

## Non-isolating 2-bondage in graphs

By Marcin KRZYWKOWSKI

(Received Dec. 14, 2010)

(Revised Apr. 30, 2011)

**Abstract.** A 2-dominating set of a graph  $G = (V, E)$  is a set  $D$  of vertices of  $G$  such that every vertex of  $V(G) \setminus D$  has at least two neighbors in  $D$ . The 2-domination number of a graph  $G$ , denoted by  $\gamma_2(G)$ , is the minimum cardinality of a 2-dominating set of  $G$ . The non-isolating 2-bondage number of  $G$ , denoted by  $b'_2(G)$ , is the minimum cardinality among all sets of edges  $E' \subseteq E$  such that  $\delta(G - E') \geq 1$  and  $\gamma_2(G - E') > \gamma_2(G)$ . If for every  $E' \subseteq E$ , either  $\gamma_2(G - E') = \gamma_2(G)$  or  $\delta(G - E') = 0$ , then we define  $b'_2(G) = 0$ , and we say that  $G$  is a  $\gamma_2$ -non-isolatingly strongly stable graph. First we discuss the basic properties of non-isolating 2-bondage in graphs. We find the non-isolating 2-bondage numbers for several classes of graphs. Next we show that for every non-negative integer there exists a tree having such non-isolating 2-bondage number. Finally, we characterize all  $\gamma_2$ -non-isolatingly strongly stable trees.

### 1. Introduction.

Let  $G = (V, E)$  be a graph. By the neighborhood of a vertex  $v$  of  $G$  we mean the set  $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ . The degree of a vertex  $v$ , denoted by  $d_G(v)$ , is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong if it is adjacent to at least two leaves. Let  $\delta(G)$  mean the minimum degree among all vertices of  $G$ . The path (cycle, respectively) on  $n$  vertices we denote by  $P_n$  ( $C_n$ , respectively). A wheel  $W_n$ , where  $n \geq 4$ , is a graph with  $n$  vertices, formed by connecting a vertex to all vertices of a cycle  $C_{n-1}$ . Let  $T$  be a tree, and let  $v$  be a vertex of  $T$ . We say that  $v$  is adjacent to a tree  $H$  if there is a neighbor of  $v$ , say  $x$ , such that the tree resulting from  $T$  by removing the edge  $vx$ , and which contains the vertex  $x$ , is a tree  $H$ . Let  $K_{p,q}$  denote a complete bipartite graph the partite sets of which have cardinalities  $p$  and  $q$ . By a star we mean a connected graph in which exactly one vertex has degree greater than one. By a double star we mean a graph that can be obtained from a star by joining a positive number of vertices to one of the leaves.

---

2010 *Mathematics Subject Classification.* Primary 05C05, 05C69.

*Key Words and Phrases.* 2-domination, bondage, non-isolating 2-bondage, graph, tree.

The research was supported by the Polish National Science Centre grant 2011/02/A/ST6/00201.

A subset  $D \subseteq V(G)$  is a dominating set of  $G$  if every vertex of  $V(G) \setminus D$  has a neighbor in  $D$ , while it is a 2-dominating set, abbreviated 2DS, of  $G$  if every vertex of  $V(G) \setminus D$  has at least two neighbors in  $D$ . The domination (2-domination, respectively) number of a graph  $G$ , denoted by  $\gamma(G)$  ( $\gamma_2(G)$ , respectively), is the minimum cardinality of a dominating (2-dominating, respectively) set of  $G$ . Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least  $k$  times for a fixed positive integer  $k$ . Multiple domination was introduced by Fink and Jacobson [3], and further studied for example in [1], [13]. For a comprehensive survey of domination in graphs, see [7], [8].

The bondage number  $b(G)$  of a graph  $G$  is the minimum cardinality among all sets of edges  $E' \subseteq E$  such that  $\gamma(G - E') > \gamma(G)$ . If for every  $E' \subseteq E$  we have  $\gamma(G - E') = \gamma(G)$ , then we define  $b(G) = 0$ , and we say that  $G$  is a  $\gamma$ -strongly stable graph. Bondage in graphs was introduced in [4], and further studied for example in [2], [5], [6], [9], [10], [11], [12] and [14].

We define the non-isolating 2-bondage number of a graph  $G$ , denoted by  $b'_2(G)$ , to be the minimum cardinality among all sets of edges  $E' \subseteq E$  such that  $\delta(G - E') \geq 1$  and  $\gamma_2(G - E') > \gamma_2(G)$ . Thus  $b'_2(G)$  is the minimum number of edges of  $G$  that have to be removed in order to obtain a graph with no isolated vertices, and with the 2-domination number greater than that of  $G$ . If for every  $E' \subseteq E$ , either  $\gamma_2(G - E') = \gamma_2(G)$  or  $\delta(G - E') = 0$ , then we define  $b'_2(G) = 0$ , and we say that  $G$  is a  $\gamma_2$ -non-isolatingly strongly stable graph.

First we discuss the basic properties of non-isolating 2-bondage in graphs. We find the non-isolating 2-bondage numbers for several classes of graphs. Next we show that for every non-negative integer there exists a tree having such non-isolating 2-bondage number. Finally, we characterize all  $\gamma_2$ -non-isolatingly strongly stable trees.

## 2. Results.

We begin with the following observations.

Observation 1: *Every leaf of a graph  $G$  is in every  $\gamma_2(G)$ -set.*

Observation 2: *If  $H \subseteq G$  and  $V(H) = V(G)$ , then  $\gamma_2(H) \geq \gamma_2(G)$ .*

Observation 3: *For every positive integer  $n$  we have  $\gamma_2(K_n) = \min\{2, n\}$ .*

Observation 4: *If  $n$  is a positive integer, then  $\gamma_2(P_n) = \lfloor n/2 \rfloor + 1$ .*

Observation 5: *For every integer  $n \geq 3$  we have  $\gamma_2(C_n) = \lfloor (n+1)/2 \rfloor$ .*



Observation 6: For every integer  $n \geq 4$  we have

$$\gamma_2(W_n) = \begin{cases} 2 & \text{if } n = 4, 5; \\ \lfloor (n+1)/3 \rfloor + 1 & \text{if } n \geq 6. \end{cases}$$

Observation 7: Let  $p$  and  $q$  be positive integers such that  $p \leq q$ . Then

$$\gamma_2(K_{p,q}) = \begin{cases} \max\{q, 2\} & \text{if } p = 1; \\ \min\{p, 4\} & \text{if } p \geq 2. \end{cases}$$

Since the definition of the non-isolating 2-bondage does not allow isolated vertices in the searched subgraphs of a given graph, in this paper, we do not consider removing edges that produces an isolated vertex.

First we find the non-isolating 2-bondage numbers of complete graphs.

REMARK 8. For every positive integer  $n$  we have

$$b'_2(K_n) = \begin{cases} 0 & \text{if } n = 1, 2, 3; \\ \lfloor 2n/3 \rfloor & \text{otherwise.} \end{cases}$$

PROOF. Of course,  $b'_2(K_1) = 0$ ,  $b'_2(K_2) = 0$ , and  $b'_2(K_3) = 0$ . Now assume that  $n \geq 4$ . Let  $E(K_n) = \{v_1, v_2, \dots, v_n\}$ . Let  $G$  be a graph with at least two vertices. Let us observe that  $\gamma_2(G) = 2$  if and only if  $G$  has two vertices which are both adjacent to every vertex other than they. Let  $E' \subseteq E(K_n)$ . Let us observe that  $\gamma_2(K_n - E') > 2$  if and only if at most one vertex of  $K_n$  is not incident to any edge of  $E'$ , and every edge of  $E'$  is adjacent to some other edge of  $E'$ . We want to choose a smallest set  $E' \subseteq E(K_n)$  satisfying the condition above while  $\delta(K_n - E') \geq 1$ . Let us observe that the most efficient way of choosing edges of  $K_n$  is to choose for example edges  $v_1v_2, v_2v_3, v_4v_5, v_5v_6$ , and so on. In this way no vertex becomes isolated. Let  $k$  be a positive integer.

If  $n = 3k$ , then we remove  $2k$  edges. Thus  $b'_2(K_{3k}) = 2k = \lfloor 2n/3 \rfloor$ . If  $n = 3k + 1$ , then we also remove  $2k$  edges as one vertex can remain universal. We have  $b'_2(K_{3k+1}) = 2k = \lfloor 2k + 2/3 \rfloor = \lfloor 2(3k + 1)/3 \rfloor = \lfloor 2n/3 \rfloor$ . Now assume that  $n = 3k + 2$ . If we remove the edges  $v_1v_2, v_2v_3, v_4v_5, v_5v_6, \dots, v_{2k-2}v_{2k-1}, v_{2k-1}v_{2k}$ , then the vertices  $v_{3k+1}$  and  $v_{3k+2}$  remain universal. Therefore  $b'_2(K_{3k+2}) > 2k$ . Let us observe that removing the edges  $v_1v_2, v_2v_3, v_4v_5, v_5v_6, \dots, v_{2k-2}v_{2k-1}, v_{2k-1}v_{2k}, v_{2k}v_{2k+1}$  increases the 2-domination number. This implies that  $b'_2(K_{3k+2}) = 2k + 1 = \lfloor 2k + 4/3 \rfloor = \lfloor 2(3k + 2)/3 \rfloor = \lfloor 2n/3 \rfloor$ .  $\square$

Now we calculate the non-isolating 2-bondage numbers of paths.



REMARK 9. If  $n$  is a positive integer, then

$$b'_2(P_n) = \begin{cases} 0 & \text{for } n = 1, 2, 3; \\ 1 & \text{for } n \geq 4. \end{cases}$$

Now we investigate the non-isolating 2-bondage in cycles.

REMARK 10. For every integer  $n \geq 3$  we have

$$b'_2(C_n) = \begin{cases} 0 & \text{if } n = 3; \\ 1 & \text{if } n \text{ is even}; \\ 2 & \text{otherwise.} \end{cases}$$

Now we calculate the non-isolating 2-bondage numbers of wheels.

REMARK 11. For every integer  $n \geq 4$  we have

$$b'_2(W_n) = \begin{cases} 1 & \text{if } n = 5; \\ 2 & \text{if } n \neq 3k + 2; \\ 3 & \text{otherwise.} \end{cases}$$

PROOF. Let  $E(W_n) = \{v_1v_2, v_1v_3, \dots, v_1v_n, v_2v_3, v_3v_4, \dots, v_{n-1}v_n, v_nv_2\}$ . Since  $W_4 = K_4$ , by Remark 8 we get  $b'_2(W_4) = b'_2(K_4) = \lfloor 8/3 \rfloor = 2$ . By Observation 6 we have  $\gamma_2(W_5) = 2$ . Let us observe that  $\gamma_2(W_5 - v_2v_3) = 3 > 2 = \gamma_2(W_5)$ . Thus  $b'_2(W_5) = 1$ . Now let us assume that  $n \geq 6$ . If we remove an edge incident with  $v_1$ , say  $v_1v_2$ , then we get  $\gamma_2(W_n - v_1v_2) = \gamma_2(W_n)$  as we can construct a  $\gamma_2(W_n)$ -set that contains the vertices  $v_1$  and  $v_2$ ; such set is also a 2DS of the graph  $W_n - v_1v_2$ . If we remove an edge non-incident with  $v_1$ , say  $v_2v_3$ , then we get  $\gamma_2(W_n - v_2v_3) = \gamma_2(W_n)$  as we can construct a  $\gamma_2(W_n)$ -set that does not contain the vertices  $v_2$  and  $v_3$ ; such set is also a 2DS of the graph  $W_n - v_2v_3$ . This implies that  $b'_2(W_n) = 0$  or  $b'_2(W_n) \geq 2$ . First assume that  $n = 3k$  or  $n = 3k + 1$ . Let us remove the edges  $v_{n-1}v_n$  and  $v_nv_2$ . We find a relation between the numbers  $\gamma_2(W_n - v_{n-1}v_n - v_nv_2)$  and  $\gamma_2(W_{n-1})$ . Let  $D$  be any  $\gamma_2(W_n - v_{n-1}v_n - v_nv_2)$ -set. By Observation 1 we have  $v_n \in D$ . Let us observe that  $D \setminus \{v_n\}$  is a 2DS of the graph  $W_{n-1}$ . Therefore  $\gamma_2(W_{n-1}) \leq \gamma_2(W_n - v_{n-1}v_n - v_nv_2) - 1$ . Using Observation 6 we get  $\gamma_2(W_n - v_{n-1}v_n - v_nv_2) \geq \gamma_2(W_{n-1}) + 1 = \lfloor n/3 \rfloor + 2 = \lfloor (n+1)/3 \rfloor + 2 > \lfloor (n+1)/3 \rfloor + 1 = \gamma_2(W_n)$ . Therefore  $b'_2(W_n) = 2$  if  $n = 3k$  or  $n = 3k + 1$ . Now assume that  $n = 3k + 2$ . It is not difficult to verify that now removing any two edges does not increase the 2-domination number. This implies

that  $b'_2(W_n) = 0$  or  $b'_2(W_n) \geq 3$ . Let us remove the edges  $v_{n-2}v_{n-1}$ ,  $v_{n-1}v_n$ , and  $v_nv_2$ . We find a relation between the numbers  $\gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2)$  and  $\gamma_2(W_{n-2})$ . Let  $D$  be any  $\gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2)$ -set. By Observation 1 we have  $v_{n-1}, v_n \in D$ . Let us observe that  $D \setminus \{v_{n-1}, v_n\}$  is a 2DS of the graph  $W_{n-2}$ . Therefore  $\gamma_2(W_{n-2}) \leq \gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2) - 2$ . Now we get  $\gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2) \geq \gamma_2(W_{n-2}) + 2 = \lfloor (n-1)/3 \rfloor + 3 = \lfloor (n+2)/3 \rfloor + 2 > \lfloor (n+1)/3 \rfloor + 1 = \gamma_2(W_n)$ . Therefore  $b'_2(W_n) = 3$  if  $n = 3k + 2$ .  $\square$

Now we investigate the non-isolating 2-bondage in complete bipartite graphs.

REMARK 12. Let  $p$  and  $q$  be positive integers such that  $p \leq q$ . Then

$$b'_2(K_{p,q}) = \begin{cases} 3 & \text{if } p = q = 3; \\ 5 & \text{if } p = q = 4; \\ p - 1 & \text{otherwise.} \end{cases}$$

PROOF. Let  $E(K_{p,q}) = \{a_i b_j : 1 \leq i \leq p \text{ and } 1 \leq j \leq q\}$ . If  $p = 1$ , then obviously  $b'_2(K_{p,q}) = 0 = p - 1$  as removing an edge gives us an isolated vertex. Now assume that  $p = 2$ . By Observation 7 we have  $\gamma_2(K_{2,q}) = 2$ . Let us observe that  $\gamma_2(K_{2,q} - a_1 b_1) = 3$  as the vertex  $b_1$  has to belong to every 2DS of the graph  $K_{2,q} - a_1 b_1$ . Thus  $b'_2(K_{2,q}) = 1 = p - 1$ .

Now let us assume that  $p = 3$ . By Observation 7 we have  $\gamma_2(K_{3,q}) = 3$ . Let us observe that removing one edge does not increase the 2-domination number. This implies that  $b'_2(K_{3,q}) = 0$  or  $b'_2(K_{3,q}) \geq 2$ . If  $q = 3$ , then it is easy to verify that removing any two edges does not increase the 2-domination number. This implies that  $b'_2(K_{3,3}) = 0$  or  $b'_2(K_{3,3}) \geq 3$ . Let us observe that  $\gamma_2(K_{3,3} - a_1 b_1 - a_1 b_2 - a_2 b_1) = 4 > 3 = \gamma_2(K_{3,3})$ . Therefore  $b'_2(K_{3,3}) = 3$ . Now assume that  $q \geq 4$ . We have  $\gamma_2(K_{3,q} - a_1 b_1 - a_2 b_1) = 4$  as the vertex  $b_1$  has to belong to every 2DS of the graph  $K_{3,q} - a_1 b_1 - a_2 b_1$ . Thus  $b'_2(K_{3,q}) = 2$  if  $q \geq 4$ .

Now assume that  $p \geq 4$ . By Observation 7 we have  $\gamma_2(K_{p,q}) = 4$ . If  $q = 4$ , then it is not difficult to verify that removing any four edges does not increase the 2-domination number. This implies that  $b'_2(K_{4,4}) = 0$  or  $b'_2(K_{4,4}) \geq 5$ . We have  $\gamma_2(K_{4,4} - a_1 b_1 - a_1 b_2 - a_1 b_3 - a_2 b_1 - a_3 b_1) = 5$  as the vertices  $a_1$  and  $b_1$  have to belong to every 2DS of the graph  $K_{4,4} - a_1 b_1 - a_1 b_2 - a_1 b_3 - a_2 b_1 - a_3 b_1$ . Thus  $b'_2(K_{4,4}) = 5$ . Now assume that  $q \geq 5$ . Let us observe that removing any  $p - 2$  edges does not increase the 2-domination number. This implies that  $b'_2(K_{p,q}) = 0$  or  $b'_2(K_{p,q}) \geq p - 1$ . We have  $\gamma_2(K_{p,q} - a_1 b_1 - a_2 b_1 - \dots - a_{p-1} b_1) = 5$  as the vertex  $b_1$  has to belong to every 2DS of the graph  $K_{p,q} - a_1 b_1 - a_2 b_1 - \dots - a_{p-1} b_1$ . Therefore  $b'_2(K_{p,q}) = p - 1$  if  $p \geq 4$  and  $q \geq 5$ .  $\square$



A paired dominating set of a graph  $G$  is a dominating set of vertices whose induced subgraph has a perfect matching. The paired domination number of  $G$ , denoted by  $\gamma_p(G)$ , is the minimum cardinality of a paired dominating set of  $G$ . The paired bondage number, denoted by  $b_p(G)$ , is the minimum cardinality among all sets of edges  $E' \subseteq E$  such that  $\delta(G - E') \geq 1$  and  $\gamma_p(G - E') > \gamma_p(G)$ . If for every  $E' \subseteq E$ , either  $\gamma_p(G - E') = \gamma_p(G)$  or  $\delta(G - E') = 0$ , then we define  $b_p(G) = 0$ , and we say that  $G$  is a  $\gamma_p$ -strongly stable graph. Raczek [11] observed that if  $H \subseteq G$ , then  $b_p(H) \leq b_p(G)$ . Let us observe that no inequality of such type is true for the non-isolating 2-bondage. Consider the complete bipartite graphs  $K_{3,3}$ ,  $K_{3,5}$ , and  $K_{4,5}$ . Of course,  $K_{3,3} \subseteq K_{3,5} \subseteq K_{4,5}$ . Using Remark 12 we get  $b'_2(K_{3,3}) = 3 > 2 = b'_2(K_{3,5}) < 3 = b'_2(K_{4,5})$ .

The authors of [4] proved that the bondage number of any tree is either one or two. Let us observe that for every non-negative integer there exists a tree having such non-isolating 2-bondage number. For positive integers  $k$  consider trees  $T_k$  of the form presented in Figure 1. It is not difficult to verify that  $b'_2(T_k) = k - 1$ .

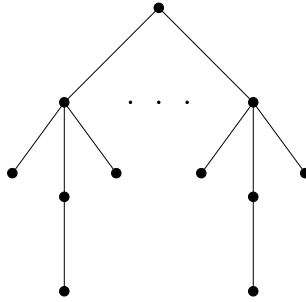


Figure 1. A tree  $T_k$  having  $5k + 1$  vertices.

Hartnell and Rall [5] characterized all trees with bondage number equaling two. We characterize all trees with the non-isolating 2-bondage number equaling zero, that is, all  $\gamma_2$ -non-isolatingly strongly stable trees.

We have the following property of  $\gamma_2$ -non-isolatingly strongly stable trees.

LEMMA 13. *Let  $T$  be a tree with  $b'_2(T) = 0$ , and let  $x$  be a vertex of  $T$  which is neither a leaf nor a support vertex. Then  $\gamma_2(T) = \gamma_2(T - x) + 1$ .*

PROOF. The neighbors of  $x$  we denote by  $y_1, y_2, \dots, y_k$ . Let  $T_i$  mean the component of  $T - x$  which contains the vertex  $y_i$ . Let  $E_0 = \{xy_i : 3 \leq i \leq k\}$ ,  $E_1 = E_0 \cup \{xy_2\}$ , and  $E_2 = E_0 \cup \{xy_1\}$ . Since  $b'_2(T) = 0$ , we have  $\gamma_2(T) = \gamma_2(T - E_0) = \gamma_2(T - E_1) = \gamma_2(T - E_2)$ . By  $T'_i$  we denote the component of  $T - E_i$  which contains the vertex  $x$ . For  $i = 1, 2$ , let  $D'_i$  be any  $\gamma_2(T'_i)$ -set. By Observation 1 we have  $x \in D'_i$ . It is easy to observe that  $D'_1 \cup D'_2$  is a 2DS of the tree  $T'_0$ . Thus



$\gamma_2(T'_0) \leq \gamma_2(T'_1) + \gamma_2(T'_2) - 1$ . Now let  $D_1$  be any  $\gamma_2(T_1)$ -set. Of course,  $D_1 \cup \{x\}$  is a 2DS of the tree  $T'_1$ . Thus  $\gamma_2(T'_1) \leq \gamma_2(T_1) + 1$ . Suppose that  $\gamma_2(T'_1) < \gamma_2(T_1) + 1$ . Now we get  $\gamma_2(T) = \gamma_2(T - E_0) = \gamma_2(T'_0) + \gamma_2(T_3) + \gamma_2(T_4) + \dots + \gamma_2(T_k) \leq \gamma_2(T'_1) + \gamma_2(T'_2) - 1 + \gamma_2(T_3) + \gamma_2(T_4) + \dots + \gamma_2(T_k) < \gamma_2(T_1) + \gamma_2(T_2) + \gamma_2(T_3) + \gamma_2(T_4) + \dots + \gamma_2(T_k) = \gamma_2(T - E_2) = \gamma_2(T)$ , a contradiction. Therefore  $\gamma_2(T'_1) = \gamma_2(T_1) + 1$ . Now we get  $\gamma_2(T) = \gamma_2(T - E_1) = \gamma_2(T'_1) + \gamma_2(T_2) + \gamma_2(T_3) + \dots + \gamma_2(T_k) = \gamma_2(T_1) + \gamma_2(T_2) + \dots + \gamma_2(T_k) + 1 = \gamma_2(T - x) + 1$ .  $\square$

We have the following sufficient condition for that a subtree of a  $\gamma_2$ -non-isolatingly strongly stable tree is also  $\gamma_2$ -non-isolatingly strongly stable.

LEMMA 14. *Let  $T$  be a  $\gamma_2$ -non-isolatingly strongly stable tree. Assume that  $T' \neq K_1$  is a subtree of  $T$  such that  $T - T'$  has no isolated vertices. Then  $b_2(T') = 0$ .*

PROOF. Let  $E_1$  mean the minimum subset of the set of edges of  $T$  such that  $T'$  is a component of  $T - E_1$ . Now let  $E'$  be a subset of the set of edges of  $T'$  such that  $\delta(T' - E') \geq 1$ . The assumption  $b_2(T') = 0$  implies that  $\gamma_2(T - E_1 - E') = \gamma_2(T)$ . We have  $T - E_1 - E' = T' - E' \cup (T - T')$ , and consequently,  $\gamma_2(T - E_1 - E') = \gamma_2(T' - E') + \gamma_2(T - T')$ . Now we get  $\gamma_2(T' - E') = \gamma_2(T - E_1 - E') - \gamma_2(T - T') = \gamma_2(T) - \gamma_2(T) + \gamma_2(T') = \gamma_2(T')$ . This implies that  $b_2(T') = 0$ .  $\square$

Now we prove that attaching a path  $P_3$  by joining it through the support vertex increases the 2-domination number of any graph by two.

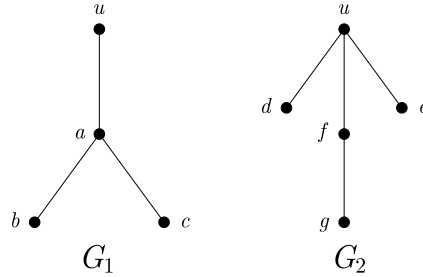
LEMMA 15. *Let  $G$  be a graph, and let  $H$  obtained from  $G$  by attaching a path  $P_3$  by joining the support vertex to any vertex of  $G$ . Then  $\gamma_2(H) = \gamma_2(G) + 2$ .*

PROOF. Let  $v_1v_2v_3$  mean the attached path. Let  $D'$  be any  $\gamma_2(G)$ -set. It is easy to see that  $D' \cup \{v_1, v_3\}$  is a 2DS of the graph  $H$ . Thus  $\gamma_2(H) \leq \gamma_2(G) + 2$ . Now let us observe that there exists a  $\gamma_2(H)$ -set that does not contain the vertex  $v_2$ . Let  $D$  be such a set. By Observation 1 we have  $v_1, v_3 \in D$ . Observe that  $D \setminus \{v_1, v_3\}$  is a 2DS of the graph  $G$ . Therefore  $\gamma_2(G) \leq \gamma_2(H) - 2$ . This implies that  $\gamma_2(H) = \gamma_2(G) + 2$ .  $\square$

Now we need to define trees  $G_1$  and  $G_2$ , see Figure 2. The tree  $G_1$  is a star  $K_{1,3}$  and the tree  $G_2$  is a double star with five vertices.

For the purpose of characterizing all  $\gamma_2$ -non-isolatingly strongly stable trees, that is, all trees  $T$  such that for every  $E' \subseteq E$ , either  $\gamma_2(T - E') = \gamma_2(T)$  or  $\delta(T - E') = 0$ , we introduce a family  $\mathcal{T}$  of trees  $T = T_k$  that can be obtained as follows. Let  $T_1 \in \{P_1, P_2, P_3\}$ . If  $k$  is a positive integer, then  $T_{k+1}$  can be obtained recursively from  $T_k$  by one of the following operations.



Figure 2. The trees  $G_1$  and  $G_2$ .

- Operation  $\mathcal{O}_1$ : Attach a vertex by joining it to a strong support vertex of  $T_k$ .
- Operation  $\mathcal{O}_2$ : Attach a path  $P_3$  by joining the support vertex to a leaf of  $T_k \neq P_3$  the neighbor of which has degree at most two.
- Operation  $\mathcal{O}_3$ : Attach a path  $P_3$  by joining the support vertex to a vertex of  $T_k$  which is not a leaf.
- Operation  $\mathcal{O}_4$ : Let  $x$  mean a vertex of  $T_k$  adjacent to a tree  $G_1$  through the vertex  $u$ . Remove that tree  $G_1$  and attach a tree  $G_2$  by joining the vertex  $u$  to the vertex  $x$ .
- Operation  $\mathcal{O}_5$ : Attach a path  $P_3$  by joining the support vertex to a leaf of  $T_k$  the neighbor of which is adjacent to at least three leaves.

Now we characterize all  $\gamma_2$ -non-isolatingly strongly stable trees.

**THEOREM 16.** *Let  $T$  be a tree. Then  $b'_2(T) = 0$  if and only if  $T \in \mathcal{T}$ .*

**PROOF.** Let  $T$  be a tree of the family  $\mathcal{T}$ . We use the induction on the number  $k$  of operations performed to construct the tree  $T$ . If  $T = P_1$ , then obviously  $b'_2(T) = 0$ . If  $T = P_2$ , then also  $b'_2(T) = 0$  as removing the edge gives us isolated vertices. Similarly,  $b'_2(P_3) = 0$ . Let  $k \geq 2$  be an integer. Assume that the result is true for every tree  $T' = T_k$  of the family  $\mathcal{T}$  constructed by  $k - 1$  operations. Let  $T = T_{k+1}$  be a tree of the family  $\mathcal{T}$  constructed by  $k$  operations.

First assume that  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_1$ . Let  $x$  mean the attached vertex, and let  $y$  mean its neighbor. Let  $D$  be any  $\gamma_2(T)$ -set. By Observation 1 we have  $x \in D$ . Let us observe that  $D \setminus \{x\}$  is a 2DS of the tree  $T'$  as the vertex  $y$  has at least two neighbors in  $D \setminus \{x\}$ . Therefore  $\gamma_2(T') \leq \gamma_2(T) - 1$ . Now let  $E'$  be a subset of the set of edges of  $T$  such that  $\delta(T - E') \geq 1$ . Since  $x$  is a leaf of  $T$ , we have  $xy \notin E'$ . The assumption  $b'_2(T') = 0$  implies that  $\gamma_2(T' - E') = \gamma_2(T')$ . Let  $D'$  be any  $\gamma_2(T' - E')$ -set. Of course,  $D' \cup \{x\}$  is a 2DS of  $T - E'$ . Thus  $\gamma_2(T - E') \leq \gamma_2(T' - E') + 1$ . Now we get  $\gamma_2(T - E') \leq$



$\gamma_2(T' - E') + 1 = \gamma_2(T') + 1 \leq \gamma_2(T)$ . On the other hand, by Observation 2 we have  $\gamma_2(T - E') \geq \gamma_2(T)$ . This implies that  $\gamma_2(T - E') = \gamma_2(T)$ , and consequently,  $b'_2(T) = 0$ .

Now assume that  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_2$ . The leaf to which is attached  $P_3$  we denote by  $x$ . Let  $y$  mean the neighbor of  $x$ . The attached path we denote by  $v_1v_2v_3$ . If  $d_T(y) = 1$ , then  $T'$  is a path  $P_2$ . It is not difficult to verify that  $b'_2(T) = 0$ . Now assume that  $d_T(y) = 2$ . The neighbor of  $y$  other than  $x$  we denote by  $z$ . Since  $T' \neq P_3$ , we have  $d_{T'}(z) \geq 2$ . Let  $t$  mean a neighbor of  $z$  other than  $y$ . By Lemma 15 we have  $\gamma_2(T) = \gamma_2(T') + 2$ . Let  $E'$  be a subset of the set of edges of  $T$  such that  $\delta(T - E') \geq 1$ . Since  $v_1$  and  $v_3$  are leaves of  $T$ , we have  $v_1v_2, v_2v_3 \notin E'$ . If  $xv_2 \in E'$ , then  $\gamma_2(T - E') = \gamma_2(P_3 \cup T' - (E' \setminus \{xv_2\})) = \gamma_2(T' - (E' \setminus \{xv_2\})) + \gamma_2(P_3) = \gamma_2(T') + 2 = \gamma_2(T)$ . Now assume that  $xv_2 \notin E'$ . If  $xy \notin E'$ , then using Lemma 15 we get  $\gamma_2(T - E') = \gamma_2(T' - E') + 2 = \gamma_2(T') + 2 = \gamma_2(T)$ . Now assume that  $xy \in E'$ . By  $T'_y$  we denote the component of  $T' - (E' \setminus \{xy\})$  which contains the vertex  $y$ . Let us observe that  $T'_y \neq P_3$ . Suppose that  $T'_y = P_3$ . Let  $E'' = E' \setminus \{xy, zt\}$  and  $E''' = E'' \cup \{yz\}$ . Since  $b'_2(T') = 0$ , we have  $\gamma_2(T' - E'') = \gamma_2(T')$  and  $\gamma_2(T' - E''') = \gamma_2(T')$ . This implies that  $\gamma_2(T' - E') = \gamma_2(T' - E''')$ . Let  $D'''$  be any  $\gamma_2(T' - E''')$ -set. By Observation 1 we have  $x, y, z \in D'''$ . Let us observe that  $D''' \setminus \{y\}$  is a 2DS of  $T' - E''$ . Consequently,  $\gamma_2(T' - E'') \leq \gamma_2(T' - E''') - 1$ , a contradiction. Therefore  $T'_y \neq P_3$ . Since  $b'_2(T') = 0$ , we have  $\gamma_2(T' - (E' \setminus \{xy\}) - yz) = \gamma_2(T')$ . Let  $D'$  be any  $\gamma_2(T' - (E' \setminus \{xy\}) - yz)$ -set. By Observation 1 we have  $x, y \in D'$ . Let us observe that  $D' \cup \{v_1, v_3\}$  is a 2DS of  $T - E'$ . Thus  $\gamma_2(T - E') \leq \gamma_2(T' - (E' \setminus \{xy\}) - yz) + 2$ . We get  $\gamma_2(T - E') \leq \gamma_2(T' - (E' \setminus \{xy\}) - yz) + 2 = \gamma_2(T') + 2 = \gamma_2(T)$ . Now we conclude that  $\gamma_2(T - E') = \gamma_2(T)$ . This implies that  $b'_2(T) = 0$ .

Now assume that  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_3$ . The vertex to which is attached  $P_3$  we denote by  $x$ . Let  $v_1v_2v_3$  mean the attached path. Let  $E'$  be a subset of the set of edges of  $T$  such that  $\delta(T - E') \geq 1$ . If  $xv_2 \in E'$ , then similarly as when considering the previous operation we get  $\gamma_2(T - E') \leq \gamma_2(T)$ . Now assume that  $xv_2 \notin E'$ . If the component of  $T - E'$  which contains the vertex  $x$  is not a star  $K_{1,3}$ , then similarly as when considering the previous operation we get  $\gamma_2(T - E') \leq \gamma_2(T)$ . Now assume that the component of  $T - E'$  which contains the vertex  $x$  is a star  $K_{1,3}$ . Let us observe that  $b'_2(T' - x) = 0$ . Suppose that  $b'_2(T' - x) > 0$ . This implies that there is a component of  $T' - x$ , say  $T_i$ , such that  $b'_2(T_i) > 0$ . Since  $x$  is not a leaf of  $T'$ , the graph  $T' - T_i$  has no isolated vertices. By Lemma 14 we have  $b'_2(T_i) = 0$ , a contradiction. Therefore  $b'_2(T' - x) = 0$ . This implies that  $\gamma_2(T' - x - (E' \cap E(T' - x))) = \gamma_2(T' - x)$ . Let  $D'$  be any  $\gamma_2(T' - x - (E' \cap E(T' - x)))$ -set. It is easy to observe that  $D' \cup \{x, v_1, v_3\}$  is a 2DS of  $T - E'$ . Thus  $\gamma_2(T - E') \leq \gamma_2(T' - x - (E' \cap E(T' - x))) + 3$ . Using Lemmas 13 and 15 we get  $\gamma_2(T - E') \leq \gamma_2(T' - x - (E' \cap E(T' - x))) + 3 = \gamma_2(T' - x) + 3 =$

$\gamma_2(T') + 2 = \gamma_2(T)$ . Now we conclude that  $\gamma_2(T - E') = \gamma_2(T)$ , and consequently,  $b'_2(T) = 0$ .

Now assume that  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_4$ . Let us observe that there exists a  $\gamma_2(T)$ -set that contains the vertex  $u$ . Let  $D$  be such a set. By Observation 1 we have  $d, e, g \in D$ . The set  $D$  is minimal, thus  $f \notin D$ . Let us observe that  $D \cup \{b, c\} \setminus \{d, e, g\}$  is a 2DS of the tree  $T'$ . Therefore  $\gamma_2(T') \leq \gamma_2(T) - 1$ . Now let  $E'$  be a subset of the set of edges of  $T$  such that  $\delta(T - E') \geq 1$ . Since  $d, e$ , and  $g$  are leaves of  $T$ , we have  $ud, ue, fg \notin E'$ . First assume that  $ux \in E'$ . The assumption  $b'_2(T') = 0$  implies that  $\gamma_2(T' - (E' \cap E(T'))) = \gamma_2(T')$ . We have  $\gamma_2(G_1) = 3$  and  $\gamma_2(G_2) = 4$ . Now we get  $\gamma_2(T - E') = \gamma_2(T' - (E' \cap E(T'))) - \gamma_2(G_1) + \gamma_2(G_2) = \gamma_2(T') - 3 + 4 = \gamma_2(T') + 1 \leq \gamma_2(T)$ . Now assume that  $ux \notin E'$ . First assume that  $x$  is a leaf of  $T' - (E' \cap E(T'))$ . Since  $b'_2(T') = 0$ , we have  $\gamma_2(T' - (E' \cap E(T'))) = \gamma_2(T')$ . Let us observe that there exists a  $\gamma_2(T' - (E' \cap E(T')))$ -set that contains the vertex  $u$ . Let  $D'$  be such a set. By Observation 1 we have  $b, c, x \in D'$ . The set  $D'$  is minimal, thus  $a \notin D'$ . Let us observe that  $D' \setminus \{u, b, c\} \cup \{d, e, f, g\}$  is a 2DS of  $T - E'$ . Thus  $\gamma_2(T - E') \leq \gamma_2(T' - (E' \cap E(T'))) + 1$ . Now we get  $\gamma_2(T - E') \leq \gamma_2(T' - (E' \cap E(T'))) + 1 = \gamma_2(T') + 1 \leq \gamma_2(T)$ . Now assume that  $x$  is not a leaf of  $T' - (E' \cap E(T'))$ . Since  $b'_2(T') = 0$ , we have  $\gamma_2(T' - (E' \cap E(T')) - ux) = \gamma_2(T')$ . Let  $D'$  be any  $\gamma_2(T' - (E' \cap E(T')) - ux)$ -set. By Observation 1 we have  $b, c, u \in D'$ . The set  $D'$  is minimal, thus  $a \notin D'$ . Let us observe that now also  $D' \setminus \{u, b, c\} \cup \{d, e, f, g\}$  is a 2DS of  $T - E'$ . Thus  $\gamma_2(T - E') \leq \gamma_2(T' - (E' \cap E(T)) - ux) + 1$ . Now we get  $\gamma_2(T - E') \leq \gamma_2(T' - (E' \cap E(T)) - ux) + 1 = \gamma_2(T') + 1 \leq \gamma_2(T)$ . We conclude that  $\gamma_2(T - E') = \gamma_2(T)$ , and consequently,  $b'_2(T) = 0$ .

Now assume that  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_5$ . The leaf to which is attached  $P_3$  we denote by  $x$ . Let  $y$  mean the neighbor of  $x$ . The attached path we denote by  $v_1v_2v_3$ . Let  $E'$  be a subset of the set of edges of  $T$  such that  $\delta(T - E') \geq 1$ . If  $xv_2 \in E'$ , then similarly as when considering the operation  $\mathcal{O}_2$  we get  $\gamma_2(T - E') \leq \gamma_2(T)$ . Now assume that  $xv_2 \notin E'$ . If the component of  $T - E'$  which contains the vertex  $x$  is not a star  $K_{1,3}$ , then similarly as when considering the operation  $\mathcal{O}_2$  we get  $\gamma_2(T - E') \leq \gamma_2(T)$ . Now assume that the component of  $T - E'$  which contains the vertex  $x$  is a star  $K_{1,3}$ . Since  $b'_2(T') = 0$ , we have  $\gamma_2(T' - (E' \setminus \{xy\})) = \gamma_2(T')$ . Let  $D'$  be any  $\gamma_2(T' - (E' \setminus \{xy\}))$ -set. By Observation 1 we have  $x \in D'$ . Let us observe that  $D' \cup \{v_1, v_3\}$  is a 2DS of  $T - E'$  as the vertex  $y$  is adjacent to at least two leaves in  $T - E'$ . Thus  $\gamma_2(T - E') \leq \gamma_2(T' - (E' \setminus \{xy\})) + 2$ . Using Lemma 15 we get  $\gamma_2(T - E') \leq \gamma_2(T' - (E' \setminus \{xy\})) + 2 = \gamma_2(T') + 2 = \gamma_2(T)$ . Now we conclude that  $\gamma_2(T - E') = \gamma_2(T)$ . Consequently,  $b'_2(T) = 0$ .

Now assume that  $T$  is a  $\gamma_2$ -non-isolatingly strongly stable tree. Let  $n$  mean the number of vertices of the tree  $T$ . We proceed by induction on this number. If  $\text{diam}(T) = 0$ , then  $T = P_1 \in \mathcal{T}$ . If  $\text{diam}(T) = 1$ , then  $T = P_2 \in \mathcal{T}$ . If

$\text{diam}(T) = 2$ , then  $T$  is a star. If  $T = P_3$ , then  $T \in \mathcal{T}$ . If  $T$  is a star different from  $P_3$ , then it can be obtained from  $P_3$  by a proper number of operations  $\mathcal{O}_1$ . Thus  $T \in \mathcal{T}$ . Now let us assume that  $\text{diam}(T) = 3$ . Thus  $T$  is a double star. Let  $a$  and  $b$  mean the support vertices of  $T$ . Without loss of generality we assume that  $d_T(a) \leq d_T(b)$ . If  $T = P_4$ , then by Remark 9 we have  $b'_2(T) = 1 \neq 0$ . Now assume that  $T$  is a double star different from  $P_4$ . First assume that  $d_T(a) = 1$ . If  $d_T(b) = 2$ , then the tree  $T$  can be obtained from  $P_2$  by operation  $\mathcal{O}_2$ . Thus  $T \in \mathcal{T}$ . If  $d_T(b) \geq 3$ , then the tree  $T$  can be obtained from  $P_2$  by first, operation  $\mathcal{O}_2$ , and then a proper number of operations  $\mathcal{O}_1$  performed on the strong support vertex. Thus  $T \in \mathcal{T}$ . Now assume that  $d_T(a) \geq 2$ . The tree  $T$  can be obtained from  $P_3$  by first, operation  $\mathcal{O}_3$  performed on the support vertex, and then possibly proper numbers of operations  $\mathcal{O}_1$  performed on the support vertices. Thus  $T \in \mathcal{T}$ .

Now assume that  $\text{diam}(T) \geq 4$ . Thus the order of the tree  $T$  is an integer  $n \geq 5$ . The result we obtain by the induction on the number  $n$ . Assume that the lemma is true for every tree  $T'$  of order  $n' < n$ .

First assume that some support vertex of  $T$ , say  $x$ , is adjacent to at least three leaves. Let  $y$  mean a leaf adjacent to  $x$ . Let  $T' = T - y$ . Let  $D'$  be any  $\gamma_2(T')$ -set. Of course,  $D' \cup \{y\}$  is a 2DS of the tree  $T$ . Thus  $\gamma_2(T) \leq \gamma_2(T') + 1$ . Now let  $E'$  be a subset of the set of edges of  $T'$  such that  $\delta(T' - E') \geq 1$ . Since  $b'_2(T) = 0$ , we have  $\gamma_2(T - E') = \gamma_2(T)$ . Let  $D$  be any  $\gamma_2(T - E')$ -set. By Observation 1 we have  $y \in D$ . Let us observe that  $D \setminus \{y\}$  is a 2DS of  $T' - E'$  as the vertex  $y$  is adjacent to at least two leaves in  $T' - E'$ . Therefore  $\gamma_2(T' - E') \leq \gamma_2(T - E') - 1$ . Now we get  $\gamma_2(T' - E') \leq \gamma_2(T - E') - 1 = \gamma_2(T) - 1 \leq \gamma_2(T')$ . On the other hand, by Observation 2 we have  $\gamma_2(T' - E') \geq \gamma_2(T')$ . This implies that  $\gamma_2(T' - E') = \gamma_2(T')$ , and consequently,  $b'_2(T') = 0$ . By the inductive hypothesis we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_1$ . Thus  $T \in \mathcal{T}$ . Henceforth, we can assume that every support vertex of  $T$  is adjacent to at most two leaves.

We now root  $T$  at a vertex  $r$  of maximum eccentricity  $\text{diam}(T)$ . Let  $t$  be a leaf at maximum distance from  $r$ ,  $v$  be the parent of  $t$ ,  $u$  be the parent of  $v$ , and  $w$  be the parent of  $u$  in the rooted tree. By  $T_x$  let us denote the subtree induced by a vertex  $x$  and its descendants in the rooted tree  $T$ .

First assume that  $d_T(v) = 2$ . Assume that among the descendants of  $u$  there is a support vertex, say  $x$ , different from  $v$ . It suffices to consider only the possibilities when  $x$  is adjacent to one or two leaves. First assume that  $x$  is adjacent to two leaves. Let  $T' = T - T_x$ . Lemma 14 implies that  $b'_2(T') = 0$ . By the inductive hypothesis we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_3$ . Thus  $T \in \mathcal{T}$ .

Now assume that  $x$  is adjacent to exactly one leaf. Let  $T' = T - T_v$ . Let us observe that there exists a  $\gamma_2(T')$ -set that contains the vertex  $u$ . Let  $D'$  be such a set. It is easy to see that  $D' \cup \{t\}$  is a 2DS of the tree  $T$ . Thus  $\gamma_2(T) \leq \gamma_2(T') + 1$ .



We have  $T - uv = T' \cup P_2$ . Now we get  $\gamma_2(T - uv) = \gamma_2(T' \cup P_2) = \gamma_2(T') + \gamma_2(P_2) = \gamma_2(T') + 2 \geq \gamma_2(T) + 1 > \gamma_2(T)$ . Therefore  $b'_2(T) = 1$ , a contradiction.

Now assume that every descendant of  $u$  excluding  $v$  is a leaf. First assume that  $u$  is adjacent to two leaves, say  $x$  and  $y$ . Let  $T'$  be a tree obtained from  $T - T_u$  by attaching a tree  $G_1$  by joining the vertex  $u$  to the vertex  $w$ . Let us observe that there exists a  $\gamma_2(T')$ -set that contains the vertex  $u$ . Let  $D'$  be such a set. By Observation 1 we have  $b, c \in D'$ . The set  $D'$  is minimal, thus  $a \notin D'$ . Let us observe that  $D' \setminus \{b, c\} \cup \{t, x, y\}$  is a 2DS of the tree  $T$ . Thus  $\gamma_2(T) \leq \gamma_2(T') + 1$ . Now let  $E'$  be a subset of the set of edges of  $T'$  such that  $\delta(T' - E') \geq 1$ . Since  $b$  and  $c$  are leaves of  $T'$ , we have  $ab, ac \notin E'$ . The assumption  $b'_2(T) = 0$  implies that  $\gamma_2(T - (E' \cap E(T))) = \gamma_2(T)$ . Let us observe that there exists a  $\gamma_2(T - (E' \cap E(T)))$ -set that contains the vertex  $u$ . Let  $D$  be such a set. By Observation 1 we have  $t, x, y \in D$ . The set  $D$  is minimal, thus  $v \notin D$ . Let us observe that  $D \cup \{b, c\} \setminus \{t, x, y\}$  is a 2DS of  $T' - E'$ . Therefore  $\gamma_2(T' - E') \leq \gamma_2(T - (E' \cap E(T))) - 1$ . Now we get  $\gamma_2(T' - E') \leq \gamma_2(T - (E' \cap E(T))) - 1 = \gamma_2(T) - 1 \leq \gamma_2(T')$ . We conclude that  $\gamma_2(T' - E') = \gamma_2(T')$ , and consequently,  $b'_2(T') = 0$ . By the inductive hypothesis we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_4$ . Thus  $T \in \mathcal{T}$ .

Now assume that  $u$  is adjacent to exactly one leaf, say  $x$ . Let  $E' = \{wu, uv\}$  and  $T' = T - T_u$ . Let  $D'$  be any  $\gamma_2(T')$ -set. It is easy to observe that  $D' \cup \{u, t, x\}$  is a 2DS of the tree  $T$ . Thus  $\gamma_2(T) \leq \gamma_2(T') + 3$ . We have  $T - E' = T' \cup P_2 \cup P_2$ . Now we get  $\gamma_2(T - E') = \gamma_2(T' \cup P_2 \cup P_2) = \gamma_2(T') + 2\gamma_2(P_2) = \gamma_2(T') + 4 \geq \gamma_2(T) + 1 > \gamma_2(T)$ . This implies that  $b'_2(T) \in \{1, 2\}$ , a contradiction.

Now assume that  $d_T(u) = 2$ . Let  $T' = T - T_v$ . Let  $D'$  be any  $\gamma_2(T')$ -set. By Observation 1 we have  $u \in D'$ . It is easy to see that  $D' \cup \{t\}$  is a 2DS of the tree  $T$ . Thus  $\gamma_2(T) \leq \gamma_2(T') + 1$ . We have  $T - uv = T' \cup P_2$ . Now we get  $\gamma_2(T - uv) = \gamma_2(T' \cup P_2) = \gamma_2(T') + \gamma_2(P_2) = \gamma_2(T') + 2 \geq \gamma_2(T) + 1 > \gamma_2(T)$ . Therefore  $b'_2(T) = 1$ , a contradiction.

Now assume that  $d_T(v) = 3$ . The leaf adjacent to  $v$  and different from  $t$  we denote by  $a$ . Assume that  $d_T(u) \geq 3$ . Let  $T' = T - T_v$ . Lemma 14 implies that  $b'_2(T') = 0$ . By the inductive hypothesis we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_3$ . Thus  $T \in \mathcal{T}$ .

Now assume that  $d_T(u) = 2$ . First assume that  $w$  is adjacent to two leaves. Let  $T' = T - T_v$ . Lemma 14 implies that  $b'_2(T') = 0$ . By the inductive hypothesis we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_5$ . Thus  $T \in \mathcal{T}$ .

Now assume that  $w$  is adjacent to exactly one leaf, say  $x$ . Let  $G'$  be a graph obtained from  $T$  by removing all edges incident to  $w$  excluding  $wx$ . Let  $D'$  be any  $\gamma_2(G')$ -set. By Observation 1 we have  $u, w, x \in D'$ . Let us observe that  $D' \setminus \{w\}$  is a 2DS of the tree  $T$ . Thus  $\gamma_2(T) \leq \gamma_2(G') - 1$ . Therefore  $b'_2(T) > 0$ , a

contradiction.

Now assume that there is a descendant of  $w$ , say  $k$ , such that the distance of  $w$  to the most distant vertex of  $T_k$  is three. It suffices to consider only the possibility when  $T_k$  is isomorphic to  $T_u$ . The descendant of  $k$  we denote by  $l$ , and the leaves adjacent to  $l$  we denote by  $m$  and  $p$ . Let  $G'$  be a graph obtained from  $T$  by removing all edges incident to  $w$  excluding  $wu$ . Let us observe that there exists a  $\gamma_2(G')$ -set that contains the vertex  $u$ . Let  $D'$  be such a set. By Observation 1 we have  $w, k \in D'$ . Let us observe that  $D' \setminus \{w\}$  is a 2DS of the tree  $T$ . Thus  $\gamma_2(T) \leq \gamma_2(G') - 1$ . This implies that  $b'_2(T) > 0$ , a contradiction.

Now assume that there is a descendant of  $w$ , say  $k$ , such that the distance of  $w$  to the most distant vertex of  $T_k$  is two. It suffices to consider only the possibilities when  $k$  is adjacent to one or two leaves. First assume that  $k$  is adjacent to two leaves. Let  $T' = T - T_k$ . Lemma 14 implies that  $b'_2(T') = 0$ . By the inductive hypothesis we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_3$ . Thus  $T \in \mathcal{T}$ .

Now assume that  $k$  is adjacent to exactly one leaf. Let  $T' = T - T_k$ . Similarly as when  $T_k$  is isomorphic to  $T_u$  we conclude that  $b'_2(T) > 0$ , a contradiction.

Now assume that  $d_T(w) = 2$ . Let  $T' = T - T_v$ . Lemma 14 implies that  $b'_2(T') = 0$ . By the inductive hypothesis we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_2$ . Thus  $T \in \mathcal{T}$ .  $\square$

ACKNOWLEDGMENTS. Thanks are due to the anonymous referee for comments that helped to make the paper correct.

## References

- [1] M. Blidia, M. Chellali and O. Favaron, Independence and 2-domination in trees, *Australas. J. Combin.*, **33** (2005), 317–327.
- [2] G. S. Domke and R. C. Laskar, The bondage and reinforcement numbers of  $\gamma_f$  for some graphs, *Discrete Math.*, **167/168** (1997), 249–259.
- [3] J. F. Fink and M. S. Jacobson,  $n$ -domination in graphs, In: *Graph Theory with Applications to Algorithms and Computer Science* (Kalamazoo, Mich., 1984), Wiley-Intersci. Publ., Wiley, New York, 1985, pp. 282–300.
- [4] J. F. Fink, M. S. Jacobson, L. F. Kinch and J. Roberts, The bondage number of a graph, *Discrete Math.*, **86** (1990), 47–57.
- [5] B. L. Hartnell and D. F. Rall, A characterization of trees in which no edge is essential to the domination number, *Ars Combin.*, **33** (1992), 65–76.
- [6] B. L. Hartnell and D. F. Rall, Bounds on the bondage number of a graph, *Discrete Math.*, **128** (1994), 173–177.
- [7] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Monogr. Textbooks Pure Appl. Math., **208**, Marcel Dekker, New York, 1998.
- [8] T. W. Haynes, S. T. Hedetniemi and P. J. Slater (eds.), *Domination in Graphs: Advanced Topics*, Monogr. Textbooks Pure Appl. Math., **209**, Marcel Dekker, New York, 1998.
- [9] L. Kang and J. Yuan, Bondage number of planar graphs, *Discrete Math.*, **222** (2000),



- 191–198.
- [10] H. Liu and L. Sun, The bondage and connectivity of a graph, *Discrete Math.*, **263** (2003), 289–293.
  - [11] J. Raczek, Paired bondage in trees, *Discrete Math.*, **308** (2008), 5570–5575.
  - [12] U. Teschner, New results about the bondage number of a graph, *Discrete Math.*, **171** (1997), 249–259.
  - [13] L. Volkmann, A Nordhaus-Gaddum-type result for the 2-domination number, *J. Combin. Math. Combin. Comput.*, **64** (2008), 227–235.
  - [14] Y. P. Wu and Q. Fan, The bondage number of four domination parameters for trees, *J. Math. (Wuhan)*, **24** (2004), 267–270.

Marcin KRZYWKOWSKI

Faculty of Electronics  
Telecommunications and Informatics  
Gdańsk University of Technology  
Narutowicza 11/12  
80–233 Gdańsk, Poland  
E-mail: marcin.krzywkowski@gmail.com