MODELING OF THE FINITE AMPLITUDE PLANE WAVE PROPAGATION IN NON-DISSIPATIVE MEDIUM

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The paper presents results of theoretical analysis of the finite amplitude plane wave propagation problem. The case of harmonic plane wave propagation in non-dissipative medium was considered. The mathematical model and some results of numerical investigations are presented. The mathematical model was built on the basis of one-dimensional continuity equation, equation of motion in differential form and state equation. The finite difference method was applied to solve the problem numerically. The pressure changes and harmonic pressure amplitude changes were analysed. The results of computer calculations were compared with solution of the Burgers equation.

INTRODUCTION

The mathematical model of finite amplitude wave propagation problem is described basing on continuity, motion and state equations. The system of these equations describes pressure, velocity and density changes in medium. They are usually converted to one nonlinear partial differential equation. The nonlinear equation of acoustics is one of them. It allows to analyse pressure changes along sound beam. However in practice equations which have easier form are used. For example, the KZK equation is very often used during theoretical studies. These equations have not analytical solutions till now and consequently it is necessary to solve them numerically. Theoretical analysis of the plane wave propagation is possible using the Westervelt equation and the Burgers equation [1, 2]. Existence of the Burgers equation solution together with knowledge of theoretical formulas of harmonic pressure amplitudes enable to verify correctness and to discus accuracy of calculations.

The aim of the paper was to present mathematical and numerical models of the finite amplitude wave propagation problem in non-dissipative medium which are built on the basis of continuum mechanics equations. The convergence and accuracy of obtained discrete equations are discussed. Numerical calculations were carried out for different values of physical and numerical parameters.

1. MATHEMATICAL MODEL

To build the mathematical model of finite amplitude plane wave propagation problem the one-dimensional continuity equation and equation of motion in differential form are considered:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0 \tag{1}$$

$$\rho\frac{\partial v}{\partial t}+\rho v\frac{\partial v}{\partial x}+\frac{\partial p}{\partial x}=0 \tag{2}$$

where p denotes pressure, ρ – density of the medium and v – velocity. To close the system state equation is added

$$\rho=\rho_0+\frac{1}{c_0^2}p'-\frac{\varepsilon-1}{c_0^4\rho_0}p'^2 \tag{3}$$

where $p'=p-p_0$ is acoustic pressure, ρ_0 – medium density at rest, c_0 – speed of sound, ε – nonlinear coefficient. Substituting equation (3) to equations (1) and (2) we obtain system of two first order partial differential equations:

$$\begin{aligned}
\frac{\partial p'}{\partial t}&=-v\frac{\partial p'}{\partial x}-\frac{\rho_0+\frac{1}{c_0^2}p'-\frac{\varepsilon-1}{c_0^4\rho_0}p'^2}{\frac{1}{c_0^2}-\frac{\varepsilon-1}{c_0^4\rho_0}2p'}\frac{\partial v}{\partial x}\\
\frac{\partial v}{\partial t}&=-v\frac{\partial v}{\partial x}-\frac{1}{\rho_0+\frac{1}{c_0^2}p'-\frac{\varepsilon-1}{c_0^4\rho_0}p'^2}\frac{\partial p'}{\partial x}
\end{aligned} \tag{4}$$

Assuming that harmonic plane wave propagates in water, the boundary condition can be written in following form:

$$\begin{aligned}
p'(x=0,t)&=p_0\sin\omega t\\
v(x=0,t)&=v_0\sin\omega t
\end{aligned} \tag{5}$$

where $\omega=2\pi f$, f - fundamental wave frequency. Moreover it is assumed that pressure p' and velocity v are periodic functions of the time coordinate.

The Burgers equation is often used to analyse finite amplitude plane wave propagation problem. If the acoustics Reynolds number is big (Re_a>>1), the dissipative term can be omitted and then the Burgers equation is written in following form:

$$\frac{\partial p'}{\partial x}-\frac{\varepsilon}{\rho_0 c_0^3}p'\frac{\partial p'}{\partial \tau}=0 \tag{6}$$

where $\tau=t-x/c_0$.

The waveform change is equivalent with spectrum change. The harmonic analysis is very often used to investigate wave distortion. The harmonic pressure amplitudes are defined in following form:

$$p_n(x) = \frac{2x_N}{nx} J_n\left(\frac{nx}{x_N}\right) \tag{7}$$

where $x_N = \dfrac{\rho_0 c_0^3}{\varepsilon \omega p_0}$ and J_n is the Bessel function of the first kind of order n.

2. NUMERICAL SOLUTION

Let us define new normalized functions $P = \dfrac{p'}{p_0}$ and $V = \dfrac{\rho_0 c_0}{p_0} v$. Substituting them into (4) we obtain system of two partial differential equations:

$$\begin{aligned}
\frac{\partial P}{\partial t} &= -\frac{p_0}{c_0 \rho_0} V \frac{\partial P}{\partial x} - \frac{1 + \dfrac{p_0}{c_0^2 \rho_0} P - \dfrac{\varepsilon - 1}{c_0^4 \rho_0^{\,2}} p_0^{\,2} P^2}{\dfrac{1}{c_0} - \dfrac{\varepsilon - 1}{c_0^3 \rho_0} 2 p_0 P} \frac{\partial V}{\partial x} \\
\frac{\partial V}{\partial t} &= -\frac{p_0}{c_0 \rho_0} V \frac{\partial V}{\partial x} - \frac{c_0}{1 + \dfrac{p_0}{c_0^2 \rho_0} P - \dfrac{\varepsilon - 1}{c_0^4 \rho_0^{\,2}} p_0^{\,2} P^2} \frac{\partial P}{\partial x}
\end{aligned} \tag{8}$$

To solve this system of equations the Lax method is used. Let we define nodal points as follows $t_m = m\Delta t,\ x_i = i\Delta x$. After approximation of derivatives equations (8) may be written in following way:

$$\begin{aligned}
\delta_t P_i^m &= -\frac{p_0}{c_0 \rho_0} V_i^m \delta_x P_i^m - \frac{A(P_i^m)}{B(P_i^m)} \delta_x V_i^m \\
\delta_t V_i^m &= -\frac{p_0}{c_0 \rho_0} V_i^m \delta_x V_i^m - \frac{c_0}{A(P_i^m)} \delta_x P_i^m
\end{aligned} \tag{9}$$

where: $P_i^m = P(x_i, t_m),\ V_i^m = V(x_i, t_m)$,

δ_t - difference operator of differential operator $\partial / \partial t$

δ_x - difference operator of differential operator $\partial / \partial x$

$$A(P_i^m) = 1 + \frac{p_0}{c_0^2 \rho_0} P_i^m - \frac{\varepsilon - 1}{c_0^4 \rho_0^{\,2}} p_0^{\,2} (P_i^m)^2$$

$$B(P_i^m) = \frac{1}{c_0} - \frac{\varepsilon - 1}{c_0^3 \rho_0} 2 p_0 P_i^m$$

3. NUMERICAL INVESTIGATIONS

Solving the system of difference equations (9) we obtain pressure and velocity at the mesh points (x_i, t_m). During finite amplitude wave propagation we observe shape changes step by step. Figure 1 presents normalized pressure as a function of time obtained for different distances. Numerical calculations were carried out assuming that harmonic wave which frequency f=1 MHz and amplitude p_0=150 kPa is propagated in water where medium density at rest ρ_0=1000 kg/m^3, speed of sound c_0=1500 m/s, nonlinear coefficient ε=3.5. Curve number 1 presents shape of primary wave curves 2 and 3 show pressure at distance x=0.3 m and x=0.6 m respectively. Similar results we obtain for velocity. Figure 2 presents four first harmonic pressure amplitudes along x axis calculated for the same values of physical and numerical parameters.

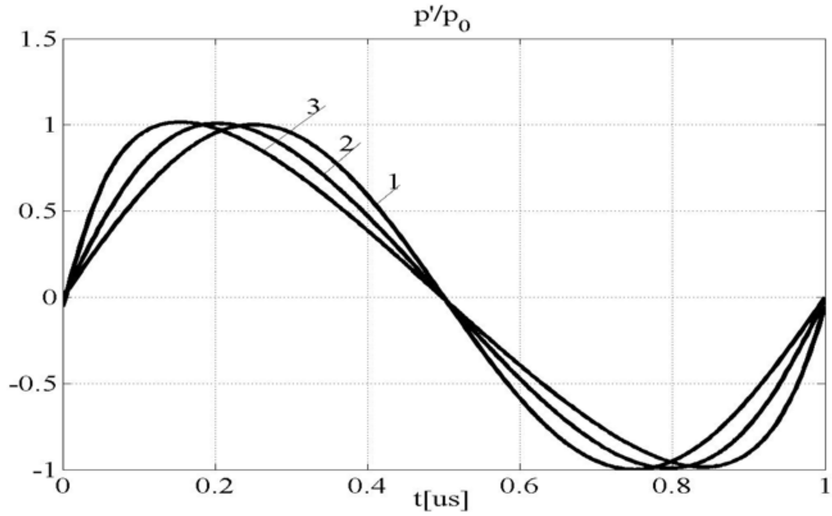

Fig.1 Normalized pressure as a function of time: 1- x=0, 2 - x=0.3m, 3 - x=0.6 m

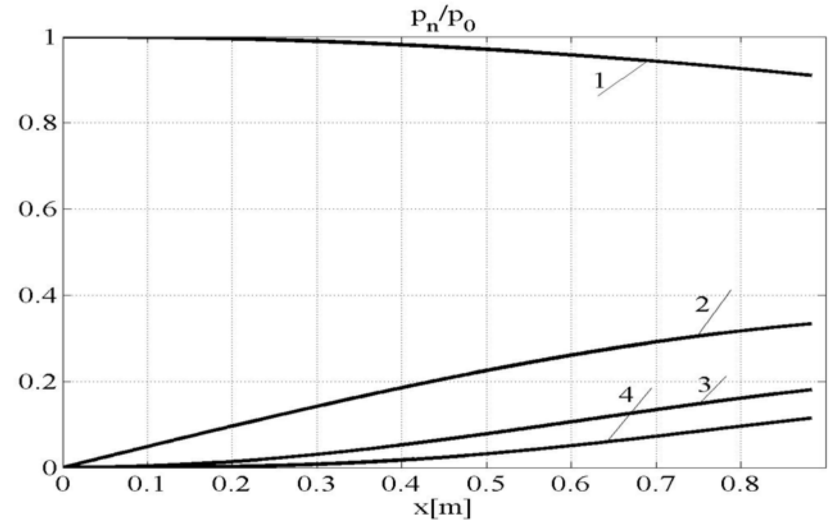

Fig.2 Normalized harmonic pressure amplitudes along x axis

The correct choice of the net spacing values is very important during numerical calculations. To analyse this problem in detail, the results of numerical calculations of pressure and their harmonic amplitudes were compared with solution of the Burgers equation.

The results of numerical calculations presented in Fig. 1 were carried out for Δt=0.0039 μs. Similar results of pressure changes obtained for Δt=0.0156 μs shows next figure. Note that in this example the time net spacing Δt and consequently distance one Δx were not small enough and for higher distances we observe numerical errors.

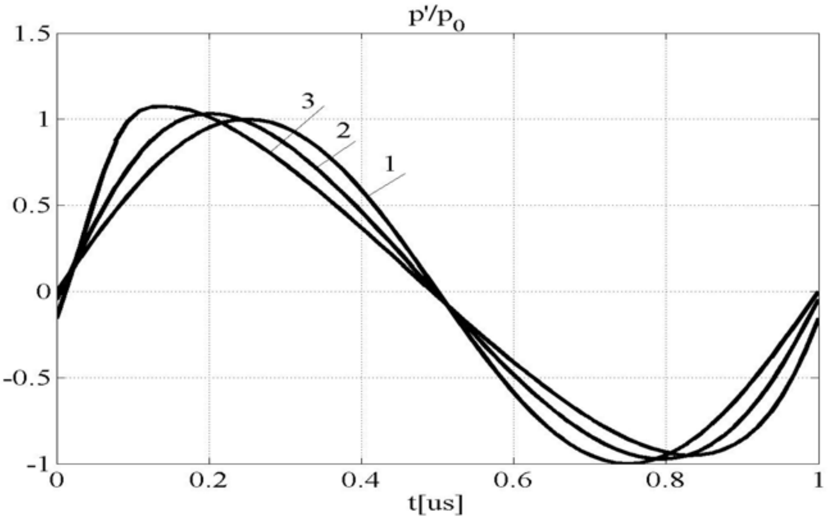

Fig.3 Normalized pressure as a function of time: 1- x=0, 2 - x=0.3m, x=0.6 m

The correct choice of numerical parameters is not only one problem during numerical calculations. It is important to remember that proposed numerical approximation of the derivatives requires continuity of solution. Solution of our problem becomes non-continuous for distances higher then x_N. For using till now physical parameters, this distance is equal x_N=1 m.

Figure 4 shows normalized pressure calculated for two different distances and two different values of time steps. Left figure presents results obtained for distance x=0.4 m, right figure presents similar results obtained for distance x=0.8 m respectively. Curve number 1 (dashed line) shows waveform obtained for Δt=0.0039 μs. Curve number 2 (solid line) was obtained for Δt=0.0156 μs.

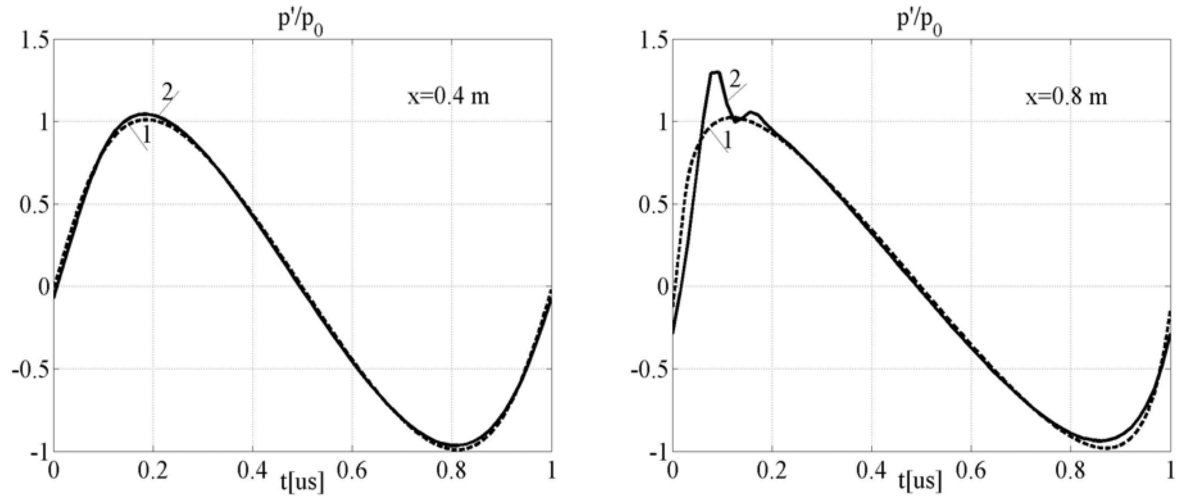

Fig.4 Normalized pressure as a function of time: 1 - Δt=0.0039 μs, 2 - Δt=0.0156 μs

Figure 5 shows four first harmonic pressure amplitudes along x axis obtained for different net spacing Δt. Left figure presents first harmonic pressure amplitude calculated for Δt=0.0156 μs (solid line) and Δt=0.0039 μs (dashed line). Similar results obtained for second, third and fourth harmonic component are shown in right figure.

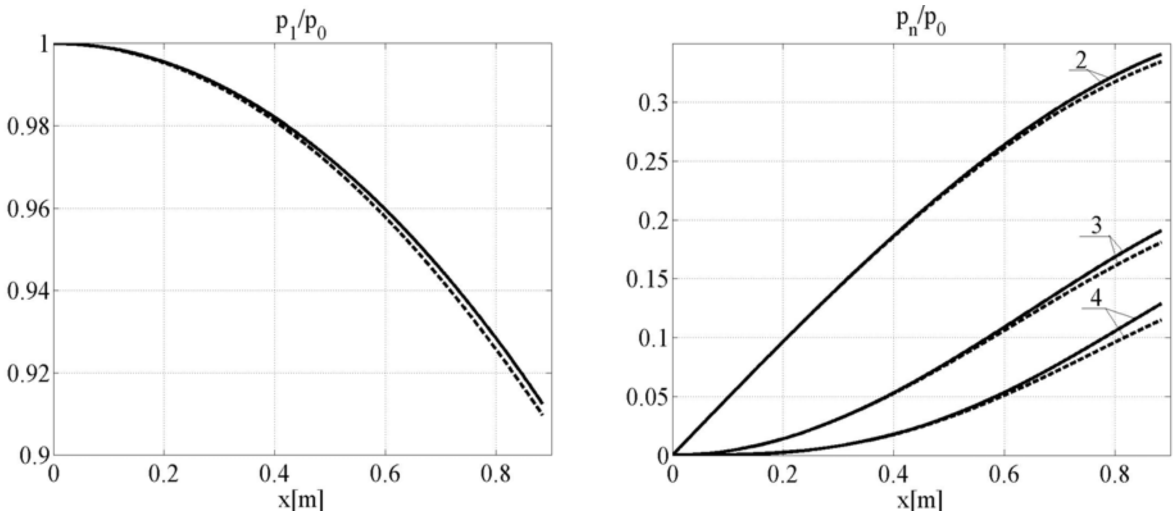

Fig.5 From first to fourth harmonic pressure amplitudes along x axis for different net spacing Δt

To analyse correctness of the results obtained numerically, the results of calculations were compared with theoretical solutions. The absolute error $E_n(x) = \left| p_n(x) - \widetilde{p}_n(x) \right|$ and relative error $R_n(x) = \left| p_n(x) - \widetilde{p}_n(x) \right| / \left| p_n(x) \right|$ were analysed. Functions p_n and $\widetilde{p}_n$ denote harmonic pressure amplitude calculated using formula (7) and numerically respectively. Numerical calculations were carried out for different values of time net spacing. The calculations were done for distances to x=0.9 m.

Tables 1 and 2 collect the maximum absolute errors and relative error of first five harmonic pressure amplitudes for fixed steps Δt respectively.

Table 1. Maximum absolute error of harmonic pressure amplitudes

	E_1	E_2	E_3	E_4	E_5
Δt=0.0156 μs	$2.95\cdot10^{-3}$	$6.63\cdot10^{-3}$	$1.05\cdot10^{-2}$	$1.44\cdot10^{-2}$	$1.85\cdot10^{-2}$
Δt=0.0078 μs	$7.25\cdot10^{-4}$	$1.54\cdot10^{-3}$	$2.31\cdot10^{-3}$	$2.98\cdot10^{-3}$	$3.59\cdot10^{-3}$
Δt=0.0039 μs	$1.82\cdot10^{-4}$	$3.83\cdot10^{-4}$	$5.66\cdot10^{-4}$	$7.20\cdot10^{-4}$	$8.49\cdot10^{-4}$

Table 2. Maximum relative error of harmonic pressure amplitudes

	R_1	R_2	R_3	R_4	R_5
Δt=0.0156 μs	$3.22\cdot10^{-3}$	$1.99\cdot10^{-2}$	$5.82\cdot10^{-2}$	$1.26\cdot10^{-1}$	$2.34\cdot10^{-1}$
Δt=0.0078 μs	$7.91\cdot10^{-4}$	$4.62\cdot10^{-3}$	$1.28\cdot10^{-2}$	$2.61\cdot10^{-2}$	$4.53\cdot10^{-2}$
Δt=0.0039 μs	$1.99\cdot10^{-4}$	$1.15\cdot10^{-3}$	$3.14\cdot10^{-3}$	$6.30\cdot10^{-3}$	$1.07\cdot10^{-2}$

Figure 6 presents relative errors of first harmonic pressure amplitude obtained for different values of time step Δt. Next figure shows the relative error for fixed harmonic pressure amplitudes calculated with step Δt=0.0039 μs. In this example the maximum relative error of waveform at distance x=0.8 m is equal 0.033.

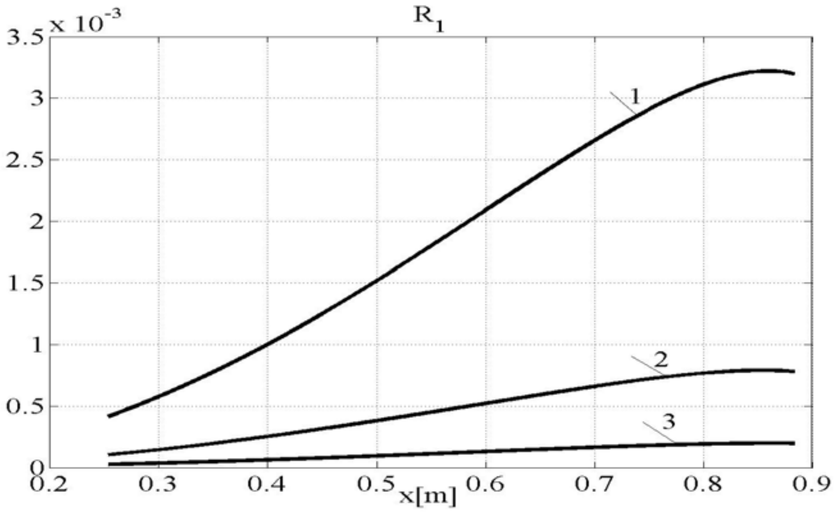

Fig.6 Relative error of first harmonic pressure amplitude:
1 - Δt=0.0156 μs, 2 - Δt=0.0078 μs, 3 - Δt=0.0039 μs

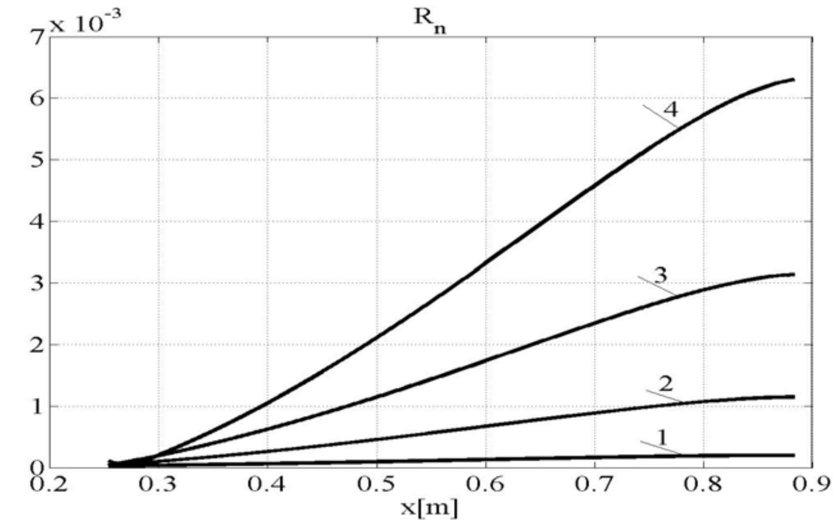

Fig.7 Relative error for different harmonic pressure amplitudes

To complete the discussion about accuracy, solution of equations (9) and solution of the Burgers equation obtained numerically were compared. The relative error of harmonic pressure amplitudes at distance x=0.8 m obtained using presented in this paper numerical model when Δt=0.0039 μs are follows:

$$R_1=1.92\cdot10^{-4},\quad R_2=1.07\cdot10^{-3},\quad R_3=2.88\cdot10^{-3},\quad R_4=5.71\cdot10^{-3}.$$

Solving numerically the Burgers equation we obtain:

$$R_1=3.01\cdot10^{-3},\quad R_2=2.09\cdot10^{-2},\quad R_3=1.19\cdot10^{-2},\quad R_4=1.94\cdot10^{-2}.$$

The finite difference method with the same values of physical and numerical parameters then earlier was used to solve equation (6). These results show that convergence of calculations is faster for proposed model than for numerical solution of the Burgers equation.

Presented till now results of computer calculations were carried out with different values of numerical parameters for the same values of primary wave frequency and pressure, i.e. for f=1MHz and p_0=150 kPa respectively. Numerical investigations were done not only for different step sizes but also for different values of physical parameters. Figure 8 shows harmonic pressure amplitudes obtained for wave which frequency f=1 MHz and pressure p_0=300 kPa. The results of computations with pressure p_0=150 kPa and frequency f=0.5 MHz presents right figure. Note that now calculations were carried out for different distances than earlier. In these examples the discontinuity appears at distance x_N=0.5 m and x_N=2 m respectively and it is the reason why investigated distances are different.

a) b)

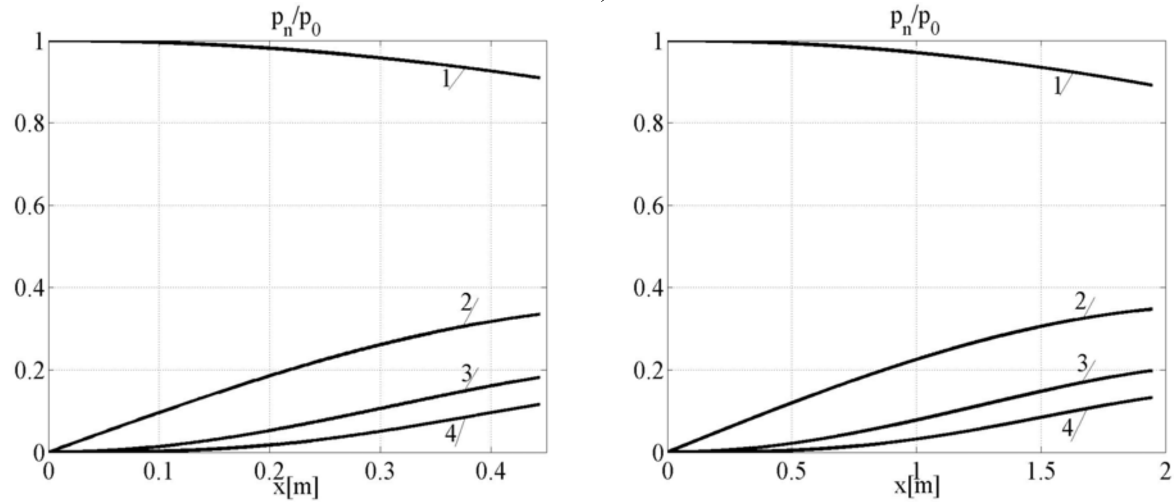

Fig.8 Normalized harmonic pressure amplitudes along x axis:
a) f=1 MHz, p_0=300 kPa, b) f=0.5 MHz, p_0=150 kPa

4. CONCLUSIONS

The finite amplitude plane wave propagation problem was considered. The mathematical model, which was worked out on the basis of mechanics equations and some results of numerical investigations have been presented. The numerical calculations were carried out using own computer program that was worked out on the basis of obtained mathematical model. The results of numerical calculations confirm that proposed numerical model is convergent. Moreover this method allows to obtain the same accuracy using higher step sizes than solving numerically the Burgers equation. However it is necessary to remember that it can be used only for continuous solutions of the problem. The presented in this paper mathematical and numerical models can be used to analyse the wave propagation in non-dissipative medium or for small distances when dissipation effects are not very big. When they cannot be omitted this model must be extended. However in this situation we obtain different type of partial differential equations and it is necessary to build more complicated difference equations.

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