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# Minimal surfaces and conservation laws for bidimensional structures

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**Abstract**
We discuss conservation laws for thin structures which could be modeled as a material minimal surface, i.e., a surface with zero mean curvatures. The models of an elastic membrane and micropolar (six-parameter) shell undergoing finite deformations are considered. We show that for a minimal surface, it is possible to formulate a conservation law similar to three-dimensional non-linear elasticity. It brings us a path-independent *J*-integral which could be used in mechanics of fracture. So, the class of minimal surfaces extends significantly a possible geometry of two-dimensional structures which possess conservation laws.

**Keywords**
Conservation law, minimal surface, membrane, micropolar shell, finite deformations

## 1. Introduction

The conservation laws play a central role in continuum physics. Indeed, it is worth to mention conservation laws of mass, energy, momentum, and moment of momentum [1,2]. In addition to these classic conservation laws, it is possible to establish other, trivial or non-trivial, conservation laws [2–5]. Let us briefly recall the definition of a conservation law. Let a problem under consideration be described through a set of functions of many variables $u_i = u_i(x_j)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, which satisfy a system of partial differential equations (PDEs):

$$l_p\left(x_j, u_i, \frac{\partial u_i}{\partial x_j}, \ldots\right) = 0, \quad p = 1, \ldots, k. \tag{1}$$

Let $\vec{P} = (P_1, \ldots, P_n) \in \mathbb{R}^n$ be a vector-valued function with components:

$$P_q = P_q\left(x_j, u_i, \frac{\partial u_i}{\partial x_j}\right), \quad q = 1, ..., n.$$

Then, if the following equation:

$$\mathrm{div}\vec{P} \equiv \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i} = 0, \tag{2}$$

holds true for any solution of equation (1), it is called a *conservation law*.

For derivation of conservation laws in the three-dimensional (3D) elasticity, one can apply various techniques including Noether's theorem and its extensions such as the Bessel–Hagen and the neutral action methods (see the previous works [5–9]). After Noether, it is known that conservation laws are closely related to invariance properties of a total energy functional that called also variational symmetries. For example, a homogeneity, i.e., local invariance with respect to infinitesimal translations, results in conservation law for the Eshelby tensor [2,4,5], which brings us well-known path-independent *J*-integral and some other invariant integrals. Conservation laws are widely used in mechanics of fracture, theory of stress-induced phase transitions, and for description of other inhomogeneity in solids [2,3,5].

Instead, in the case of two-dimensional (2D) structures such as shells, one faces a problem of homogeneity as a shell is an inhomogeneous 2D medium, since its geometry is point-dependent, in general. The 3D conservation laws could be transformed for plane geometry, i.e., for plates (see results for first-order shear-deformable linear plates [10,11], linear second-order plate theory [12], and von Kármán plates [13]). As a result, unlike to plate theory conservation laws for shells were established for particular geometries, such as spherical, cylindrical, or shells of revolution (see the previous works [5,14,15]). Path-independent integrals were introduced for cylindrical shells and shells of revolution within the Sanders–Koiter variant of linear shell theory and non-linear membrane theory in Lo [16]. Conservation laws are also known for linear shallow shell model with applications to cracked cylindrical shell [17], and non-linear shallow shell models [18] including the Marguerre–von Kármán theory [19]. In fact, the concept of shallow shell inherits its plane geometry from plates.

The aim of this paper is to discuss new conservation laws for 2D structures which could be modeled using a minimal surface as a base surface carrying physical properties of the structure. The principal property of a minimal surface is zero mean curvature (see the previous works [20–22]) for basic properties of the minimal surfaces. Recently, some structures based on minimal surface geometry were proposed for advanced composites (see, e.g., the previous works [23–26]). Let us note that one can easily meet minimal surfaces in natures, e.g., as seashells [27,28].

The paper is organized as follows. First, in section 2, we briefly recall necessary information from differential geometry including the surface divergence theorems. In section 3, we discuss the kinematics of a material surface considering membrane theory [29] and enriched (Cosserat-like) surfaces. The latter model has straightforward relation to micropolar shells [30,31] called also six-parameter shell model [32]. It could be treated as 2D Cosserat continuum, i.e., a 2D medium with translational and rotational degrees of freedom, and with surface stresses and surface couple stresses. In sections 4 and 5, we introduced the Eshelby tensors for these models and present the corresponding conservation laws and invariant integrals. Modeling stress-induced phase transformations in micropolar shells, the 2D Eshelby tensor was introduced in Eremeyev and Pietraszkiewicz [33], whereas its relation to the 3D counterpart was discussed in Eremeyev and Konopińska-Zmysłowska [34]. Recently, the 2D Eshelby tensor was also used for modeling of adhesion of thin structures [35]. In section 6, we briefly discuss 3D-to-2D reduction as an alternative way of derivation of 2D conservation laws.

In what follows, we almost always use the direct (index-free) tensor calculus as in the previous works [36,37].

## 2. Preliminaries

First, let us briefly introduce some formulae of differential geometry. Let $\Sigma \in \mathbb{R}^3$ be a smooth enough surface with a boundary $\Gamma = \partial\Sigma$. $\Sigma$ could be parameterized with a position vector given as a function of two surface coordinates $s^1$ and $s^2$:

$$\mathbf{X} = \mathbf{X}(s^1, s^2) = X_1(s^1, s^2)\mathbf{i}_1 + X_2(s^1, s^2)\mathbf{i}_2 + X_3(s^1, s^2)\mathbf{i}_3, \tag{3}$$

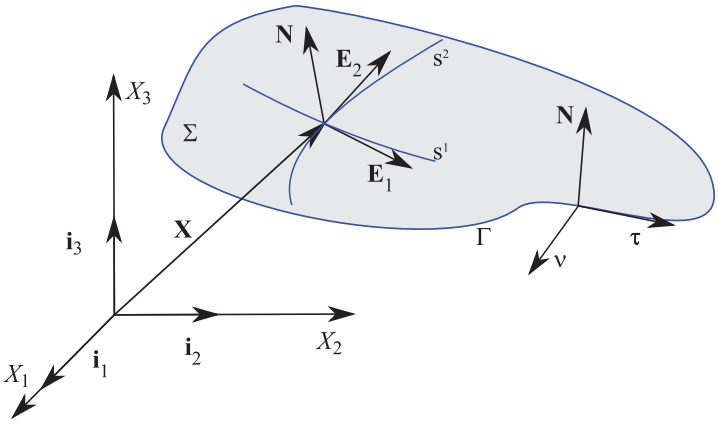


**Figure 1.** Surface with parametrization.

where $X_j$ and $\mathbf{i}_j$ are the Cartesian coordinates and corresponding unit base vectors, respectively (see Figure 1). We introduce the surface nabla-operator $\nabla$ and the natural and reciprocal base vectors as follows:

$$\nabla = \mathbf{E}^\alpha \frac{\partial}{\partial s^\alpha}, \quad \mathbf{E}^\alpha \cdot \mathbf{E}_\beta = \delta^\alpha_\beta, \quad \mathbf{E}^\alpha \cdot \mathbf{N} = 0, \quad \mathbf{E}_\beta = \frac{\partial \mathbf{X}}{\partial s^\beta}, \quad \alpha, \beta = 1, 2,$$

where $\delta^\alpha_\beta$ is the Kronecker symbol, $\mathbf{N} = \frac{\mathbf{E}^1 \times \mathbf{E}^2}{|\mathbf{E}^1 \times \mathbf{E}^2|}$ is the unit normal to $\Sigma$, and "$\cdot$" and "$\times$" denote the dot and cross products, respectively. Hereinafter, Greek indices take values 1 and 2, whereas Latin indices will take values 1, 2, and 3, and Einstein's summation rule is used.

For any differentiable surface field $\mathbf{T}$, we introduce the surface divergence theorem (the Gauss–Ostrogradsky theorem) [29,37]:

$$\iint_\Sigma (\nabla \cdot \mathbf{T} + H\,\mathbf{N} \cdot \mathbf{T})\, d\Sigma = \oint_\Gamma \boldsymbol{\nu} \cdot \mathbf{T}\, ds. \tag{4}$$

where $H \equiv -\frac{1}{2}\nabla \cdot \mathbf{N}$ is the mean curvature of $\Sigma$, $\boldsymbol{\nu}$ is the external unit normal to $\Gamma = \partial\Sigma$ such that $\boldsymbol{\nu} \cdot \mathbf{N} = 0$. Let us note that $\mathbf{T}$ could be a vector-valued or tensor-valued surface field of any order. There are other forms of the surface divergence theorem:

$$\iint_\Sigma (\nabla \mathbf{T} + H\,\mathbf{N} \otimes \mathbf{T})\, d\Sigma = \oint_\Gamma \boldsymbol{\nu} \otimes \mathbf{T}\, ds, \tag{5}$$

$$\iint_\Sigma (\nabla \times \mathbf{T} + H\,\mathbf{N} \times \mathbf{T})\, d\Sigma = \oint_\Gamma \boldsymbol{\nu} \times \mathbf{T}\, ds, \tag{6}$$

$$\iint_\Sigma \nabla \times (\mathbf{N} \otimes \mathbf{T})\, d\Sigma = \oint_\Gamma \boldsymbol{\tau} \otimes \mathbf{T}\, ds. \tag{7}$$

In equations (5) and (7), "$\otimes$" is the dyadic product, $\boldsymbol{\tau}$ denotes the unit vector tangent to $\Gamma$, $\boldsymbol{\tau} \times \boldsymbol{\nu} = \mathbf{N}$ (see Figure 1).

Obviously, the form of surface divergence theorems (4)–(6) differs from its 3D counterparts due to presence of terms related to the mean curvature. If the mean curvature of $\Sigma$ vanishes, that is if:

$$H = 0, \tag{8}$$

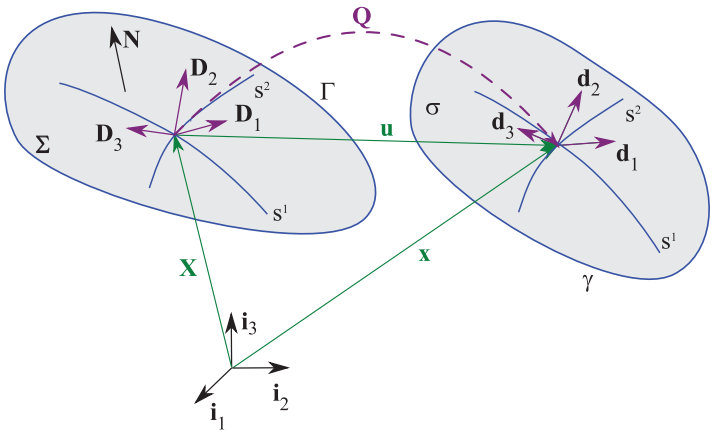


**Figure 2.** Deformation of a material surface $\mathcal{S}$.

then there is no such difference. A surface whose mean curvature is zero at any point is called a *minimal surface*. Basic properties of the minimal surfaces can be found in the previous works [20–22].

## 3. Kinematics of a material surface

In what follows, we utilize the concept of a material surface [29]. In the theory of plates and shells, it is also called the direct approach [38]. We introduce a deformation of a material surface $\mathcal{S}$ as a differentiable mapping from a reference placement $\kappa$ into a current placement $\chi$. Let $\Sigma$ and $\sigma$ be surfaces describing $\mathcal{S}$ in $\kappa$ and $\chi$, respectively. Within the Lagrangian description, we introduce a displacement vector $\mathbf{u}$ of a point $z \in \mathcal{S}$ with coordinates $s^1$ and $s^2$ defined on $\Sigma$ as follows:

$$\mathbf{u} = \mathbf{u}(s^1, s^2) = \mathbf{x} - \mathbf{X}, \tag{9}$$

where $\mathbf{x} = \mathbf{x}(s^1, s^2)$ and $\mathbf{X} = \mathbf{X}(s^1, s^2)$ are the position vectors of $z$ in $\kappa$ and $\chi$, respectively (see Figure 2).

In order to describe deformations of kinematically enriched (Cosserat-like) material surfaces, in addition to position vector of $z \in \mathcal{S}$, we consider two triples of unit orthogonal vectors called directors. So, we have two triples $\{\mathbf{D}_k\}$ and $\{\mathbf{d}_k\}$, $k = 1, 2, 3$, defined in reference and current placements, respectively. Using these triples, we introduce an orthogonal tensor:

$$\mathbf{Q} = \mathbf{Q}(s^1, s^2) = \mathbf{D}_k \otimes \mathbf{d}_k, \tag{10}$$

as a complementary kinematical descriptor (see the previous works [30,37] for more details). As a result, for enriched material surface, we have two kinematical descriptors $\mathbf{u}$ (or $\mathbf{x}$) and $\mathbf{Q}$ which could be treated as translational and rotational degrees of freedom used in the theory of shells [30–32].

## 4. Eshelby tensor and conservation laws: elastic membrane

In order to discuss a derivation of conservation laws for thin structures modeled using the minimal surface property, first let us study a simple case, that is an elastic membrane.

### 4.1. Finite deformations

For a hyperelastic membrane, there exists a surface strain energy $W$. In what follows, we restrict ourselves to homogeneous membranes, so $W$ does not depend on $\mathbf{X} \in \Sigma$. So, it is a function of the surface deformation gradient $\mathbf{F} = \nabla \mathbf{x}$:

$$W = W(\mathbf{F}). \tag{11}$$

Applying to equation (11), the material frame-indifference principle [39], we came to the dependence:

$$W = W(\mathbf{C}), \tag{12}$$

where $\mathbf{C} = \mathbf{F} \cdot \mathbf{F}^T$ is the surface Cauchy–Green strain measure [29]. Note that for simplicity, we keep in equation (12) the same notation for the energy function.

Neglecting surface forces, we have the following Lagrangian equilibrium equation:

$$\nabla \cdot \mathbf{P} = \mathbf{0}, \tag{13}$$

where $\mathbf{P}$ is the surface first Piola–Kirchhoff stress tensor. It is given by the formulae:

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}} = \mathbf{S} \cdot \mathbf{F}, \quad \mathbf{S} = 2\frac{\partial W}{\partial \mathbf{C}}.$$

where $\mathbf{S}$ is the surface second Piola–Kirchhoff stress tensor. Note that $\mathbf{N} \cdot \mathbf{P} = \mathbf{0}$. Equations (11)–(13) constitute a 2D counterpart of governing equations of the 3D non-linear elasticity. So, an elastic membrane model could be treated as a 2D Cauchy continuum, or as a 2D simple medium in sense of Noll and his colleagues [39,40].

For an elastic membrane, the Eshelby tensor is defined as follows:

$$\mathbf{B}_m = W\mathbf{A} - \mathbf{P} \cdot \mathbf{F}^T = W\mathbf{A} - \mathbf{S} \cdot \mathbf{C}, \tag{14}$$

where $\mathbf{A} = \mathbf{I} - \mathbf{N} \otimes \mathbf{N}$ and $\mathbf{I}$ is the 3D unit tensor. So, by definition, $\mathbf{N} \cdot \mathbf{B}_m = \mathbf{0}$. $\mathbf{B}_m$ has also the following property:

$$\nabla \cdot \mathbf{B}_m = 2HW\mathbf{N}. \tag{15}$$

Indeed, using the identities:

$$\nabla \cdot \mathbf{A} = \nabla \cdot (\mathbf{I} - \mathbf{N} \otimes \mathbf{N}) = -(\nabla \cdot \mathbf{N})\mathbf{N} = 2H\mathbf{N}, \tag{16}$$

$$\nabla W = \mathbf{E}^\beta \frac{\partial W}{\partial s^\beta} = \nabla \mathbf{F} : \mathbf{P}, \tag{17}$$

we have that:

$$\begin{aligned}
\nabla \cdot \mathbf{B}_m &= \nabla \cdot (W\mathbf{A} - \mathbf{P} \cdot \mathbf{F}^T) \\
&= (\nabla W) \cdot \mathbf{A} + W\nabla \cdot \mathbf{A} - (\nabla \cdot \mathbf{P}) \cdot \mathbf{F}^T - \mathbf{E}^\alpha \cdot \mathbf{P} \cdot \frac{\partial \mathbf{F}^T}{\partial s^\alpha} \\
&= \nabla \mathbf{F} : \mathbf{P} + 2HW\mathbf{N} - \mathbf{P} : \nabla \mathbf{F}^T = 2HW\mathbf{N}.
\end{aligned}$$

where " : " stands for the double dot product. For dyads and triads of vectors, it could be defined as follows:

$$\begin{aligned}
(\mathbf{a} \otimes \mathbf{b}) : (\mathbf{c} \otimes \mathbf{d}) &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}), \\
(\mathbf{a} \otimes \mathbf{b}) : (\mathbf{c} \otimes \mathbf{d} \otimes \mathbf{e}) &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})\mathbf{e}, \\
(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) : (\mathbf{d} \otimes \mathbf{e}) &= (\mathbf{b} \cdot \mathbf{d})(\mathbf{c} \cdot \mathbf{e})\mathbf{a},
\end{aligned}$$

and by linearity could be extended for tensors of any order.

Using the surface divergence theorem (4) for $\mathbf{B}_m$, we get the identity:

$$\iint_{\Sigma_Y} 2H\,W\,\mathbf{N}\,d\Sigma = \oint_Y \boldsymbol{\nu} \cdot \mathbf{B}_m\,ds, \tag{18}$$

for any part $\Sigma_\Upsilon \subset \Sigma$ with the boundary $\Upsilon$. Equation (18) shows that the right side of equation (18) does not constitute a path-independent integral, in general.

Instead, for a minimal surface $H = 0$ and we get the conservation law:

$$\nabla \cdot \mathbf{B}_m = \mathbf{0}, \tag{19}$$

and with equation (4), we came to the path-independent $J$-integral:

$$J_m \equiv \oint_\Upsilon \boldsymbol{\nu} \cdot \mathbf{B}_m \, ds = \mathbf{0}. \tag{20}$$

Using equations (13) and (19), we can derive another useful integral identity. First, we have two relations:

$$\nabla \cdot (\mathbf{B}_m \cdot \mathbf{X}) = \operatorname{tr} \mathbf{B}_m, \quad \operatorname{tr} \mathbf{B}_m = 2W - \mathbf{P} : \mathbf{F}. \tag{21}$$

Then, we can see that:

$$\oint_\Upsilon \boldsymbol{\nu} \cdot \mathbf{B}_m \cdot \mathbf{X} \, ds = \iint_{\Sigma_\Upsilon} \nabla \cdot (\mathbf{B}_m \cdot \mathbf{X}) \, d\Sigma = 2 \iint_{\Sigma_\Upsilon} W \, d\Sigma - \iint_{\Sigma_\Upsilon} \mathbf{P} : \mathbf{F} \, d\Sigma. \tag{22}$$

where $\Sigma_\Upsilon \subset \Sigma$ is an area bounded by $\Upsilon$. The last integral in equation (22) could be transformed as follows:

$$\iint_{\Sigma_\Upsilon} \mathbf{P} : \mathbf{F} \, d\Sigma = - \iint_{\Sigma_\Upsilon} \nabla \cdot (\mathbf{P} \cdot \mathbf{x}) \, d\Sigma + \oint_\Upsilon \boldsymbol{\nu} \cdot \mathbf{P} \cdot \mathbf{x} \, ds = \oint_\Upsilon \boldsymbol{\nu} \cdot \mathbf{P} \cdot \mathbf{x} \, ds. \tag{23}$$

As a result, equation (22) takes the form:

$$\oint_\Upsilon \boldsymbol{\nu} \cdot \mathbf{B}_m \cdot \mathbf{X} \, ds = 2 \iint_{\Sigma_\Upsilon} W \, d\Sigma - \oint_\Upsilon \boldsymbol{\nu} \cdot \mathbf{P} \cdot \mathbf{x} \, ds. \tag{24}$$

So, we came to:

$$\oint_\Upsilon \boldsymbol{\nu} \cdot [\mathbf{B}_m \cdot \mathbf{X} + \mathbf{P} \cdot \mathbf{x}] \, ds = 2 \iint_{\Sigma_\Upsilon} W \, d\Sigma. \tag{25}$$

### 4.2. Infinitesimal deformations

Surface integral in equation (25) could be transformed into a contour one only for very particular cases such as small deformations. Let us consider it in more detail. For infinitesimal deformations, $W$ takes the form:

$$W = W(\boldsymbol{\varepsilon}), \quad \boldsymbol{\varepsilon} = \frac{1}{2} \left( \nabla \mathbf{u} \cdot \mathbf{A} + \mathbf{A} \cdot (\nabla \mathbf{u})^T \right), \tag{26}$$

where $\boldsymbol{\varepsilon}$ is a linear surface strain tensor. Equations of equilibrium transform into:

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}, \quad \boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}}, \tag{27}$$

where $\boldsymbol{\sigma}$ is the symmetric surface stress tensor. The Eshelby tensor is modified as follows:

$$\mathbf{B}_{sm} = W\mathbf{A} - \boldsymbol{\sigma} \cdot (\nabla \mathbf{u})^T. \tag{28}$$

Repeating derivations (21)–(24), we came to the identity:

$$\oint_{\Upsilon} \boldsymbol{\nu} \cdot [\mathbf{B}_{sm} \cdot \mathbf{X} + \boldsymbol{\sigma} \cdot \mathbf{u}]\, ds = 2 \iint_{\Sigma_\Upsilon} W\, d\Sigma. \tag{29}$$

For a linear membrane $W = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon}$ and $\iint_{\Sigma_\Upsilon} W\, d\Sigma$ could be represented as a contour integral. Indeed, we have:

$$\begin{aligned} \iint_{\Sigma_\Upsilon} W\, d\Sigma &= \frac{1}{2} \iint_{\Sigma_\Upsilon} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, d\Sigma = \frac{1}{2} \iint_{\Sigma_\Upsilon} \boldsymbol{\sigma} : \nabla \mathbf{u}\, d\Sigma, \\ &= -\frac{1}{2} \iint_{\Sigma_\Upsilon} (\nabla \cdot \boldsymbol{\sigma}) \cdot \mathbf{u}\, d\Sigma + \frac{1}{2} \oint_{\Upsilon} \boldsymbol{\nu} \cdot \boldsymbol{\sigma} \cdot\ ds, \\ &= \frac{1}{2} \oint_{\Upsilon} \boldsymbol{\nu} \cdot \boldsymbol{\sigma} \cdot\ ds. \end{aligned} \tag{30}$$

So, instead of equation (29), we get the formula:

$$\oint_{\Upsilon} \boldsymbol{\nu} \cdot \mathbf{B}_{sm} \cdot \mathbf{X}\, ds = 0. \tag{31}$$

As in the case of plane stress state [5], in the theory of linear membranes, path-independent integral (31) could be called $M$-integral.

An $M$-integral-type identity could be also derived for a power-law constitutive relation, that is for $W$ given by:

$$W = \frac{1}{2m} (\boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon})^m, \tag{32}$$

where $\mathbf{C}$ and $m$ are the material parameters, and a fourth-order tensor $\mathbf{C}$ has the same symmetry properties as in the case of linear plane stress elasticity. So, we have that:

$$\boldsymbol{\sigma} = (\boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon})^{m-1} \mathbf{C} : \boldsymbol{\varepsilon}, \quad W = \frac{1}{2m} \boldsymbol{\sigma} : \boldsymbol{\varepsilon},$$

and instead of equation (31), we came to another $M$-integral:

$$\oint_{\Upsilon} \boldsymbol{\nu} \cdot \left[ \mathbf{B}_{sm} \cdot \mathbf{X} + \frac{m-1}{m} \boldsymbol{\sigma} \cdot \mathbf{u} \right] ds = 0. \tag{33}$$

Constitutive equation (32) could be useful for modeling of some hardening phenomena in inelastic materials. In fact, in plasticity, they used power law-type constitutive equations like $\sigma = K\varepsilon^n$, where $K$ is a strength coefficient and $n$ is an exponent (see, e.g., Besseling and Giessen [41, p. 94]). Power-law constitutive relations such as Norton's law are also widely used the theory of plasticity and creep (see the previous works [42,43], and the references therein). For example, $J$- and $M$-integrals for power-law materials were used in the previous works [44–46], in order to estimate a stress concentration in vicinity of crack tips.

## 5. Eshelby tensor and conservation laws: micropolar (six-parameter) shell

As an example of more complex 2D model, we consider micropolar or six-parameter shells [30–32]. Within the model, we have an extended kinematics which includes two kinematically independent fields of translations and rotations.

### 5.1. Finite deformations

For a hyperelastic micropolar (six-parameter) shell, a surface strain energy density $U$ depends on two surface strain measures $\mathbf{E}$ and $\mathbf{K}$ [30,37]:

$$U = U(\mathbf{E}, \mathbf{K}), \tag{34}$$

where

$$\mathbf{E} = \mathbf{F} \cdot \mathbf{Q}^T - \mathbf{A}, \quad \mathbf{K} = \frac{1}{2}\mathbf{E}^\alpha \otimes \left(\frac{\partial \mathbf{Q}}{\partial s^\alpha} \cdot \mathbf{Q}^T\right)_\times, \tag{35}$$

where $\mathbf{T}_\times$ is the vectorial invariant of a second-order tensor $\mathbf{T}$ (see, e.g., Eremeyev et al. [37]). For a dyad of two vectors $\mathbf{a}$ and $\mathbf{b}$, we have $(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \times \mathbf{b}$. In what follows, we consider only homogeneous shells that is shells whose strain energy density depends only on the strain measures $\mathbf{E}$ and $\mathbf{K}$.

Without surface forces and couples, the Lagrangian equations of statics take the form:

$$\nabla \cdot \mathbf{T} = \mathbf{0}, \quad \nabla \cdot \mathbf{M} + \left[\mathbf{F}^T \cdot \mathbf{T}\right]_\times = \mathbf{0}, \tag{36}$$

where

$$\mathbf{T} = \frac{\partial U}{\partial \mathbf{F}} = \mathbf{S}_1 \cdot \mathbf{Q}, \quad \mathbf{M} = \mathbf{S}_2 \cdot \mathbf{Q}, \quad \mathbf{S}_1 = \frac{\partial U}{\partial \mathbf{E}}, \quad \mathbf{S}_2 = \frac{\partial U}{\partial \mathbf{K}}, \tag{37}$$

where $\mathbf{T}$ and $\mathbf{M}_\varkappa$ are the surface stress and couple stress tensors of the first Piola–Kirchhoff type, whereas the stress measures $\mathbf{S}_1$ and $\mathbf{S}_2$ are the referential stress and couple stress tensors similar to the respective second Piola–Kirchhoff stress tensors of 3D non-linear elasticity.

Within the six-parameter shell model, the Eshelby tensor $\mathbf{B}$ was introduced in Eremeyev and Pietraszkiewicz [33] for description of stress-induced phase transitions. More precisely, using $\mathbf{B}$, the thermodynamic compatibility condition along a phase interface was formulated. $\mathbf{B}$ is defined as follows:

$$\mathbf{B} = U\mathbf{A} - \mathbf{T} \cdot \mathbf{F}^T - \mathbf{M} \cdot \mathbf{Q}^T \cdot \mathbf{K}^T, \tag{38}$$

or as:

$$\mathbf{B} = U\mathbf{A} - \mathbf{S}_1 \cdot \mathbf{E}^T - \mathbf{S}_2 \cdot \mathbf{K}^T, \tag{39}$$

In Eremeyev and Konopińska-Zmysłowska [34], it was shown that under some conditions, $\mathbf{B}$ could be obtained from its 3D counterpart using the through-the-thickness integration similar to derivation of stress resultants [32].

For $\mathbf{B}$, we have the identity:

$$\nabla \cdot \mathbf{B} = 2HU\mathbf{N}, \tag{40}$$

which could be proven similar to equation (15). For brevity, we omit awkward calculations here.

As a result, we came to the integral identity:

$$\iint\limits_{\Sigma_\Upsilon} 2HU\mathbf{N}\, d\Sigma = \oint\limits_{\Upsilon} \boldsymbol{\nu} \cdot \mathbf{B}\, ds, \tag{41}$$

Again, for a minimal surface, we get the conservation law and $J$-integral:

$$\nabla \cdot \mathbf{B} = \mathbf{0}, \quad J \equiv \oint\limits_{\Upsilon} \boldsymbol{\nu} \cdot \mathbf{B}\, ds = \mathbf{0}. \tag{42}$$

This conservation law is the 2D counterpart of the 3D one derived for non-linear micropolar continua in the previous works [47,48] with the use of Noether's theorem.

### 5.2. Small deformations

In the case of small deformations, we can provide a similar study of conservation laws. For small rotations instead of the microrotation tensor $\mathbf{Q}$, one can use the infinitesimal vector $\boldsymbol{\phi}$ since $\mathbf{Q}$ can be approximated as follows [30,49]:

$$\mathbf{Q} \approx \mathbf{I} + \boldsymbol{\phi} \times \mathbf{I}.$$

As a result, $\mathbf{E}$ and $\mathbf{K}$ could be replaced by the linear strain measures $\mathbf{e}$ and $\mathbf{k}$:

$$\mathbf{e} = \nabla \mathbf{u} - \mathbf{I} \times \boldsymbol{\phi}, \quad \mathbf{k} = \nabla \boldsymbol{\phi}.$$

Equilibrium equations take the form:

$$\nabla \cdot \mathbf{T} = \mathbf{0}, \quad \nabla \cdot \mathbf{M} + \mathbf{T}_{\times} = \mathbf{0}, \tag{43}$$

with surface stress $\mathbf{T}$ and couple stress $\mathbf{M}$ tensors. The latter relate to a surface strain energy density through the formulae:

$$\mathbf{T} = \frac{\partial U}{\partial \mathbf{e}}, \quad \mathbf{M} = \frac{\partial U}{\partial \mathbf{k}}, \quad U = U(\mathbf{e}, \mathbf{k}). \tag{44}$$

The Eshelby tensor $\mathbf{B}_s$ has the form:

$$\mathbf{B}_s = U\mathbf{A} - \mathbf{T} \cdot (\nabla \mathbf{u})^T - \mathbf{M} \cdot (\nabla \boldsymbol{\phi})^T. \tag{45}$$

Let us note that equation (45) is symmetrized with respect to translations and rotations. Indeed, here we face a full symmetry under replacements:

$$\mathbf{u} \leftrightarrows \boldsymbol{\phi}, \quad \mathbf{T} \leftrightarrows \mathbf{M}.$$

We can again prove the identity:

$$\nabla \cdot \mathbf{B}_s = 2HU\mathbf{N}. \tag{46}$$

Unlike the case of finite deformations, here the calculations are more simple. Indeed, similar to equation (17), we have:

$$\nabla U = \nabla \mathbf{e} : \mathbf{T} + \nabla \mathbf{k} : \mathbf{M} = \nabla\nabla \mathbf{u} : \mathbf{T} + \nabla\nabla \boldsymbol{\phi} : \mathbf{M} - \nabla(\mathbf{I} \times \boldsymbol{\phi}) : \mathbf{T},$$

and we came to a series of identities:

$$\begin{aligned} \nabla \cdot \mathbf{B}_s &= \nabla U + 2HU\mathbf{N} - (\nabla \cdot \mathbf{T}) \cdot (\nabla \mathbf{u})^T - (\nabla \cdot \mathbf{M}) \cdot (\nabla \boldsymbol{\phi})^T \\ &\quad - \mathbf{T} : \nabla(\nabla \mathbf{u})^T - \mathbf{M} : \nabla(\nabla \boldsymbol{\phi})^T \\ &= \nabla\nabla \mathbf{u} : \mathbf{T} - \mathbf{T} : \nabla(\nabla \mathbf{u})^T + \nabla\nabla \boldsymbol{\phi} : \mathbf{M} - \mathbf{M} : \nabla(\nabla \boldsymbol{\phi})^T \\ &\quad - \nabla(\mathbf{I} \times \boldsymbol{\phi}) : \mathbf{T} - \mathbf{T}_{\times} \cdot (\nabla \boldsymbol{\phi})^T + 2HU\mathbf{N} = 2HU\mathbf{N}, \end{aligned}$$

that results in equation (46). Thus, for a minimal surface, we get the conservation law and corresponding $J$-integral for small deformations:

$$\nabla \cdot \mathbf{B}_s = \mathbf{0}, \quad J \equiv \oint_{\Upsilon} \boldsymbol{\nu} \cdot \mathbf{B}_s \, ds = \mathbf{0}. \tag{47}$$

This conservation law is similar to 3D analogous in linear micropolar elasticity (see, e.g., Lubarda and Markenscoff [50]). Unlike the case of elastic membrane, as for micropolar 3D solids [50], $M$-integral for micropolar shell cannot be derived, in general. It could be possible for a particular class of

constitutive equations with symmetric stress resultant tensor, $\mathbf{T}_{\times} = \mathbf{0}$, or for decoupled relations with an energy in the form $U = U(\boldsymbol{\varepsilon}, \mathbf{k})$, see Neff [51] for a discussion on this class of 3D constitutive relations.

## 6. On 3D-to-2D reduction of conservation laws

We have discussed the so-called direct approach to bidimensional structures. Within the approach, the basic governing equations, i.e., equations of equilibrium and constitutive equations, are formulated as for a 2D continuum. As a result, additional non-trivial conservation laws could be derived using these 2D governing equations as was demonstrated above. In other words, derived 2D conservation laws are *exact* consequence of 2D equilibrium and constitutive equations.

An alternative way could be a 3D-to-2D reduction also applied to 3D conservation laws. Any 3D-to-2D reduction results in 2D equations, so one could also apply it to 3D conservation laws. Let us note that any reduction procedure results in an *approximate* representation of a 3D state through its 2D counterpart. Using such an approach, one should be aware of the following:

Obtained conservation law could be reduction-dependent, i.e., could depend on the chosen reduction procedure. Indeed, in the literature, there are known various shell models which have different conservative laws, in general.
Reduction of a conservation law could result in an identity which is not a 2D conservation law according to definitions (1) and (2).

Let us discuss this matter in more detail using the through-the-thickness procedure. This 3D-to-2D reduction leads to the non-linear resultants six-parameter shell theory [32]. For finite deformations of a non-linear elastic solid, we have the Lagrangian equitation of equilibrium and the conservation law for the Eshelby tensor $\mathbf{b}$ in the form [2]:

$$\operatorname{Div} \boldsymbol{\Sigma} = \mathbf{0}, \tag{48}$$

$$\operatorname{Div} \mathbf{b} = \mathbf{0}, \qquad \mathbf{b} = V\,\mathbf{I} - \boldsymbol{\Sigma} \cdot \mathbf{G}^{T}, \tag{49}$$

where $\boldsymbol{\Sigma}$ is the first Piola–Kirchhoff stress tensor, $\mathbf{G} = \operatorname{Grad} \mathbf{r}$ is the deformation gradient, $V$ is a strain energy function, $\mathbf{r}$ is a 3D position vector in a current placement, and Div and Grad are the Lagrangian divergence and gradient operators, respectively.

Following [32,37], we consider deformations of a shell-like body $\mathcal{B}$ as a differentiable invertible mapping from a reference placement into a current one. Let $V = \{(s^1, s^2, \zeta) : (s^1, s^2) \in \Sigma, \zeta \in [-h^-, h^+]\}$ be a volume of $\mathcal{B}$ in the reference placement, where $\Sigma$ is a referential base surface, and $h = h^- + h^+$ is the shell total thickness (see Figure 3). So, the position vector of a given point $z$ of $\mathcal{B}$ in the reference placement is given by:

$$\mathbf{R} = \mathbf{X}(s^1, s^2) + \zeta \mathbf{N}.$$

In a current placement, $z$ could be represented through its position vector:

$$\mathbf{r} = \mathbf{x}(s^1, s^2) + \mathbf{z}(s^1, s^2, \zeta),$$

where $\mathbf{z} = \mathbf{r} - \mathbf{x}$ is called the base reference deviation vector [32].

Integrating equation (48) through the thickness, we came to equation $(43)_1$ with $\mathbf{T}$ defined as follows [37]:

$$\mathbf{T} = \int_{-h^-}^{h^+} (\mathbf{A} - \zeta \mathbf{H})^{-1} \cdot \boldsymbol{\Sigma} \mu \, d\zeta, \tag{50}$$

where $\mathbf{H} = -\nabla \mathbf{N}$ is the curvature tensor of $\Sigma$, and $\mu$ is the scale factor defined by the formulae:

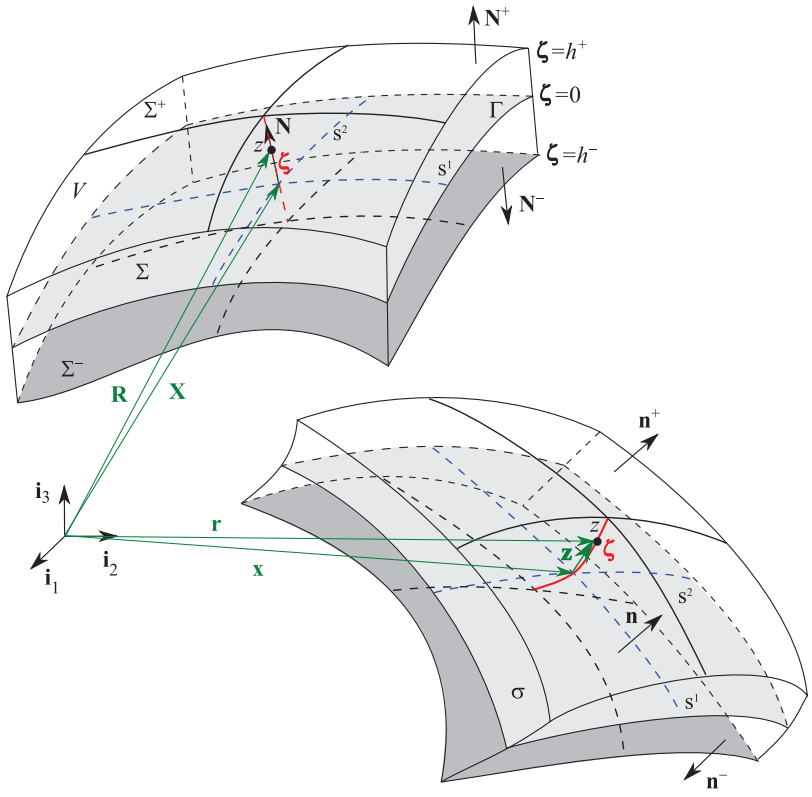


**Figure 3.** Deformation of a shell-like body $\mathcal{B}$.

$$dV = \mu\, d\zeta\, d\Sigma, \quad \mu \equiv \det(\mathbf{A} - \zeta\mathbf{H}) = 1 - 2H\zeta + K\zeta^2,$$

$K \equiv \det \mathbf{H}$ is the Gaussian curvature of $\Sigma$. Note that here we have used the assumption:

$$\mathbf{N} \cdot \boldsymbol{\Sigma}|_{\Sigma^{\pm}} = \mathbf{0}, \tag{51}$$

i.e., we assumed that faces $\Sigma^-$ and $\Sigma^+$ are free.

To derive equation $(43)_2$, we cross-multiply equation (48) by $\mathbf{z}$ from the left and integrate the result through the thickness. We get equation $(43)_2$ with $\mathbf{M}$ defined as:

$$\mathbf{M} = \int_{-h^-}^{h^+} (\mathbf{A} - \zeta\mathbf{H})^{-1} \cdot \boldsymbol{\Sigma} \times \mathbf{z}\, \mu\, d\zeta. \tag{52}$$

So, equation (43) is the exact consequence of the 3D equilibrium equations.

Similarly, the 2D strain energy density $U$ could be introduced as follows:

$$U = \int_{-h^-}^{h^+} V\, \mu\, d\zeta. \tag{53}$$

Obviously, within this 3D-to-2D reduction, some part of an energy stored in $\mathcal{B}$ could be lost (see Libai and Simmonds [32], for more details).

Let us now repeat the same integration technique to equation (49). Similarly, we came to:

$$\nabla \cdot \widehat{\mathbf{B}} + \mu^+ V^+ \mathbf{N}^+ + \mu^- V^- \mathbf{N}^- = \mathbf{0}, \tag{54}$$

$$\widehat{\mathbf{B}} = \int_{-h^-}^{h^+} (\mathbf{A} - \zeta \mathbf{H})^{-1} \cdot \mathbf{b}\, \mu\, d\zeta, \tag{55}$$

where we have used equation (51), $\mu^{\pm}$ and $V^{\pm}$ are the values of $\mu$ and $V$ taken on $\Sigma^{\pm}$, respectively, i.e., at $\zeta = \pm h^{\pm}$, and $\mathbf{N}^-$ and $\mathbf{N}^+$ are the unit outward normals to $\Sigma^{\pm}$ (see Figure 3). Obviously, $\widehat{\mathbf{B}}$ does not coincide nor with $\mathbf{B}_s$ neither with $\mathbf{B}$. Moreover, equation (54) does not constitute a conservation law, in general. So, one has to apply additional assumptions of kinematical and/or smallness type to get a conservation law.

As an example, let us transform equations (54) and (55) to the case of an elastic non-linear membrane. First, we restrict ourselves to a symmetric case $h^+ = h^- = h/2$. In addition, we assume that $\mathbf{G}$ does not depend on $\zeta$ or that such dependence is negligible. So, for $\mathbf{G}$, we use an approximation $\mathbf{G} = \mathbf{F} + \mathbf{N} \otimes \mathbf{n}$, where $\mathbf{n}$ is a normal to $\sigma$. As a result, $V$ does not depend on $\zeta$. For a thin enough structure, we also assume that $\mathbf{N}^{\pm} = \pm \mathbf{N}$. As a result, we get the formulae:

$$\mu^{\pm} = 1 \mp Hh + \frac{1}{4} K h^2, \quad \mu^+ V^+ \mathbf{N}^+ + \mu^- V^- \mathbf{N}^- = 2HhV\mathbf{N}.$$

With these assumptions, $\widehat{\mathbf{B}}$ transforms into:

$$\widehat{\mathbf{B}} = hV\mathbf{A} - \int_{-h/2}^{h/2} \mathbf{A} \cdot \boldsymbol{\Sigma}\, d\zeta \cdot \mathbf{F}^T. \tag{56}$$

Thus, introducing $W$ and $\mathbf{P}$ as follows:

$$W = hV, \quad \mathbf{P} = \int_{-h/2}^{h/2} \mathbf{A} \cdot \boldsymbol{\Sigma}\, d\zeta,$$

we came to equation (15) with $\mathbf{B}_m = \widehat{\mathbf{B}}$. For derivation of 2D equations of elastic membranes, we also refer to the previous works [32,52].

Transformation of equations (54) and (55) to the case of six-parameter (micropolar) shells can be provided similarly. It requires additional and more complex kinematical assumptions (see equation (28)) in Eremeyev and Konopińska-Zmysłowska [34] for normal components of $\mathbf{b}$ and $\mathbf{B}$.

## 7. Conclusion

We have discussed a few conservation laws for thin-walled structures, i.e., elastic membranes and six-parameter shells, modeled using material minimal surface. Let us underline that minimal surfaces significantly extended a class of geometry of shells and membranes for which it is possible to introduce such conservation laws. Using the property of a minimal surface (8), we have demonstrated that conservation laws for 2D systems are similar to the case of 3D non-linear elasticity. With conservation laws, one can derive invariant (path-independent) integrals such as $J$-integrals. The latter could be useful in mechanics of fracture, e.g., for estimation of stress concentration in the vicinity of geometrical singularities such as holes, crack tips, notches, and rigid inclusions. Moreover, they could be related to the energy release rate for quasistatically propagating defects in thin structures.


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