us that we can estimate two independent quantities for each of the three moments  $\Pi_{\{2,3,4\}}$ , which after normalization become the probabilities  $p_+ = p_{-,-} + p_{+,+}$  and  $p_- = p_{-,+} + p_{+,-}$ . Consequently, the output of the setup provides six parameters that are generally independent. Note that there are nine parameters if the normalizations are included.

The moments  $\Pi_{\{2,3,4\}}$  are interrelated. Each higher moment is a function of lower moments and some additional parameters. After tedious calculations, we showed that  $\Pi_{\{2,3,4\}}$  are functions of nine fundamental Makhlin invariants [30] of  $\rho$ . The relevant invariants are  $I_{\{1,2,3,4,5,7,8,12,14\}}$  (see Sec. IV). Under closer inspection we discovered that only six fundamental quantities are needed to estimate the values of  $\Pi_{\{2,3,4\}}$ , i.e., to detect entanglement via  $\det \rho^\Gamma$  (or to measure the negativity of an arbitrary two-qubit state [31]). These are  $y_1 = I_2$ ,  $y_2 = I_4$ ,  $y_3 = I_7$ ,  $y_4 = I_1 + I_{12}$ ,  $y_5 = I_5 + I_8 + I_{14}$ , and  $y_6 = I_3$ . Thus, our setup estimates no more quantities than those.

Note that local qubit unitary operations have three relevant independent real parameters (excluding global phase). Thus, the number of parameters of  $U_A \otimes U_B$  is 6, while the total number of parameters of a mixed two-qubit state is 15. The resulting  $9 = 15 - 6$  parameters are exactly all the relevant ones after introducing the  $U_A \otimes U_B$  invariant equivalence classes and they correspond to the nine fundamental invariants. This number of parameters can be further reduced by swapping the subsystems of  $\rho$ .

This is probably the reason behind the minimalist character of this method. Indeed, it requires no unitary operations, which may reflect the symmetry of the problem under local unitary operations. Because of its simplicity, we believe that the presented setup paves the way for the experimental realization of a necessary and sufficient universal test of entanglement.

Finally, let us underscore that the key feature responsible for the success of the proposed approach is the sequential character of measurements. It seems that this property of the setup reframes the paradigm for entanglement, correlations, and any other nonlocal (i.e., not depending solely on the reduced

density matrices of subsystems) property of quantum-state detection and/or estimation in practice. As a result a more general problem can be conceived of. Given only very specific measurement modules (analogous to a beam splitter in the Hong-Ou-Mandel interference experiment), which can be repeated in different subsystems, is it possible to estimate nonlocal quantities and, if so, what setup would minimize the number of required measurements?

We developed a general method of the measurement of invariant-based moments of partially transposed density matrices. Our detailed description of the method is focused on the detection of the entanglement of two qubits. This method can be generalized and applied to measure (at least some) moments of partially transposed density matrices of higher-dimensional systems too. Nevertheless, such detection setups can be, in general, more complicated than the setups for quantum-state tomography. However, let us stress again the main result of our paper, which is a proposal of an experimental entanglement detection without performing a complete quantum-state tomography. Our method enables us to detect the entanglement between two qubits in an arbitrary state if and only if these qubits are entangled. The main idea of our method is based on the measurement of the universal entanglement witness, which is a necessary and sufficient entanglement condition for an arbitrary state of only two qubits. In this sense, our universal two-qubit entanglement detection is not directly scalable for two-qudit or multiqubit systems.

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## APPENDIX: DERIVATION OF EQ. (10)

In order to derive Eq. (10), let us start by noting that

$$\text{tr}(B_n A_n \rho^{\otimes n}) = \text{tr}[B_n A_n (\rho^{\otimes n})'], \quad (\text{A1})$$

where the statistical operator

$$(\rho^{\otimes n})' = \frac{1}{2}(\rho^{\otimes n} + A_n \rho^{\otimes n} A_n) \quad (\text{A2})$$

can be implemented by alternating between the input state  $\rho^{\otimes n}$  and  $A_n \rho^{\otimes n} A_n$  (qubits are swapped according to the definition of  $A_n$ ). The newly obtained density matrix  $(\rho^{\otimes n})'$  has a very important property, i.e., it commutes with the operator  $A_n$ :

$$[A_n, (\rho^{\otimes n})'] = \frac{1}{2}([A_n, \rho^{\otimes n}] + [\rho^{\otimes n}, A_n]) = 0. \quad (\text{A3})$$

Thus, the two operators have a common set of eigenvectors  $|\psi_m^{(n)}\rangle$  for  $m = 1, 2, \dots, 4^n$ , i.e., they are both diagonal in the same basis. Let us expand the expression  $\text{tr}[B_n A_n (\rho^{\otimes n})']$  using the diagonal representations of the operators  $A_n = \sum_k a_k^{(n)} |\psi_k^{(n)}\rangle \langle \psi_k^{(n)}|$ ,  $B_n = \sum_l b_l^{(n)} |\phi_l\rangle \langle \phi_l|$  and  $(\rho^{\otimes n})' = \sum_m r_m^{(n)} |\psi_m^{(n)}\rangle \langle \psi_m^{(n)}|$ . By doing so, we arrive

at

$$\begin{aligned} & \text{tr}[B_n A_n (\rho^{\otimes n})'] \\ &= \sum_{k,l,m} \text{tr}(a_k^{(n)} b_l^{(n)} r_m^{(n)} |\phi_l^{(n)}\rangle\langle\phi_l^{(n)}| |\psi_k^{(n)}\rangle\langle\psi_k^{(n)}| |\psi_l^{(n)}\rangle\langle\psi_l^{(n)}|) \\ &= \sum_{k,l} a_k^{(n)} b_l^{(n)} r_k^{(n)} |\langle\phi_l^{(n)}|\psi_k^{(n)}\rangle|^2. \end{aligned} \quad (\text{A4})$$

This is equivalent to a measurement strategy consisting of measuring  $A_n$  first and then measuring  $B_n$ , which can be expressed as

$$\begin{aligned} & \sum_k \text{tr}[B_n a_k^{(n)} |\psi_k^{(n)}\rangle\langle\psi_k^{(n)}| (\rho^{\otimes n})' |\psi_k^{(n)}\rangle\langle\psi_k^{(n)}|] \\ &= \sum_{k,l} a_k^{(n)} b_l^{(n)} |\langle\phi_l^{(n)}|\psi_k^{(n)}\rangle|^2 |\langle\psi_k^{(n)}|(\rho^{\otimes n})'|\psi_k^{(n)}\rangle| \\ &= \sum_{k,l} a_k^{(n)} b_l^{(n)} r_k^{(n)} |\langle\phi_l^{(n)}|\psi_k^{(n)}\rangle|^2. \end{aligned} \quad (\text{A5})$$

The operator  $A_n$  has degenerated eigenvalues, which makes its set of eigenvectors not unique, i.e., any linear combination of two eigenvectors associated with the same eigenvalue is an eigenvector itself. Thus, finding the basis  $\{|\psi_k^{(n)}\rangle\}$ , in which both  $A_n$  and  $(\rho^{\otimes n})'$  are diagonal, seems to be a difficult problem that depends on the particular form of  $\rho$ . So this approach is not universal. However, now we can express  $A_n = P_n^+ - P_n^-$  in terms of the projectors onto the symmetric ( $P_n^+$ ) and antisymmetric ( $P_n^-$ ) subspaces. From the above it follows that

$$\Pi_n = \text{tr}[B_n P_n^+ (\rho^{(n)})' P_n^+] - \text{tr}[B_n P_n^- (\rho^{(n)})' P_n^-]. \quad (\text{A6})$$

Moreover, we can see that

$$P_n^\pm (\rho^{(n)})' P_n^\pm = P_n^\pm \rho^{(n)} P_n^\pm. \quad (\text{A7})$$

Thus, we derive

$$\Pi_n = \text{tr}[B_n P_n^+ \rho^{(n)} P_n^+] - \text{tr}[B_n P_n^- \rho^{(n)} P_n^-], \quad (\text{A8})$$

which corresponds to Eq. (10).

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