

# Non-isolating Bondage in Graphs

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**Abstract** A dominating set of a graph  $G = (V, E)$  is a set  $D$  of vertices of  $G$  such that every vertex of  $V(G) \setminus D$  has a neighbor in  $D$ . The domination number of a graph  $G$ , denoted by  $\gamma(G)$ , is the minimum cardinality of a dominating set of  $G$ . The non-isolating bondage number of  $G$ , denoted by  $b'(G)$ , is the minimum cardinality among all sets of edges  $E' \subseteq E$  such that  $\delta(G - E') \geq 1$  and  $\gamma(G - E') > \gamma(G)$ . If for every  $E' \subseteq E$  we have  $\gamma(G - E') = \gamma(G)$  or  $\delta(G - E') = 0$ , then we define  $b'(G) = 0$ , and we say that  $G$  is a  $\gamma$ -non-isolatingly strongly stable graph. First we discuss various properties of non-isolating bondage in graphs. We find the non-isolating bondage numbers for several classes of graphs. Next we show that for every non-negative integer, there exists a tree having such non-isolating bondage number. Finally, we characterize all  $\gamma$ -non-isolatingly strongly stable trees.

**Keywords** Domination · Bondage · Non-isolating bondage · Graph · Tree

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## 1 Introduction

Let  $G = (V, E)$  be a graph. By the neighborhood of a vertex  $v$  of  $G$ , we mean the set  $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ . The degree of a vertex  $v$ , denoted by  $d_G(v)$ , is the cardinality of its neighborhood. Let  $\delta(G)$  mean the minimum degree among all vertices of  $G$ . By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph  $G$ , denoted by  $\text{diam}(G)$ , is the maximum eccentricity among all vertices of  $G$ . We denote the path (cycle, respectively) on  $n$  vertices by  $P_n$  ( $C_n$ , respectively). A wheel  $W_n$ , where  $n \geq 4$ , is a graph with  $n$  vertices, formed by connecting a vertex to all vertices of a cycle  $C_{n-1}$ . Let  $T$  be a tree, and let  $v$  be a vertex of  $T$ . We say that  $v$  is adjacent to a path  $P_n$  if there is a neighbor of  $v$ , say  $x$ , of degree two such that the tree resulting from  $T$  by removing the edge  $vx$ , and which contains the vertex  $x$ , is a path  $P_n$ . Let  $K_{p,q}$  denote a complete bipartite graph the partite sets of which have cardinalities  $p$  and  $q$ . By a star we mean a connected graph in which exactly one vertex has degree greater than one.

A subset  $D \subseteq V(G)$  is a dominating set, abbreviated DS, of  $G$  if every vertex of  $V(G) \setminus D$  has a neighbor in  $D$ . The domination number of a graph  $G$ , denoted by  $\gamma(G)$ , is the minimum cardinality of a dominating set of  $G$ . For a comprehensive survey of domination in graphs, see for example [5].

The bondage number  $b(G)$  of a graph  $G$  is the minimum cardinality among all sets of edges  $E' \subseteq E$  such that  $\gamma(G - E') > \gamma(G)$ . The concept of bondage in graphs was introduced in [2] and further studied for example in [1, 3, 4, 6–9].

We define the non-isolating bondage number of a graph  $G$ , denoted by  $b'(G)$ , to be the minimum cardinality among all sets of edges  $E' \subseteq E$  such that  $\delta(G - E') \geq 1$  and  $\gamma(G - E') > \gamma(G)$ . Thus  $b'(G)$  is the minimum number of edges of  $G$  that have to be removed in order to obtain a graph with no isolated vertices, and with the domination number greater than that of  $G$ . If for every  $E' \subseteq E$  we have  $\gamma(G - E') = \gamma(G)$  or  $\delta(G - E') = 0$ , then we define  $b'(G) = 0$ , and we say that  $G$  is a  $\gamma$ -non-isolatingly strongly stable graph.

First we discuss various properties of non-isolating bondage in graphs. We find the non-isolating bondage numbers for several classes of graphs. Next we show that for every non-negative integer, there exists a tree having such non-isolating bondage number. Finally, we characterize all  $\gamma$ -non-isolatingly strongly stable trees.

## 2 Results

We begin with the following well known observations.

For every graph  $G$  of diameter at least two there exists a  $\gamma(G)$ -set that contains all support vertices.

If  $H$  is a subgraph of  $G$  such that  $V(H) = V(G)$ , then  $\gamma(H) \geq \gamma(G)$ .

If  $n$  is a positive integer, then  $\gamma(P_n) = \lfloor (n + 2)/3 \rfloor$ .

For every integer  $n \geq 3$  we have  $\gamma(C_n) = \lfloor (n + 2)/3 \rfloor$ .

**Observation 1** *If  $n$  is a positive integer, then  $\gamma(K_n) = 1$ .*

**Observation 2** *For every integer  $n \geq 4$  we have  $\gamma(W_n) = 1$ .*

**Observation 3** *Let  $p$  and  $q$  be positive integers such that  $p \leq q$ . Then*

$$\gamma(K_{p,q}) = \begin{cases} 1 & \text{if } p = 1; \\ 2 & \text{otherwise.} \end{cases}$$

First we calculate the non-isolating bondage numbers of paths.

**Lemma 4** *For any positive integer  $n$  we have*

$$b'(P_n) = \begin{cases} 0 & \text{if } n = 1, 2, 3, 4, 5, 7; \\ 1 & \text{if } n \geq 6 \text{ and } n \neq 3k + 1; \\ 2 & \text{if } n \geq 10 \text{ and } n = 3k + 1. \end{cases}$$

*Proof* Let us observe that if a path has at most five or exactly seven vertices, then removing any edges does not increase the domination number, or gives an isolated vertex. Assume that  $n = 6$  or  $n \geq 8$ . First assume that  $n = 3k$ . We have  $\gamma(P_n) = \lfloor (n + 2)/3 \rfloor = \lfloor (3k + 2)/3 \rfloor = k$ . We also have  $\gamma(P_{n-2}) + \gamma(P_2) = \lfloor n/3 \rfloor + 1 = k + 1 > \gamma(P_n)$ . Thus  $b'(P_n) = 1$  if  $n = 3k$  and  $n \geq 6$ . Now assume that  $n = 3k + 2$ . We have  $\gamma(P_n) = \lfloor (n + 2)/3 \rfloor = \lfloor (3k + 4)/3 \rfloor = k + 1$ . We also have  $\gamma(P_{n-4}) + \gamma(P_4) = \lfloor n/3 \rfloor + 2 = k + 2 > \gamma(P_n)$ . Thus  $b'(P_n) = 1$  if  $n = 3k + 2$  and  $n \geq 8$ . Now assume that  $n = 3k + 1$ . We have  $\gamma(P_n) = \lfloor (n + 2)/3 \rfloor = \lfloor (3k + 3)/3 \rfloor = k + 1$ . Let us observe that removing any edge does not increase the domination number. We have  $\gamma(P_{n-6}) + \gamma(P_4) + \gamma(P_2) = \lfloor (n - 4)/3 \rfloor + 3 = \lfloor (3k - 3)/3 \rfloor + 3 = k + 2 > \gamma(P_n)$ . Therefore  $b'(P_n) = 2$  if  $n = 3k + 1$  and  $n \geq 10$ .  $\square$

We now investigate the non-isolating bondage in cycles.

**Lemma 5** *For every integer  $n \geq 3$  we have*

$$b'(C_n) = \begin{cases} 0 & \text{if } b'(P_n) = 0; \\ b'(P_n) + 1 & \text{if } b'(P_n) \neq 0. \end{cases}$$

*Proof* We have  $\gamma(P_n) = \gamma(C_n)$ . Clearly,  $C_n - e = P_n$ . This implies that  $b'(C_n) = 0$  if  $b'(P_n) = 0$ , while  $b'(C_n) = b'(P_n) + 1$  if  $b'(P_n) \neq 0$ .  $\square$

We now find the non-isolating bondage numbers of complete graphs.

**Proposition 6** *If  $n$  is a positive integer, then*

$$b'(K_n) = \begin{cases} 0 & \text{for } n = 1, 2, 3; \\ \lfloor (n + 1)/2 \rfloor & \text{for } n \geq 4. \end{cases}$$



*Proof* Obviously,  $b'(K_1) = 0$  and  $b'(K_2) = 0$ . We have  $K_3 - e = C_3$  and  $b'(C_3) = 0$ . This implies that  $b'(K_3) = 0$ . Now assume that  $n \geq 4$ . By Observation 1 we have  $\gamma(K_n) = 1$ . Let us observe that the domination number of a graph equals one if and only if the graph has a universal vertex. Given a complete graph, we increase the domination number if and only if for every vertex we remove at least one incident edge. If  $n$  is even, then we remove  $n/2 = \lfloor (n+1)/2 \rfloor$  edges. If  $n$  is odd, then we remove  $(n-1)/2 + 1 = (n+1)/2 = \lfloor (n+1)/2 \rfloor$  edges.  $\square$

We now calculate the non-isolating bondage numbers of wheels.

**Proposition 7** For integers  $n \geq 4$  we have

$$b'(W_n) = \begin{cases} 2 & \text{if } n = 4; \\ 1 & \text{if } n \geq 5. \end{cases}$$

*Proof* Since  $W_4 = K_4$ , using Proposition 6 we get  $b'(W_4) = b'(K_4) = \lfloor 5/2 \rfloor = 2$ . Now assume that  $n \geq 5$ . By Observation 2 we have  $\gamma(W_n) = 1$ . The domination number of a graph equals one if and only if it has a universal vertex. Removing an edge of  $W_n$  incident to the vertex of maximum degree gives a graph without universal vertices. Therefore  $b'(W_n) = 1$  for  $n \geq 5$ .  $\square$

We now investigate the non-isolating bondage in complete bipartite graphs.

**Proposition 8** Let  $p$  and  $q$  be positive integers such that  $p \leq q$ . Then

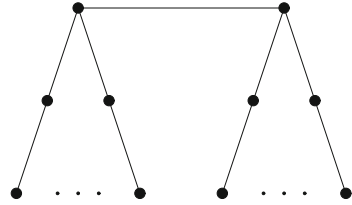
$$b'(K_{p,q}) = \begin{cases} 0 & \text{if } p = 1, 2; \\ 4 & \text{if } p = 3; \\ p & \text{otherwise.} \end{cases}$$

*Proof* Let  $E(K_{p,q}) = \{a_i b_j : 1 \leq i \leq p \text{ and } 1 \leq j \leq q\}$ . If  $p = 1$ , then obviously  $b'(K_{p,q}) = 0$  as removing any edge produces an isolated vertex. Now assume that  $p \geq 2$ . By Observation 3 we have  $\gamma(K_{p,q}) = 2$ . Let  $E'$  be a subset of the set of edges of  $K_{2,q}$  such that  $\delta(K_{2,q} - E') \geq 1$ . Each vertex  $b_i$  is adjacent to  $a_1$  or  $a_2$  in the graph  $K_{2,q} - E'$ . Observe that the vertices  $a_1$  and  $a_2$  form a dominating set of  $K_{2,q} - E'$ . Therefore  $b'(K_{2,q}) = 0$ . Now assume that  $p = 3$ . It is not very difficult to verify that removing any three edges does not increase the domination number while not producing an isolated vertex. We have  $\gamma(K_{3,q} - a_1 b_2 - a_1 b_3 - a_2 b_1 - a_3 b_1) = 3 > 2 = \gamma(K_{3,q})$ . Therefore  $b'(K_{3,q}) = 4$ . Now assume that  $p \geq 4$ . If we remove at most  $p-1$  edges, then there are vertices  $a_i$  and  $b_j$  which have degrees  $q$  and  $p$ , respectively. It is easy to observe that the vertices  $a_i$  and  $b_j$  still form a dominating set. Let us observe that  $\gamma(K_{p,q} - a_1 b_1 - a_2 b_1 - a_3 b_2 - a_4 b_2 - a_5 b_2 - \dots - a_p b_2) = 3 > 2 = \gamma(K_{p,q})$ . Therefore  $b'(K_{p,q}) = p$  if  $p \geq 4$ .  $\square$

The authors of [2] proved that the bondage number of any tree is either one or two.

**Theorem 9** ([2]) For every tree  $T$  we have  $b(T) \in \{1, 2\}$ .

**Fig. 1** A tree  $T_k$  having  $4k + 2$  vertices, where both central vertices are of degree  $k + 1$



Let us observe that for every non-negative integer there exists a tree with such non-isolating bondage number. We have  $b'(P_4) = 0$ . For positive integers  $k$ , consider trees  $T_k$  of the form presented in Fig. 1. It is not difficult to verify that  $b'(T_k) = k$ .

Hartnell and Rall [3] characterized all trees with bondage number equal to two. We characterize all trees with the non-isolating bondage number equal to zero, that is, all  $\gamma$ -non-isolatingly strongly stable trees.

We now show that joining two  $\gamma$ -non-isolatingly strongly stable trees gives us also a  $\gamma$ -non-isolatingly strongly stable tree.

**Lemma 10** *Let  $T_1$  and  $T_2$  be vertex-disjoint  $\gamma$ -non-isolatingly strongly stable trees. Let  $x$  be a support vertex of  $T_1$  and let  $y$  be a leaf of  $T_2$ . Let  $T$  be a tree obtained by joining the vertices  $x$  and  $y$ . If  $\gamma(T) = \gamma(T_1) + \gamma(T_2)$ , then the tree  $T$  is also  $\gamma$ -non-isolatingly strongly stable.*

*Proof* Let  $E_1$  be a subset of the set of edges of  $T$  such that  $\delta(T - E_1) \geq 1$ . If  $xy \in E_1$ , then we get  $\gamma(T - E_1) = \gamma(T_1 - E_1 \cap E(T_1)) + \gamma(T_2 - E_1 \cap E(T_2)) = \gamma(T_1) + \gamma(T_2) = \gamma(T)$ . Now assume that  $xy \notin E_1$ . Let  $z$  be the neighbor of  $y$  other than  $x$ . If  $yz \notin E_1$ , then let  $E_2 = E_1 \cup \{xy\}$ . Similarly as earlier we get  $\gamma(T - E_2) = \gamma(T)$ . We have  $\gamma(T - E_1) \leq \gamma(T - E_2)$ , and consequently,  $\gamma(T - E_1) = \gamma(T)$ . Now assume that  $yz \in E_1$ . Let  $E_3 = E_1 \cup \{xy\} \setminus \{yz\}$ . Similarly as earlier we get  $\gamma(T - E_3) = \gamma(T)$ . Let  $D_2$  be a  $\gamma(T - E_3)$ -set that contains the vertices  $x$  and  $z$ . It is easy to observe that  $D_2$  is also a DS of the graph  $T - E_1$ . Therefore  $\gamma(T - E_1) \leq \gamma(T - E_3)$ . This implies that  $\gamma(T - E_1) = \gamma(T)$ . We now conclude that  $b'(T) = 0$ .  $\square$

We next show that a subtree of a  $\gamma$ -non-isolatingly strongly stable tree is also  $\gamma$ -non-isolatingly strongly stable.

**Lemma 11** *Let  $T$  be a  $\gamma$ -non-isolatingly strongly stable tree. Assume that  $T'$  is a subtree of  $T$  such that  $T - T'$  has no isolated vertices. Then  $b'(T') = 0$ .*

*Proof* If  $T'$  consists of a single vertex, then obviously  $b'(T') = 0$ . Thus assume that  $T' \neq K_1$ . Let  $E_1$  be the minimum subset of  $E(T)$  such that  $T'$  is a component of  $T - E_1$ . Now let  $E'$  be a subset of  $E(T')$  such that  $\delta(T' - E') \geq 1$ . Notice that  $\delta(T - E_1 - E') \geq 1$ . The assumption  $b'(T) = 0$  implies that  $\gamma(T - E_1) = \gamma(T)$  and  $\gamma(T - E_1 - E') = \gamma(T)$ . We have  $T - E_1 - E' = T' - E' \cup (T - T')$  and  $T - E_1 = T' \cup (T - T')$ . We now get  $\gamma(T' - E') = \gamma(T - E_1 - E') - \gamma(T - T') = \gamma(T) - \gamma(T - E_1) + \gamma(T') = \gamma(T')$ . This implies that  $b'(T') = 0$ .  $\square$

For the purpose of characterizing all  $\gamma$ -non-isolatingly strongly stable trees, we introduce a family  $\mathcal{T}$  of trees  $T = T_k$  that can be obtained as follows. Let  $T_1 \in \{P_1, P_2\}$ .

If  $k$  is a positive integer, then  $T_{k+1}$  can be obtained recursively from  $T_k$  by one of the following operations.

- Operation  $\mathcal{O}_1$ : Attach a vertex by joining it to any support vertex of  $T_k$ .
- Operation  $\mathcal{O}_2$ : Attach a path  $P_2$  by joining one of its vertices to a vertex of  $T_k$ , which is adjacent to a path  $P_1$  or  $P_4$ , or is not a leaf and is adjacent to a support vertex.
- Operation  $\mathcal{O}_3$ : Attach a path  $P_3$  by joining one of its leaves to a vertex of  $T_k$  adjacent to a path  $P_1$  or  $P_3$ .
- Operation  $\mathcal{O}_4$ : Attach a path  $P_5$  by joining one of its leaves to any support vertex of  $T_k$ .

We now prove that every tree of the family  $\mathcal{T}$  is  $\gamma$ -non-isolatingly strongly stable.

**Lemma 12** *If  $T \in \mathcal{T}$ , then  $b'(T) = 0$ .*

*Proof* We use induction on the number  $k$  of operations performed to construct the tree  $T$ . If  $T = P_1$ , then obviously  $b'(T) = 0$ . If  $T = P_2$ , then  $b'(T) = 0$  as removing the edge gives isolated vertices. Let  $k$  be a positive integer. Assume that the result is true for every tree  $T' = T_k$  of the family  $\mathcal{T}$  constructed by  $k - 1$  operations. Let  $T = T_{k+1}$  be a tree of the family  $\mathcal{T}$  constructed by  $k$  operations.

First assume that  $T$  is obtained from  $T'$  by Operation  $\mathcal{O}_1$ . Let  $x$  be the attached vertex, and let  $y$  be its neighbor. Let  $z$  be a leaf adjacent to  $y$  and different from  $x$ . Let  $D$  be a  $\gamma(T)$ -set that contains all support vertices. The set  $D$  is minimal, thus  $x \notin D$ . Obviously,  $D$  is a DS of the tree  $T'$ . Therefore  $\gamma(T') \leq \gamma(T)$ . Now let  $E'$  be a subset of the set of edges of  $T$  such that  $\delta(T - E') \geq 1$ . Since both  $x$  and  $z$  are leaves of  $T$ , we have  $xy \notin E'$  and  $yz \notin E'$ . The assumption  $b'(T) = 0$  implies that  $\gamma(T' - E') = \gamma(T')$ . Let us observe that there exists a  $\gamma(T' - E')$ -set that contains the vertex  $y$ . Let  $D'$  be such a set. It is easy to see that  $D'$  is a DS of the graph  $T - E'$ . Thus  $\gamma(T - E') \leq \gamma(T' - E')$ . We now get  $\gamma(T - E') \leq \gamma(T' - E') = \gamma(T') \leq \gamma(T)$ . On the other hand, we have  $\gamma(T - E') \geq \gamma(T)$ . This implies that  $\gamma(T - E') = \gamma(T)$ , and consequently,  $b'(T) = 0$ .

Now assume that  $T$  is obtained from  $T'$  by Operation  $\mathcal{O}_2$ . The vertex to which is attached  $P_2$  we denote by  $x$ . Let  $v_1 v_2$  be the attached path. Let  $v_1$  be joined to  $x$ . If  $x$  is adjacent to a leaf or a support vertex, say  $a$ , then let  $D$  be a  $\gamma(T)$ -set that contains all support vertices. We have  $v_2 \notin D$  as the set  $D$  is minimal. It is easy to observe that  $D \setminus \{v_1\}$  is a DS of the tree  $T'$ . If  $x$  is adjacent to a path  $P_4$ , then we denote it by  $abcd$ . Let  $a$  and  $x$  be adjacent. Let us observe that there exists a  $\gamma(T)$ -set that contains the vertices  $v_1, c$ , and  $x$ . Let  $D$  be such a set. It is easy to observe that  $D \setminus \{v_1\}$  is a DS of the tree  $T'$ . We conclude that  $\gamma(T') \leq \gamma(T) - 1$ . Now let  $E'$  be a subset of the set of edges of  $T$  such that  $\delta(T - E') \geq 1$ . Since  $v_2$  is a leaf of  $T$ , we have  $v_1 v_2 \notin E'$ . If  $x v_1 \in E'$ , then  $\delta(T' - (E' \cap E(T'))) \geq 1$ . We get  $\gamma(T - E') = \gamma(P_2 \cup T' - (E' \setminus \{x v_1\})) = \gamma(T' - (E' \cap E(T'))) + \gamma(P_2) = \gamma(T') + 1 \leq \gamma(T)$ . Now assume that  $x v_1 \notin E'$ . By  $T'_x$  ( $T'_x$ , respectively), we denote the component of  $T - E'$  ( $T' - E'$ , respectively) which contains the vertex  $x$ . If  $\delta(T' - (E' \cap E(T'))) \geq 1$ , then let  $D'_x$  be any  $\gamma(T'_x)$ -set. It is easy to see that  $D'_x \cup \{v_1\}$  is a DS of the tree  $T_x$ . Thus  $\gamma(T_x) \leq \gamma(T'_x) + 1$ . We now get  $\gamma(T - E') = \gamma(T - E' - T_x) + \gamma(T_x) \leq \gamma(T - E' - T_x) + \gamma(T'_x) + 1 = \gamma(T' - E' - T'_x) + \gamma(T'_x) + 1 = \gamma(T' - E') + 1 = \gamma(T') + 1 \leq \gamma(T)$ . Now



assume that  $\delta(T' - (E' \cap E(T'))) = 0$ . This implies that  $x$  is the only isolated vertex of  $T' - (E' \cap E(T'))$ , and so  $x$  is not adjacent to any leaf in the trees  $T'$  and  $T$ . Consequently,  $T'_x$  consists only of the vertex  $x$ , and  $T_x$  is a path  $P_3$ . Let us observe that  $\delta(T' - (E' \setminus \{xa\})) \geq 1$ . Let  $T'_a$  be the component of  $T' - E'$ , which contains the vertex  $a$ . Now let  $T''_a$  be a tree obtained from  $T'_a$  by attaching a vertex to the vertex  $a$ . We now get  $\gamma(T - E') = \gamma(T - E' - T_x) + \gamma(P_3) = \gamma(T' - E' - T'_x) + 1 = \gamma(T' - E' - T'_x - T'_a) + \gamma(T'_a) + 1 \leq \gamma(T' - E' - T'_x - T'_a) + \gamma(T''_a) + 1 = \gamma((T' - E' - T'_x - T'_a) \cup T''_a) + 1 = \gamma(T' - (E' \setminus \{xa\})) + 1 = \gamma(T' - E') + 1 = \gamma(T') + 1 \leq \gamma(T)$ . We conclude that  $\gamma(T - E') = \gamma(T)$ , and consequently,  $b'(T) = 0$ .

Now assume that  $T$  is obtained from  $T'$  by Operation  $\mathcal{O}_3$ . The vertex to which is attached  $P_3$  we denote by  $x$ . If  $x$  is a support vertex, then using Lemma 10, for  $T_1 = T'$  and  $T_2 = P_3$ , we get  $b'(T) = 0$ . Now assume that  $x$  is adjacent to a path  $P_3$ , say  $abc$ . Let  $a$  and  $x$  be adjacent. The attached path we denote by  $v_1 v_2 v_3$ . Let  $v_1$  be joined to  $x$ . Let us observe that there exists a  $\gamma(T)$ -set that contains all support vertices and does not contain the vertex  $v_1$ . Let  $D$  be such a set. We have  $v_3 \notin D$  as the set  $D$  is minimal. Observe that  $D \setminus \{v_2\}$  is a DS of the tree  $T'$ . Therefore  $\gamma(T') \leq \gamma(T) - 1$ . Now let  $E'$  be a subset of the set of edges of  $T$  such that  $\delta(T - E') \geq 1$ . We have  $v_2 v_3 \notin E'$  as the vertex  $v_3$  is a leaf. If  $x v_1 \in E'$ , then  $v_1 v_2 \notin E'$ ; otherwise we get an isolated vertex. Let us observe that  $\delta(T' - (E' \cap E(T'))) \geq 1$ . We get  $\gamma(T - E') = \gamma(P_3 \cup T - (E' \setminus \{x v_1\})) = \gamma(T' - (E' \cap E(T'))) + \gamma(P_3) = \gamma(T') + 1 \leq \gamma(T)$ . Now assume that  $x v_1 \notin E'$ . Because of the similarity between the paths  $abc$  and  $v_1 v_2 v_3$  adjacent to the vertex  $x$ , it suffices to consider only the possibility when  $x a \notin E'$ . Let us observe that  $\delta(T' - (E' \cap E(T'))) \geq 1$ . By  $T_x$  ( $T'_x$ , respectively), we denote the component of  $T - E'$  ( $T' - (E' \cap E(T'))$ , respectively) which contains the vertex  $x$ . If  $v_1 v_2 \notin E'$ , then let  $D'_x$  be any  $\gamma(T'_x)$ -set. It is easy to see that  $D'_x \cup \{v_2\}$  is a DS of the tree  $T_x$ . Thus  $\gamma(T_x) \leq \gamma(T'_x) + 1$ . We now get  $\gamma(T - E') = \gamma(T - E' - T_x) + \gamma(T_x) \leq \gamma(T - E' - T_x) + \gamma(T'_x) + 1 = \gamma(T' - E' - T'_x) + \gamma(T'_x) + 1 = \gamma(T' - E') + 1 = \gamma(T') + 1 \leq \gamma(T)$ . Now assume that  $v_1 v_2 \in E'$ . Because of the similarity between the paths  $abc$  and  $v_1 v_2 v_3$ , it suffices to consider only the possibility when  $ab \in E'$ . Let  $D'_x$  be a  $\gamma(T'_x)$ -set that contains all support vertices (so  $x \in D'_x$ ). It is easy to see that  $D'_x$  is a DS of the tree  $T_x$ . Thus  $\gamma(T_x) \leq \gamma(T'_x)$ . We get  $\gamma(T - E') = \gamma(T - E' - T_x) + \gamma(T_x) \leq \gamma(T - E' - T_x) + \gamma(T'_x) = \gamma(T' - E' - T'_x) + \gamma(T'_x) = \gamma(T' - E') = \gamma(T') \leq \gamma(T)$ . We now conclude that  $\gamma(T - E') = \gamma(T)$ , and consequently,  $b'(T) = 0$ .

Now assume that  $T$  is obtained from  $T'$  by Operation  $\mathcal{O}_4$ . By Lemma 4 we have  $b'(P_5) = 0$ . Using Lemma 10, for  $T_1 = T'$  and  $T_2 = P_5$ , we get  $b'(T) = 0$ . □

We now prove that if a tree is  $\gamma$ -non-isolatingly strongly stable, then it belongs to the family  $\mathcal{T}$ .

**Lemma 13** *Let  $T$  be a tree. If  $b'(T) = 0$ , then  $T \in \mathcal{T}$ .*

*Proof* If  $\text{diam}(T) \in \{0, 1\}$ , then  $T \in \{P_1, P_2\} \subseteq \mathcal{T}$ . If  $\text{diam}(T) = 2$ , then  $T$  is a star. The tree  $T$  can be obtained from  $P_2$  by an appropriate number of Operations  $\mathcal{O}_1$ . Thus  $T \in \mathcal{T}$ . Now assume that  $\text{diam}(T) \geq 3$ . Thus the order  $n$  of the tree  $T$  is at least four. We obtain the result by the induction on the number  $n$ . Assume that the lemma is true for every tree  $T'$  of order  $n' < n$ .



First assume that some support vertex of  $T$ , say  $x$ , is strong. Let  $y$  be a leaf adjacent to  $x$ . Let  $T' = T - y$ . Let  $D'$  be a  $\gamma(T')$ -set that contains all support vertices. It is easy to see that  $D'$  is a DS of the tree  $T$ . Thus  $\gamma(T) \leq \gamma(T')$ . Now let  $E'$  be a subset of the set of edges of  $T'$  such that  $\delta(T' - E') \geq 1$ . Since  $b'(T) = 0$ , we have  $\gamma(T - E') = \gamma(T)$ . Let us observe that there exists a  $\gamma(T - E')$ -set that contains the vertex  $x$ . Let  $D$  be such a set. The set  $D$  is minimal, thus  $y \notin D$ . Obviously,  $D$  is a DS of the graph  $T' - E'$ . Therefore  $\gamma(T' - E') \leq \gamma(T - E')$ . We now get  $\gamma(T' - E') \leq \gamma(T - E') = \gamma(T) \leq \gamma(T')$ . On the other hand, we have  $\gamma(T' - E') \geq \gamma(T')$ . This implies that  $\gamma(T' - E') = \gamma(T')$ , and consequently,  $b'(T') = 0$ . By the inductive hypothesis, we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by Operation  $\mathcal{O}_1$ . Thus  $T \in \mathcal{T}$ . Henceforth, we assume that every support vertex of  $T$  is weak.

We now root  $T$  at a vertex  $r$  of maximum eccentricity  $\text{diam}(T)$ . Let  $t$  be a leaf at maximum distance from  $r$ ,  $v$  be the parent of  $t$ , and  $u$  be the parent of  $v$  in the rooted tree. If  $\text{diam}(T) \geq 4$ , then let  $w$  be the parent of  $u$ . If  $\text{diam}(T) \geq 5$ , then let  $d$  be the parent of  $w$ . If  $\text{diam}(T) \geq 6$ , then let  $e$  be the parent of  $d$ . By  $T_x$  we denote the subtree induced by a vertex  $x$  and its descendants in the rooted tree  $T$ .

Assume that  $d_T(u) \geq 3$ . Thus some child of  $u$  is a leaf or a support vertex other than  $v$ . Let  $T' = T - T_v$ . By Lemma 11 we have  $b'(T') = 0$ . By the inductive hypothesis, we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by Operation  $\mathcal{O}_2$ . Thus  $T \in \mathcal{T}$ .

Now assume that  $d_T(u) = 2$ . Assume that  $d_T(w) \geq 3$ . First assume that there is a child of  $w$  other than  $u$ , say  $k$ , such that the distance of  $w$  to the most distant vertex of  $T_k$  is three. It suffices to consider only the possibility when  $T_k$  is a path  $P_3$ , say  $klm$ . Let  $T' = T - T_u$ . By Lemma 11 we have  $b'(T') = 0$ . By the inductive hypothesis, we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by Operation  $\mathcal{O}_3$ . Thus  $T \in \mathcal{T}$ .

Now assume that some child of  $w$  is a leaf. Let  $T' = T - T_u$ . By Lemma 11 we have  $b'(T') = 0$ . By the inductive hypothesis, we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by Operation  $\mathcal{O}_3$ . Thus  $T \in \mathcal{T}$ .

Thus there is a child of  $w$ , say  $k$ , such that the distance of  $w$  to the most distant vertex of  $T_k$  is two. Consequently,  $k$  is a support vertex of degree two. Due to the earlier analysis of the children of the vertex  $u$ , it suffices to consider only the possibility when  $d_T(w) = 3$ . Let  $T' = T - T_w$ . It is easy to observe that  $D' \cup \{v, k\}$  is a DS of the tree  $T$ . Thus  $\gamma(T) \leq \gamma(T') + 2$ . We have  $\delta(T - dw - uv - wk) \geq 1$ . We now get  $\gamma(T - dw - uv - wk) = \gamma(T' \cup P_2 \cup P_2 \cup P_2) = \gamma(T') + 3\gamma(P_2) = \gamma(T') + 3 \geq \gamma(T) + 1 > \gamma(T)$ . This implies that  $b'(T) \neq 0$ , a contradiction.

If  $d_T(w) = 1$ , then  $T = P_4$ . Let  $T' = P_2 \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by Operation  $\mathcal{O}_2$ . Thus  $T \in \mathcal{T}$ . Now assume that  $d_T(w) = 2$ . First assume that there is a child of  $d$  other than  $w$ , say  $k$ , such that the distance of  $d$  to the most distant vertex of  $T_k$  is four or one. It suffices to consider only the possibilities when  $T_k$  is a path  $P_4$ , or  $k$  is a leaf. Let  $T' = T - T_w$ . Let us observe that there exists a  $\gamma(T')$ -set that contains the vertex  $d$ . Let  $D'$  be such a set. It is easy to observe that  $D' \cup \{v\}$  is a DS of the tree  $T$ . Thus  $\gamma(T) \leq \gamma(T') + 1$ . We have  $\delta(T - dw - uv) \geq 1$ . We now get  $\gamma(T - dw - uv) = \gamma(T' \cup P_2 \cup P_2) = \gamma(T') + 2\gamma(P_2) = \gamma(T') + 2 \geq \gamma(T) + 1 > \gamma(T)$ . This implies that  $b'(T) \neq 0$ , a contradiction.

Now assume that there is a child of  $d$ , say  $k$ , such that the distance of  $d$  to the most distant vertex of  $T_k$  is three. It suffices to consider only the possibility when  $T_k$  is a



path  $P_3$ , say  $klm$ . Let  $T' = T - T_l$ . Due to the similarity of  $T'$  to the tree  $T$  from the previous case when  $d$  is adjacent to a leaf, we conclude that  $b'(T') \neq 0$ . On the other hand, by Lemma 11 we have  $b'(T') = 0$ , a contradiction.

Now assume that there is a child of  $d$ , say  $k$ , such that the distance of  $d$  to the most distant vertex of  $T_k$  is two. Thus  $k$  is a support vertex of degree two. Let  $T' = T - T_k$ . By Lemma 11 we have  $b'(T') = 0$ . By the inductive hypothesis, we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by Operation  $\mathcal{O}_2$ . Thus  $T \in \mathcal{T}$ .

If  $d_T(d) = 1$ , then  $T = P_3$ . Let  $T' = P_2 \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by Operation  $\mathcal{O}_3$ . Thus  $T \in \mathcal{T}$ .

Now assume that  $d_T(d) = 2$ . First assume that  $e$  is adjacent to a leaf, say  $k$ . Let  $T' = T - T_d$ . By Lemma 11 we have  $b'(T') = 0$ . By the inductive hypothesis, we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by Operation  $\mathcal{O}_4$ . Thus  $T \in \mathcal{T}$ .

Now assume that  $e$  is not adjacent to any leaf. Let  $E'$  be the set of edges incident with  $e$  excluding  $ed$ . Let  $G' = T - T_d - e$ . Let  $D'$  be any  $\gamma(G')$ -set. It is easy to observe that  $D' \cup \{d, v\}$  is a DS of the tree  $T$ . Thus  $\gamma(T) \leq \gamma(G') + 2$ . We have  $\delta(T - (E' \cup \{dw, uv\})) \geq 1$ . We now get  $\gamma(T - (E' \cup \{dw, uv\})) = \gamma(G' \cup P_2 \cup P_2) = \gamma(G') + 3\gamma(P_2) = \gamma(G') + 3 \geq \gamma(T) + 1 > \gamma(T)$ . This implies that  $b'(T) \neq 0$ , a contradiction.  $\square$

As an immediate consequence of Lemmas 12 and 13, we have the following characterization of all  $\gamma$ -non-isolatingly strongly stable trees.

**Theorem 14** *Let  $T$  be a tree. Then  $b'(T) = 0$  if and only if  $T \in \mathcal{T}$ .*

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