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Minimization of the number of periodic points for smooth self-maps of simply-connected manifolds with periodic sequence of Lefschetz numbers

Research Article

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Abstract: Let f be a smooth self-map of m-dimensional, $m \geq 4$, smooth closed connected and simply-connected manifold, r a fixed natural number. For the class of maps with periodic sequence of Lefschetz numbers of iterations the authors introduced in [Graff G., Kaczkowska A., Reducing the number of periodic points in smooth homotopy class of self-maps of simply-connected manifolds with periodic sequence of Lefschetz numbers, Ann. Polon. Math. (in press)] the topological invariant $J[f]$ which is equal to the minimal number of periodic points with the periods less or equal to r in the smooth homotopy class of f.

In this paper the invariant $J[f]$ is computed for self-maps of 4-manifold M with $\dim H_2(M;\mathbb{Q}) \leq 4$ and estimated for other types of manifolds. We also use $J[f]$ to compare minimization of the number of periodic points in smooth and in continuous categories.

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1. Introduction

Let M be a compact m-dimensional manifold and f a self-map of M. The problem of reducing fixed or periodic points in the homotopy class is a classical one and goes back to Nielsen. In dimension $m \geq 3$ the Nielsen number of f, $N(f)$, is the best lower estimate for the number of fixed points for all maps homotopic to f [19]. If f is smooth, then it turns out that the same is also true for all maps smoothly homotopic to f [18]. However, in general minimization of periodic points instead of fixed points, depends strongly on whether one concerns smooth or continuous category.

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Let us define

$$\mathrm{MF}_r(f) = \min\{\#\,\mathrm{Fix}(g^r) : g \sim f\}, \tag{1}$$

where $g \sim f$ means that the maps g and f are homotopic. We also define its smooth counterpart $\mathrm{MF}_r^{\mathrm{diff}}(f)$ by considering smooth maps and smooth homotopies in the formula (1). The value of $\mathrm{MF}_r(f)$ is given by Jiang invariant $\mathrm{NF}_r(f)$, cf. [15, 19], while $\mathrm{MF}_r^{\mathrm{diff}}(f)$ by $\mathrm{NJD}_r[f]$, the invariant introduced in [9].

In this paper we study the related problem of finding the minimal number of periodic points with the periods less or equal to r in the smooth homotopy class of f, i.e. we seek for the value of $\mathrm{MF}_{\leq r}^{\mathrm{diff}}(f)$, which is defined as

$$\mathrm{MF}_{\leq r}^{\mathrm{diff}}(f) = \min\left\{\#\bigcup_{k\leq r}\mathrm{Fix}(g^k) : g \overset{s}{\sim} f\right\}, \tag{2}$$

where $g \overset{s}{\sim} f$ means that g and f are smoothly (i.e. C^1) homotopic.

In general, finding $\mathrm{MF}_{\leq r}^{\mathrm{diff}}(f)$ is a very challenging problem, and this is the reason why we restrict our attention to some particular classes of manifolds and their self-maps. We consider simply-connected manifolds and their smooth self-maps with Lefschetz numbers of iterations, $\{L(f^n)\}_{n=1}^{\infty}$, periodic. In the simply-connected case classical (i.e. continuous) Nielsen theory is trivial, i.e. $\mathrm{MF}_{\leq r}(f)$, obtained by taking continuous maps and homotopies in (2), always is ≤ 1. However, $\mathrm{MF}_{\leq r}^{\mathrm{diff}}(f)$ is usually greater than 1 [8]. In [13] we considered smooth self-maps with $\{L(f^n)\}_{n=1}^{\infty}$ periodic and defined a topological invariant $J_r[f]$ in terms of Lefschetz numbers of iterations and local fixed point indices of iterations. We proved that for manifolds of dimension $m \geq 4$, $J_r[f]$ is independent of the choice of r for all sufficiently large r and that then $J_r[f] = J[f] = \mathrm{MF}_{\leq r}^{\mathrm{diff}}(f)$.

In other words, $J[f]$ is equal to the minimal number of all periodic points with the periods less or equal to r in a smooth homotopy class of f for all sufficiently large r. What is more, for a given manifold M we can easily determine such R_M that $J_r[f]$ is constantly equal to $J[f]$ for $r \geq R_M$.

One of the advantages of $J[f]$ in comparison to the other, mentioned above invariants, is that it is quite easy to compute, especially for manifolds with all homology groups being low dimensional. The aim of this paper is to show the methods of computing or estimating $J[f]$. In particular, we illustrate the whole theory by determining the invariant for self-maps of some class of 4-manifolds.

Let us point out that the estimate for $J[f]$ also provides important information about the minimization of periodic points. If $J[f] \leq A$, then we can reduce to A the number of periodic points with the periods less or equal to r in the smooth homotopy class of f. In other words, there is a map g smoothly homotopic to f with no more than A k-periodic points for $k \leq r$.

The article is organized in the following way. In Section 2 we describe the construction of $J[f]$ which is combinatorial in its nature and give a geometrical interpretation of the invariant. Section 3 is devoted to determination of the invariant for some 4-dimensional manifolds. In Section 4 we provide the estimate from above for $J[f]$ which is valid for all self-maps of any manifold of dimension at least 4. Finally, in Section 5 we find, by an application of $J[f]$, a class of manifolds for which minimizations of periodic points in smooth and continuous categories coincide (for the considered class of maps).

2. Construction of $J[f]$

In this section we sketch the definition of $J[f]$, for the details the reader may consult [13]. We start with introducing the so-called basic sequences.

Definition 2.1.
For a given $k \in \mathbb{N}$ we define a *basic sequence*

$$\mathrm{reg}_k(n) = \begin{cases} k & \text{if} \quad k \mid n, \\ 0 & \text{if} \quad k \nmid n. \end{cases}$$

We consider the class of maps with periodic sequence of Lefschetz numbers of iteration. For such maps $\{L(f^n)\}_{n=1}^{\infty}$ can be represented uniquely as a finite integral combination of basic sequences, called the periodic expansion [17]:

$$L(f^n) = \sum_{k \in O} b_k \operatorname{reg}_k(n), \tag{3}$$

where $O = \{k : b_k \neq 0\}$ is finite and $b_k \in \mathbb{Z}$.

By a p-orbit we will understand an orbit consisting of points with minimal period equal to p.

Definition 2.2.

A sequence of integers $\{c_n\}_{n=1}^{\infty}$ is called a $\mathrm{DD}^m(p)$ sequence if there are a C^1 map $\phi\colon U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^m$ is open; and P, an isolated p-orbit of ϕ, such that

$$c_n = \operatorname{ind}(\phi^n, P)$$

(notice that $c_n = 0$ if n is not a multiple of p).

Definition 2.3.

Let $\{L(f^n)\}_{n=1}^{\infty}$ be a periodic sequence of Lefschetz numbers. We decompose $\{L(f^n)\}_{n=1}^{\infty}$ into the sum

$$L(f^n) = c_1(n) + \cdots + c_s(n),$$

where c_i is a $\mathrm{DD}^m(1)$ sequence for $i = 1, \ldots, s$, $m \geq 4$. Each such decomposition determines the number $l = l_1 + \cdots + l_s$. We define the number $J[f]$ as the smallest l which can be obtained in this way. We will use the convention that in the case the sequence of Lefschetz numbers consists only of zero elements, then it is a sum of zero $\mathrm{DD}^m(1)$ sequences.

Let us remind that

$$\mathrm{MF}^{\mathrm{diff}}_{\leq r}(f) = \min \left\{ \# \bigcup_{k \leq r} \operatorname{Fix}(g^k) \, : \, g \overset{s}{\sim} f \right\},$$

where $g \overset{s}{\sim} f$ means that g and f are smoothly homotopic. The invariant $J[f]$ does not depend on r and is equal to $\mathrm{MF}^{\mathrm{diff}}_{\leq r}(f)$ for all sufficiently large r:

Theorem 2.4 ([13]).

Let f be a self-map of a smooth closed connected and simply-connected manifold M of dimension $m \geq 4$ and $R = \max\{k : k \in O\}$ in the formula (3). Then for every $r \geq R$,

$$J[f] = \mathrm{MF}^{\mathrm{diff}}_{\leq r}(f).$$

As a consequence, minimization of the number of periodic points with periods less or equal to r in a smooth homotopy class of f is equivalent to finding the value of $J[f]$.

Below we describe the consecutive steps of the construction of $J[f]$, which reduces the computation of $J[f]$ to a combinatorial procedure. First of all, we introduce some notation and definitions.

For a compact connected manifold M of dimension m we will consider $H_i(M;\mathbb{Q})$, where $i = 0, 1, \ldots, m$, the homological groups with coefficients in $\mathbb{Q}$, which are then finite dimensional linear spaces over $\mathbb{Q}$. For a self-map f of M we denote by f_{*i} the linear map induced by f on $H_i(M;\mathbb{Q})$ and by f_* the self-map $\bigoplus_{i=0}^{m} f_{*i}$ of $\bigoplus_{i=0}^{m} H_i(M;\mathbb{Q})$.

Definition 2.5 ([17]).
Let M be an m-dimensional compact connected manifold. For integers $i \geq 0$ and $f\colon M \to M$, let $e_i(\lambda)$ be the number of eigenvalues of f_{*i} equal to λ (i.e. the dimension of the eigenspace in $H_i(M;\mathbb{Q})$ corresponding to λ). Define

$$e(\lambda) = \sum_{i=0}^{m} (-1)^i e_i(\lambda).$$

We will call an eigenvalue $\lambda \neq 0$ *essential* provided $e(\lambda) \neq 0$.

Let $\varepsilon_1, \dots, \varepsilon_{\varphi(d)}$ be all dth primitive roots of unity, where φ denotes the Euler function, i.e. $\varphi(d)$ is the number of positive integers less than or equal to d that are coprime to d. For a given d we define $L_d(n) = \varepsilon_1^n + \cdots + \varepsilon_{\varphi(d)}^n$. Let P_d denote the set of all dth primitive roots of unity and $\sigma_{\mathrm{es}}(f)$ be the set of all essential eigenvalues of f. We define

$$e(d) = \sum_{\lambda \in P_d \cap \sigma_{\mathrm{es}}(f)} e(\lambda).$$

2.1. Construction of $J[f]$

I If f is a map with $\{L(f^n)\}_{n=1}^{\infty}$ periodic, then all its essential eigenvalues are (primitive) roots of unity [13, Theorem 1.8]. As a consequence, we may represent Lefschetz numbers of iterations in the form

$$L(f^n) = \sum_d \frac{e(d)}{\varphi(d)} L_d(n).$$

II We represent each $\{L_d(n)\}_{n=1}^{\infty}$ as a finite combination of basic sequences

$$L_d(n) = \sum_{k|n} a_k^d \operatorname{reg}_k(n), \tag{4}$$

where

$$a_k^d = \begin{cases} 0 & k \nmid d, \\ \mu\left(\dfrac{d}{k}\right) & k \mid d. \end{cases} \tag{5}$$

In such a way we find the periodic expansion of $\{L(f^n)\}_{n=1}^{\infty}$, cf. also [13].

III A full list of possible sequences of local indices of iterations in a given dimension m (i.e. $\mathrm{DD}^m(1)$ sequences) in the form of periodic expansion has been recently provided in [12].

IV We decompose the periodic expansion of $\{L(f^n)\}_{n=1}^{\infty}$ into the minimal number of $\mathrm{DD}^m(1)$ sequences, obtaining the value of $J[f]$.

Remark 2.6.
The combinatorial procedure described in subsection 2.1 has clear geometrical interpretation. Namely, let f be a smooth self-map of a manifold M of dimension at least 4 and r a fixed natural number. In the smooth homotopy class of f one can create fixed points so that the sum of their indices of iterations is equal to the Lefschetz numbers of iterations and then remove all other r-periodic points [8]. This strong result was obtained by the use of powerful Nielsen technics (Canceling and Creating Procedures proved by Jezierski in [16]). On the other hand, the created fixed points have indices that are $\mathrm{DD}^m(1)$ sequences and it is known that $\mathrm{MF}_{\leq r}^{\mathrm{diff}}(f)$ can be realized by a map with all r-periodic points being fixed points [11]. Thus, the minimal number of r-periodic points in the smooth homotopy class of f is given by item IV.

Remark 2.7.
Let us mention that both sequences $\{L(f^n)\}_{n=1}^{\infty}$ and $\{\mathrm{ind}(f^n, x_0)\}_{n=1}^{\infty}$ which were used to define $J[f]$ are powerful instruments that recently have found many various applications in dynamical systems, differential equations and mathematical physics, cf. [3, 5, 7, 20, 22].

3. Determination of $J[f]$ for 4-dimensional manifold with $\dim H_2(M;\mathbb{Q}) \leq 4$

Remind that we consider a closed connected and simply-connected manifold M of dimension at least 4. In this section we assume that $\dim M = 4$. As M is connected, $H_0(M;\mathbb{Q}) = \mathbb{Q}$; simply-connectedness and duality imply that $H_1(M;\mathbb{Q}) = H_3(M;\mathbb{Q}) = 0$; and by the dimension argument $H_4(M;\mathbb{Q}) = \mathbb{Q}$. Thus, only the second homology group can take different forms: $H_2(M;\mathbb{Q}) = \mathbb{Q}^k$, where $k \geq 0$ is an integer.

In order to obtain some examples of such manifolds for a given k, one can take connected sums of the appropriate number of copies of $\mathbb{C}P^2$ or $S^2 \times S^2$, e.g. for $k = 4$ we can take $M = S^2 \times S^2 \# S^2 \times S^2$, cf. [6].

We will give the complete list of all possible values of $J[f]$ in terms of the eigenvalues of f for $k \leq 4$ (i.e. for M such that $\dim H_2(M;\mathbb{Q}) \leq 4$).

Remark 3.1.

Let us point out that not all possibilities enlisted in Theorem 3.4 could be realized. For a given manifold M there are further geometrical restrictions that prevent the appearance of some combinations of eigenvalues. The ways in which one can apply Theorem 3.4 for finding the value of $J[f]$ for self-maps of a given manifold is illustrated by Theorem 3.6.

Remark 3.2.

As it was shown in [13] that for (connected) manifolds of dimension at least 4 satisfying $\dim H_i(M;\mathbb{Q}) \leq 1$ the invariant $J[f] \leq 1$, and $J[f] = 0$ for manifolds with vanishing odd-dimensional homology groups [14], we will deal with the cases of $k \in \{2, 3, 4\}$.

Let us remind that by item **I** of subsection 2.1 all essential eigenvalues of f_* are (primitive) roots of unity. In particular, f induces identity on $H_0(M;\mathbb{Q})$ and D with $D \in \{-1, 0, 1\}$ on $H_4(M;\mathbb{Q})$. The eigenvalues of f_{*2} are various groups of primitive roots of unity of degree d such that $\varphi(d) \leq k$. We denote their contribution to $\{L(f^n)\}_{n=1}^{\infty}$ as $C^{(k)}(n)$. We get three possible cases in dependence of the value of D:

$$D = 1, \quad \text{then} \quad L(f^n) = 1 + C^{(k)}(n) + 1 = 2\mathrm{reg}_1(n) + C^{(k)}(n), \tag{6a}$$

$$D = -1, \quad \text{then} \quad L(f^n) = 1 + C^{(k)}(n) + (-1)^n = \mathrm{reg}_2(n) + C^{(k)}(n), \tag{6b}$$

$$D = 0, \quad \text{then} \quad L(f^n) = 1 + C^{(k)}(n) = \mathrm{reg}_1(n) + C^{(k)}(n). \tag{6c}$$

We will make use of the list of all possible sequences of local indices of iterations of smooth maps in dimension 4 (i.e. $\mathrm{DD}^4(1)$ sequences) provided in [10, Theorem 3.3], see also [4, 12].

Theorem 3.3.

In dimension $m = 4$ there are six patterns of possible local indices of iterations $\{\mathrm{ind}(f^n, x_0)\}_{n=1}^{\infty}$ at an isolated fixed point x_0, which are enlisted in Table 1.

Table 1. $\mathrm{DD}^4(1)$ sequences, i.e. all sequences of local indices of iterations of smooth maps in dimension $m = 4$; $p, q > 2$.

Case	$m = 4$
(A)	$a_1\mathrm{reg}_1(n) + a_2\mathrm{reg}_2(n)$
(B)	$a_1\mathrm{reg}_1(n) + a_p\mathrm{reg}_p(n)$
(C)	$\mathrm{reg}_1(n) + a_2\mathrm{reg}_2(n) + a_p\mathrm{reg}_p(n) + a_{[p,2]}\mathrm{reg}_{[p,2]}(n)$
(D)	$-\mathrm{reg}_1(n) + a_2\mathrm{reg}_2(n) + a_p\mathrm{reg}_p(n) + a_{[p,2]}\mathrm{reg}_{[p,2]}(n)$
(E)	$a_2\mathrm{reg}_2(n) + a_p\mathrm{reg}_p(n) + a_{[p,2]}\mathrm{reg}_{[p,2]}(n)$
(F)	$\mathrm{reg}_1(n) + a_p\mathrm{reg}_p(n) + a_q\mathrm{reg}_q(n) + a_{[p,q]}\mathrm{reg}_{[p,q]}(n)$

By $[p, q]$ we denote the least common multiple of p and q. Now we formulate the main result of this section.

Theorem 3.4.

Let f be a smooth self-map of a smooth closed connected and simply-connected manifold M of the dimension 4. Assume that the sequence of Lefschetz numbers of iterations of f is periodic. Then $J[f] \in \{1, 2\}$. What is more, the value of $J[f]$ can be expressed in dependence on the eigenvalues of f_* in the following way. For $k = 2$:

(i) *if there is an eigenvalue of f_{*2} which is a primitive root of unity of degree $d \in \{3, 4, 6\}$, then the values of $J[f]$ are given in Table 2;*

(ii) *otherwise $J[f] = 1$.*

For $k = 3$:

(iii) *if 0 is an eigenvalue of f_{*2}, then this case reduces to the case $k = 2$;*

(iv) *if 0 is not an eigenvalue of f_{*2} but there is an eigenvalue of f_{*2} which is a primitive root of unity of degree $d \in \{3, 4, 6\}$, then the values of $J[f]$ are given in Table 3;*

(v) *otherwise $J[f] = 1$.*

For $k = 4$:

(vi) *if 0 is an eigenvalue of f_{*2}, then this case reduces to the case $k = 3$;*

(vii) *if 0 is not an eigenvalue of f_{*2} but there is an eigenvalue of f_{*2} which is a primitive root of unity of degree $d > 2$, then the values of $J[f]$ are given in Table 4;*

(viii) *otherwise $J[f] = 1$.*

In Tables 2, 3, 4, d denotes degrees of primitive roots of unity being eigenvalues of f_{*2} and (a), (b), (c) refer to the different values of D enlisted in (6).

Table 2. $J[f]$ in the case (i), $k = 2$.

d	(a)	(b)	(c)
3	**1**	**1**	**1**
4	2	**1**	**1**
6	2	**1**	2

Table 3. $J[f]$ in the case (iv), $k = 3$.

d	(a)	(b)	(c)
3, 2	**1**	2	**1**
3, 1	**1**	**1**	**1**
4, 2	**1**	**1**	**1**
4, 1	2	**1**	2
6, 2	2	**1**	**1**
6, 1	2	2	2

Before we give the proof of Theorem 3.4 let us illustrate its geometrical meaning by the following example.

Example 3.5.

Assume that $k = 2$ in Theorem 3.4 and consider a smooth self-map f of M such that the induced map f_{*2} has the primitive roots of unity of degree 4 and D is equal to 1. Then, by Table 2, case $d = 4$ (a), we find that $J[f] = 2$. This means that for each $r \geq 4$ (r can be determined by the formula in Theorem 2.4) we can reduce the number of periodic points in smooth homotopy class (less or equal to r) to 2, but not any more. In other words, there is such g in the smooth homotopy class of f that g has only two elements in the set $\bigcup_{k \leq r} \mathrm{Fix}(g^k)$. On the other hand, there is no h smoothly homotopic to f with less than 2 elements in $\bigcup_{k \leq r} \mathrm{Fix}(h^k)$.

Table 4. $J[f]$ in the case (vii), $k = 4$.

Case	d	(a)	(b)	(c)	Case	d	(a)	(b)	(c)
(α)	5	**1**	**1**	**1**	(β)	3, 3	**1**	2	**1**
	8	2	2	**1**		4, 4	2	**1**	**1**
	10	2	**1**	2		6, 6	2	2	2
	12	2	2	2		3, 4	2	2	2
						3, 6	2	**1**	**1**
						4, 6	2	2	2
(γ)	3, 1, 1	**1**	**1**	**1**					
	3, 1, 2	**1**	**1**	**1**					
	3, 2, 2	**1**	2	2					
	4, 1, 1	2	**1**	2					
	4, 1, 2	**1**	**1**	**1**					
	4, 2, 2	**1**	2	**1**					
	6, 1, 1	2	2	2					
	6, 1, 2	2	**1**	2					
	6, 2, 2	**1**	**1**	**1**					

Proof of Theorem 3.4. For $k = 2$, f_{*2} has two eigenvalues. They can be either

(i) two conjugated primitive roots of unity of degree d, where $\varphi(d) = 2$, i.e. $d \in \{3, 4, 6\}$; *or*

(ii) both of them belong to the set $\{-1, 0, 1\}$, i.e. each of them is either 0 or a root of degree $d \in \{1, 2\}$.

Case (i): We notice that $C^{(k)}(n)$, the contribution to $\{L(f^n)\}_{n=1}^{\infty}$ which comes from the second homology group, is equal to $L_d(n)$, which we calculate using the formula (4). Then by (6) we determine $\{L(f^n)\}_{n=1}^{\infty}$ in dependence of D. The resulting sequences of Lefschetz numbers are presented in Table 5. Now, for a given set of eigenvalues we calculate $J[f]$. In order to do that, according to the item IV of subsection 2.1, we have to find the minimal number of $\mathrm{DD}^4(1)$ sequences that in sum give $\{L(f^n)\}_{n=1}^{\infty}$. Thus, the calculation of $J[f]$ reduces to the following task: for each sequence $\{c(n)\}_n$ determined by the pair (d, D) in Table 5, we find the minimal number of sequences from Table 1 that in sum give $\{c(n)\}_n$.

Table 5. All possible sequences $\{L(f^n)\}_{n=1}^{\infty}$ in the case (i), $k = 2$.

d	(a)	(b)	(c)
3	$\mathrm{reg}_1 + \mathrm{reg}_3$	$-\mathrm{reg}_1 + \mathrm{reg}_2 + \mathrm{reg}_3$	reg_3
4	$2\mathrm{reg}_1 - \mathrm{reg}_2 + \mathrm{reg}_4$	reg_4	$\mathrm{reg}_1 - \mathrm{reg}_2 + \mathrm{reg}_4$
6	$3\mathrm{reg}_1 - \mathrm{reg}_2 - \mathrm{reg}_3 + \mathrm{reg}_6$	$\mathrm{reg}_1 - \mathrm{reg}_3 + \mathrm{reg}_6$	$2\mathrm{reg}_1 - \mathrm{reg}_2 - \mathrm{reg}_3 + \mathrm{reg}_6$

We illustrate the method of computing $J[f]$ for $d = 4$ and $D = 1$ which corresponds to (6a). We have that $L(f^n) = 2\mathrm{reg}_1(n) + C^{(k)}(n)$, and $C^{(k)}(n) = L_4(n) = \sum_{l|4} \mu(4/l)\mathrm{reg}_l(n) = -\mathrm{reg}_2(n) + \mathrm{reg}_4(n)$. Thus $L(f^n) = 2\mathrm{reg}_1(n) - \mathrm{reg}_2(n) + \mathrm{reg}_4(n)$. This sequence is not listed in Table 1, because the coefficient at $\mathrm{reg}_1(n)$ is bigger than 1. As a consequence, $J[f] > 1$. On the other hand, this sequence is a sum of two sequences listed in Table 1, namely of the type (A) and (C) with $p = 2$,

$$2\mathrm{reg}_1(n) - \mathrm{reg}_2(n) + \mathrm{reg}_4(n) = \mathrm{reg}_1(n) + [\mathrm{reg}_1(n) - \mathrm{reg}_2(n) + \mathrm{reg}_4(n)].$$

Finally, $J[f] = 2$.

We repeat the same analysis for all other pairs (d, D) and observe that we need in each case either one or two $\mathrm{DD}^4(1)$ sequences to obtain $\{L(f^n)\}_{n=1}^{\infty}$. All the cases are described in Table 6, where the needed $\mathrm{DD}^4(1)$ sequences are specified, and from which the values of $J[f]$ are exported into Table 3.

Table 6. $J[f]$ for the case (i), $k = 2$.

d	(a)	(b)	(c)	$J[f]$
3	B	D	B	1
4	A and C	B	F	1 or 2
6	A and C	F	A and F	1 or 2

Case (ii): Here, it is easy to see that $\{L(f^n)\}_{n=1}^{\infty} = a_1\mathrm{reg}_1(n) + a_2\mathrm{reg}_2(n)$, with some integer coefficients a_1 and a_2, not both equal to zero. Thus, the sequence $\{L(f^n)\}_{n=1}^{\infty}$ can be represented as one $\mathrm{DD}^4(1)$ sequence, namely of the type (A), which implies that $J[f] = 1$. This completes the proof of the case $k = 2$.

For $k = 3$ the eigenvalues of f_{*2} are primitive roots of degree d, where $d \leq 2$, because there is no d for which $\varphi(d) = 3$. There are three possible cases:

(iii) 0 is one of the eigenvalues of f_{*2}. This case obviously reduces to $k = 2$.

(iv) There is one group of conjugated primitive roots of unity of degree d, where $\varphi(d) = 2$ and one primitive root of unity with $\varphi(d) = 1$, $d \in \{1, 2\}$.

(v) There are three primitive roots of degree d, where $\varphi(d) = 1$.

The list of all possible forms of Lefschetz numbers for the case (iv) is presented in Table 7. Comparing it with the Table 1 we find in each case the decomposition of $\{L(f^n)\}_{n=1}^{\infty}$ into $\mathrm{DD}^4(1)$ sequences. The resulting decompositions are given in Table 8, as well as the respective values of $J[f]$.

Table 7. All possible sequences $\{L(f^n)\}_{n=1}^{\infty}$ in the case (iv), $k = 3$.

d	(a)	(b)	(c)
3, 2	$\mathrm{reg}_2 + \mathrm{reg}_3$	$-2\mathrm{reg}_1 + 2\mathrm{reg}_2 + \mathrm{reg}_3$	$-\mathrm{reg}_1 + \mathrm{reg}_2 + \mathrm{reg}_3$
3, 1	$2\mathrm{reg}_1 + \mathrm{reg}_3$	$\mathrm{reg}_2 + \mathrm{reg}_3$	$\mathrm{reg}_1 + \mathrm{reg}_3$
4, 2	$\mathrm{reg}_1 + \mathrm{reg}_4$	$-\mathrm{reg}_1 + \mathrm{reg}_2 + \mathrm{reg}_4$	reg_4
4, 1	$3\mathrm{reg}_1 - \mathrm{reg}_2 + \mathrm{reg}_4$	$\mathrm{reg}_1 + \mathrm{reg}_4$	$2\mathrm{reg}_1 - \mathrm{reg}_2 + \mathrm{reg}_4$
6, 2	$2\mathrm{reg}_1 - \mathrm{reg}_3 + \mathrm{reg}_6$	$\mathrm{reg}_2 - \mathrm{reg}_3 + \mathrm{reg}_6$	$\mathrm{reg}_1 - \mathrm{reg}_3 + \mathrm{reg}_6$
6, 1	$4\mathrm{reg}_1 - \mathrm{reg}_2 - \mathrm{reg}_3 + \mathrm{reg}_6$	$2\mathrm{reg}_1 - \mathrm{reg}_3 + \mathrm{reg}_6$	$3\mathrm{reg}_1 - \mathrm{reg}_2 - \mathrm{reg}_3 + \mathrm{reg}_6$

Table 8. $J[f]$ for the case (iv), $k = 3$.

d	(a)	(b)	(c)	$J[f]$
3, 2	E	A and C	D	1 or 2
3, 1	B	E	B	1
4, 2	B	D	B	1
4, 1	A and C	B	A and C	1 or 2
6, 2	A and C	E	C	1 or 2
6, 1	A and C	A and C	A and C	2

In the case (v), by the same argument as in (ii) we obtain that $J[f] = 1$. This completes the proof of the case $k = 3$.

For $k = 4$ we concentrate on the case (vii), as the remaining cases are analogous to the respective ones considered before. We analyze the following subcases:

(vii.α) The eigenvalues of f_{*2} are primitive roots of unity of degree d, where $\varphi(d) = 4$, $d \in \{5, 8, 10, 12\}$.

(vii.β) The eigenvalues of f_{*2} are two groups of primitive roots of degree d, where $\varphi(d) = 2$, $d \in \{3, 4, 6\}$.

(vii.γ) The eigenvalues of f_{*2} are one group of primitive roots of degree d, where $\varphi(d) = 2$, $d \in \{3, 4, 6\}$, and two roots with $\varphi(d) = 1$, $d \in \{1, 2\}$.

The list of all possible forms of Lefschetz numbers for the case (vii.α) is presented in Table 9. Again, to calculate $J[f]$ we compare Table 9 with Table 1, finding the decompositions of $\{L(f^n)\}_{n=1}^{\infty}$ into $\mathrm{DD}^4(1)$ sequences. The results are presented in Table 10 for all the situations which appear in the case (vii.α).

Table 9. All possible sequences $\{L(f^n)\}_{n=1}^{\infty}$ in the case (vii.α), $k = 4$.

d	(a)	(b)	(c)
5	$\mathrm{reg}_1 + \mathrm{reg}_5$	$-\mathrm{reg}_1 + \mathrm{reg}_2 + \mathrm{reg}_5$	reg_5
8	$2\mathrm{reg}_1 - \mathrm{reg}_4 + \mathrm{reg}_8$	$\mathrm{reg}_2 - \mathrm{reg}_4 + \mathrm{reg}_8$	$\mathrm{reg}_1 - \mathrm{reg}_4 + \mathrm{reg}_8$
10	$3\mathrm{reg}_1 - \mathrm{reg}_2 - \mathrm{reg}_5 + \mathrm{reg}_{10}$	$\mathrm{reg}_1 - \mathrm{reg}_5 + \mathrm{reg}_{10}$	$2\mathrm{reg}_1 - \mathrm{reg}_2 - \mathrm{reg}_5 + \mathrm{reg}_{10}$
12	$2\mathrm{reg}_1 + \mathrm{reg}_2 - \mathrm{reg}_4 - \mathrm{reg}_6 + \mathrm{reg}_{12}$	$2\mathrm{reg}_2 - \mathrm{reg}_4 - \mathrm{reg}_6 + \mathrm{reg}_{12}$	$\mathrm{reg}_1 + \mathrm{reg}_2 - \mathrm{reg}_4 - \mathrm{reg}_6 + \mathrm{reg}_{12}$

Table 10. $J[f]$ in the case (vii.α), $k = 4$.

d	(a)	(b)	(c)	$J[f]$
5	B	D	B	1
8	B and F	A and F	F	1 or 2
10	A and F	C	A and F	1 or 2
12	A and F	A and F	A and F	2

In the cases (vii.β) and (vii.γ) we get by the formula (4) that $\{L(f^n)\}_{n=1}^{\infty} = a_1\mathrm{reg}_1 + a_2\mathrm{reg}_2 + a_3\mathrm{reg}_3 + a_4\mathrm{reg}_4 + a_6\mathrm{reg}_6$ with not all coefficients a_i equal to zero, which can always be represented as a sum of $\mathrm{DD}^4(1)$ sequences of the type (B) (for $p = 4$) and (C) (for $p = 3$) and thus $1 \leq J[f] \leq 2$. We omit the detailed calculations for the cases (vii.β) and (vii.γ), which are analogous to the previous ones, presenting the final results in Table 4. This ends the proof of the case $k = 4$ and the proof of the whole theorem. □

Now, we illustrate the method of applying Theorem 3.4 to find $J[f]$, considering the case of self-maps of $S^2 \times S^2$.

Theorem 3.6.

Let $f\colon S^2 \times S^2 \to S^2 \times S^2$ be a smooth map. Then $J[f] = 1$.

Proof. By Theorem 3.4, $J[f] \in \{1, 2\}$. We show that there are no self-maps of $S^2 \times S^2$ for which $J[f] = 2$. First notice that $k = 2$ here, which implies that the values of $J[f]$ are described by the cases (i) and (ii). By Table 2, $J[f]$ can be equal to 2 only in the following cases:

($\star$) $d \in \{4, 6\}$ and $D = 1$,

($\star\star$) $d = 6$ and $D = 0$.

We exclude both of the above items by a use of cohomological ring of $S^2 \times S^2$. The non-zero cohomology groups (as well as the corresponding homology groups) of $M = S^2 \times S^2$ are the following: $H^0(M; \mathbb{Q}) = \mathbb{Q}$, $H^2(M; \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$, $H^4(M; \mathbb{Q}) = \mathbb{Q}$. Notice that the induced maps on homology and cohomology have the same eigenvalues. Let f^{*4} be a multiplication by a constant $D \in \mathbb{Z}$, where D is the degree of f. Assume that the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

represents the induced homomorphism f^{*2}. Let us consider the cohomology ring of $S^2 \times S^2$: $H^*(M;\mathbb{Q}) = \bigoplus_{i\in\{0,2,4\}} H^i(M;\mathbb{Q})$. If α and β are the generators of $H^2(M;\mathbb{Q})$, then $\alpha \cup \beta$ is a generator of $H^4(M;\mathbb{Q})$.
Using the fact that $\alpha^2 = \beta^2 = 0$ and well-known properties of cohomology ring, cf. [14] for details, we obtain

$$f^{*4}(\alpha\cup\beta) = f^{*2}(\alpha)\cup f^{*2}(\beta) = (a\alpha + c\beta)\cup(b\alpha+d\beta) = ad(\alpha\cup\beta) + bc(\beta\cup\alpha) = (ad+bc)(\alpha\cup\beta) = D(\alpha\cup\beta)$$

and therefore

$$ad + bc = D. \tag{7}$$

On the other hand, the eigenvalues λ_1, λ_2 of A satisfy the equation $x^2 - (a+d)x + ad - bc = 0$. As they are conjugated roots of unity, we get $\lambda_1\lambda_2 = |\lambda_1| = |\lambda_2| = 1$. Thus, by Vieta's formula,

$$\lambda_1\lambda_2 = ad - bc = 1. \tag{8}$$

Now consider the cases ($\star$) and ($\star\star$) separately.

Case ($\star$): For $D = 1$ we have by (7), $ad + bc = 1$ and by (8), $ad - bc = 1$, so $ad = 1$ and $bc = 0$. The corresponding eigenvalues of A are then equal either 1 or -1 while roots of degree 4 or 6 could not appear.

Case ($\star\star$): For $D = 0$, similarly we have $ad + bc = 0$ and $ad - bc = 1$, which is however not possible for any $a, b, c, d \in \mathbb{Z}$.

Finally, there are no self-maps of $S^2 \times S^2$ for which $J[f] = 2$, and thus always $J[f] = 1$. □

4. Estimate for $J[f]$

Assume that a manifold M satisfies $\dim H_i(M;\mathbb{Q}) \le k$ for each i. In this section we find an estimate from above for $J[f]$ in dependence of k for manifolds of arbitrary dimension $m \ge 4$. By $\Theta(n)$, where $n \in \mathbb{N}$, we will denote the number of distinct divisors of n and by $\lceil x \rceil$ the smallest integer not less than x.

Theorem 4.1.
Let M be an m-dimensional smooth closed connected and simply-connected manifold with $m \ge 4$. Consider a smooth self-map f of M having periodic sequence of Lefschetz numbers of iterations. Let $k > 1$, if $\dim H_i(M;\mathbb{Q}) \le k$ for each i, then

$$J[f] \le k + \left\lceil \frac{k}{2\log^2 k} \right\rceil.$$

Proof. The proof consists of two parts. First, we determine the maximal number of basic sequences which may appear in the periodic expansion of $\{L(f^n)\}_{n=1}^{\infty}$, i.e. the maximal cardinality of the set O in formula (3). Next, we use Theorem 3.3 to estimate the number of $\mathrm{DD}^4(1)$ sequences that have to be used to realize all these basic sequences.

Each group of primitive roots of unity of a prescribed degree d can produce, by the formula (5), at most $\Theta(d)$ non-zero basic sequences of the form $a_s \mathrm{reg}_s(n)$, where $s \,|\, d$. As a consequence, taking into account the contributions from all such groups, we obtain

$$\#O \le \#\{s : s \,|\, d,\ \varphi(d) \le k\}.$$

On the other hand, if $s \,|\, d$ then $\phi(s) \le \phi(d)$. Thus for $s \,|\, d$, if d satisfies the inequality $\varphi(d) \le k$, then also $\varphi(s) \le k$. We get

$$\#O \le \#\{s : \varphi(s) \le k\}.$$

Let $F(k) = \#\{s : \varphi(s) \le k\}$. Then, $F(k)$ can be represented in the following form:

$$F(k) = \frac{\zeta(2)\,\zeta(3)}{\zeta(6)} \cdot k + R(k),$$

where $R(k) < k/\log^2 k$ and $\zeta(z) = \sum_{n=1}^{\infty} 1/n^z$ is the Riemann zeta function [21] and $\zeta(3)$ (Apéry's constant) satisfies $\zeta(3) \in (1{,}202; 1{,}203)$ [2]. Taking into account that $\zeta(2) = \pi^2/6$ and $\zeta(6) = \pi^6/945$, cf. [1], we obtain

$$\#O \leq F(k) < \frac{1{,}203 \cdot \pi^2/6}{\pi^6/945} \cdot k + \frac{k}{\log^2 k} < 2k + \frac{k}{\log^2 k}. \tag{9}$$

Now, to estimate $J[f]$ from above by a number A, we have to represent the sum of $2k + k/\log^2 k$ basic sequences as a sum of A $\mathrm{DD}^4(1)$ sequences. First of all, let us notice that 1 and 2 belong to the set $\{s : \varphi(s) \leq k\}$ and $a_1\mathrm{reg}_1(n) + a_2\mathrm{reg}_2(n)$ is a $\mathrm{DD}^4(1)$ sequence of the type (A), see Table 1. Furthermore, observe that by Theorem 3.3 there is a $\mathrm{DD}^4(1)$ sequence which consists of at least two basic sequences $a_p\mathrm{reg}_p$, $a_q\mathrm{reg}_q$ with arbitrary coefficients a_p and a_q, $p, q > 2$, namely the sequence of the type (F). Thus, we can always represent r basic sequences as a sum of one sequence of the type (A) and $\lceil r/2 \rceil - 1$, $\mathrm{DD}^4(1)$ sequences of the type (F). What is more, every $\mathrm{DD}^4(1)$ sequence is also a $\mathrm{DD}^m(1)$ sequence for $m \geq 4$ [8]. We get finally, by the formula (9), that for any manifold of dimension $m \geq 4$,

$$J[f] \leq k + \left\lceil \frac{k}{2\log^2 k} \right\rceil. \qquad \square$$

5. Continuous and smooth categories: $J[f]$ equal to one

As it was mentioned in Introduction, for simply-connected manifolds the value of $\mathrm{MF}_{\leq r}(f) = \min\left\{\# \bigcup_{k \leq r} \mathrm{Fix}(g^k) : g \sim f\right\}$ is always equal either to 0 or to 1 and the first case holds if and only if all Lefschetz numbers of iterations are equal to zero, cf. [13, 16].

If $\{L(f^n)\}_{n=1}^{\infty}$ is constantly equal to zero, then by the definition $J[f] = 0$, thus minimizations in the smooth and continuous categories are equivalent. We ask in which other cases this is still true. This question might be expressed by the use of the invariant $J[f]$ in the following way: which conditions on the manifold M should be satisfied to get $J[f] = 1$ for all smooth self-maps of M (with periodic Lefschetz numbers of iterations). We will prove that if for all i, $\dim H_i(M; \mathbb{Q}) \leq k$, then, under the assumption that dimension of M is high enough, $J[f] = 1$. By a prime power we mean a positive integer being a power of a prime number. We will denote by $\pi_*(l)$ the number of prime powers not exceeding l, and by $[x]$ the integer part of x.

Definition 5.1.

Let H be a subset of natural numbers. By $\mathrm{LCM}(H)$ we mean the least common multiple of all elements in H with the convention that $\mathrm{LCM}(\emptyset) = 1$. We define the set $\overline{H}$ by: $\overline{H} = \{\mathrm{LCM}(Q) : Q \subset H\}$. Next, for natural s we denote by $L_2(s)$ any set of natural numbers of the form $\overline{L}$, where $\#L = s + 1$ and $1 \notin L$, $2 \in L$.

Theorem 5.2.

Let M be an m-dimensional, $m \geq 4$, smooth closed connected and simply-connected manifold such that for each $0 \leq i \leq m$, $\dim H_i(M; \mathbb{Q}) \leq k$ for a prescribed number k. If $[(m-3)/2] + 1 \geq \pi_*(k^2)$, and $\{L(f^n)\}_{n=1}^{\infty}$ is not constantly equal to zero, then $J[f] = 1$ for all smooth self-maps of M with periodic sequence of Lefschetz numbers.

Proof. The case $k = 1$ holds by Remark 3.2. Let $k > 1$. If $\dim H_i(M; \mathbb{Q}) \leq k$, then we consider a representation of Lefschetz numbers in the form

$$L(f^n) = \sum_{l \in \mathbb{N}} b_l \mathrm{reg}_l(n).$$

We notice that $b_l \neq 0$ implies $\varphi(l) \leq k$. Now let $l \in B = \{s : \varphi(s) \leq k\}$, we consider

$$L(f^n) = \sum_{l \in B} b_l \mathrm{reg}_l(n). \tag{10}$$

We ask for which m the sequence (10) is a $\mathrm{DD}^m(1)$ sequence, which is equivalent to the statement that $J[f]=1$. Let p be a prime number, observe that

$$p^\alpha \in B \quad \Longrightarrow \quad p^\alpha \le k^2. \tag{11}$$

Indeed, by the equality $\sqrt{s} \le \varphi(s)$ which holds for $s>6$, cf. [21], we get that if $s \in B$, then $\sqrt{s} \le \varphi(s) \le k$, thus $s \le k^2$. We verify that for $p^\alpha = s \le 6$ the implication (11) also holds.

Let $\mathcal{P}$ be the set of all primes. Define the set

$$G = \{p^\alpha : p \in \mathcal{P},\, p^\alpha \le k^2,\, \alpha \in \mathbb{N}\}.$$

By (11) we notice that $B \subset \overline{G}$. On the other hand, it was proved in [12] that there exists a $\mathrm{DD}^m(1)$ sequence of the form

$$c_A(n) = \sum_{k \in L_2([(m-3)/2])} a_k \operatorname{reg}_k(n),$$

where the coefficients a_k could be arbitrary integers.

Because $2 \in G$ ($k \ge 1$), we may take $L_2([(m-3)/2]) = G$, realizing by c_A the sequence $\{L(f^n)\}_{n=1}^\infty$ under the condition that $[(m-3)/2]+1 \ge \# G = \pi_*(k^2)$. This ends the proof. □

Remark 5.3.

The estimate for m (which guarantees that $J[f]=1$) given in Theorem 5.2 is rather rough. One can obtain a much better estimate in the following way. Observe that if $p^\alpha \in B$ then also $p \in B$, so under the condition $p^\alpha \in B$ we get that $\varphi(p) \le k$ and $\varphi(p^\alpha) \le k$, or equivalently $p \le k+1$ and $p^\alpha - p^{\alpha-1} \le k$. The last inequality implies

$$\alpha \le \left[\log_p \frac{k}{p-1} + 1\right]. \tag{12}$$

Let us denote the right-hand side of the inequality (12) by α_p. Then, instead of the dependence (11) we can use the following one:

$$p^\alpha \in B \qquad \Longrightarrow \qquad p \le k+1 \quad \text{and} \quad \alpha \le \alpha_p.$$

As a consequence, $B \subset \overline{G}_1$, where

$$G_1 = \{p^\alpha : p \in \mathcal{P},\, p \le k+1,\, \alpha \le \alpha_p\}.$$

Finally, we can change the condition for m in Theorem 5.2, demanding that the following inequality is satisfied:

$$\left[\frac{m-3}{2}\right] + 1 \ge \# G_1. \tag{13}$$

The condition (13) provides a better estimate for m because $\# G_1$ is much less than $\# G = \pi_*(k^2)$.

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