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## NOTE ON THE VARIANCE OF THE SUM OF GAUSSIAN FUNCTIONALS

Abstract. Let  $(X_i, i = 1, 2, ...)$  be a Gaussian sequence with  $X_i \in N(0, 1)$  for each i and suppose its correlation matrix  $R = (\rho_{ij})_{i,j\geq 1}$  is the matrix of some linear operator  $R: l_2 \to l_2$ . Then for  $f_i \in L^2(\mu), i = 1, 2, ...$ , where  $\mu$  is the standard normal distribution, we estimate the variation of the sum of the Gaussian functionals  $f_i(X_i), i = 1, 2, ...$ 

**1. Introduction.** Let (X, Y) be a Gaussian random vector such that  $X, Y \in N(0, 1)$  and  $E(XY) = \rho$ ,  $(|\rho| < 1)$ . We denote by  $\mu$  the normalized one-dimensional Gaussian measure, i.e.

$$\mu(dx) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx.$$

In  $L^2(\mu)$  we have the scalar product

$$(f,g)_{\mu} = \int_{\mathbb{R}} f(x)g(x)\,\mu(dx).$$

Introducing a random variable  $Z \in N(0,1)$  such that Z, Y are independent, we find that the Gaussian vectors (X,Y) and (U,Y) with  $U = \rho Y + \sqrt{1-\rho^2} Z$  have the same joint distribution. Thus, for  $f,g \in L^2(\mu)$  we have (1.1)  $E(f(X)g(Y)) = E(f(U)g(Y)) = E(P_{\rho}(Y)g(Y)),$ 

where

$$P_{\rho}f(y) = E(f(U) | Y = y) = \int_{\mathbb{R}} f(\rho y + \sqrt{1 - \rho^2} z) d\mu(z), \quad y \in \mathbb{R},$$

is called the *Ornstein–Uhlenbeck operator*. The Ornstein–Uhlenbeck operator has a representation in terms of orthonormal Hermite polynomials

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 ${h_n}_{n\geq 0} \subset L^2(\mu)$ , namely

(1.2) 
$$P_{\rho}f = \sum_{n=0}^{\infty} \rho^{n}(f, h_{n})_{\mu} h_{n}, \quad f \in L^{2}(\mu).$$

In particular,

$$P_{\rho}h_n = \rho^n h_n, \quad n \ge 0$$

From (1.2) we obtain Gebelein's inequality (see [G] and [DK]):

PROPOSITION 1.1. If 
$$f \in L^2$$
 and  $(f, 1)_{\mu} = 0$ , then

(1.3)  $||P_{\rho}f||_{2} \leq |\rho| \cdot ||f||_{2},$ or equivalently for any  $g \in L^{2}$  and f as above,

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$$(P_{\rho}f,g)_{\mu} \leq |\rho| \cdot ||f||_2 \cdot ||g||_2.$$

In both inequalities we have equality if and only if f(x) = cx.

Consider a Gaussian sequence  $(X_i, i = 1, 2, ...)$  of random variables with  $X_i \in N(0, 1)$  for each *i*. It is assumed that the correlation matrix  $R = (\rho_{ij})_{i,j\geq 1}$ , where  $\rho_{ij} = E(X_iX_j)$ , i, j = 1, 2, ..., satisfies

(1.4) 
$$C = \sup_{i \ge 1} \sum_{j \ge 1} |\rho_{ij}| < \infty.$$

It is evident that  $C \ge 1$ . The Frobenius Theorem (see [HLP]) implies that R is the matrix (in the standard basis) of a continuous linear operator (which we denote by the same letter)  $R: l_p \to l_p$  for  $1 \le p \le \infty$  with  $||R|| \le C$ . Hence, it is easily seen that for C < 2 the linear operator R is invertible. Using Gebelein's inequality (1.3), one can prove (see [BC1], [BC2], [V])

LEMMA 1.1. Let the Gaussian sequence  $(X_i, i = 1, 2, ...)$  with  $X_i \in N(0,1)$  for each *i* satisfy the hypothesis (1.4) and let  $(f_i, i = 1, 2, ...) \subset L^2(\mu)$ . Then for each  $n \geq 1$  we have

(1.5) 
$$(2-C)\sum_{i=1}^{n} \operatorname{Var}(f_i(X_i)) \le \operatorname{Var}\left(\sum_{i=1}^{n} f_i(X_i)\right) \le C\sum_{i=1}^{n} \operatorname{Var}(f_i(X_i)).$$

For  $C \ge 2$  the left inequality in (1.5) holds trivially. In fact, we can say more: an inequality of the form

(1.6) 
$$M\sum_{i=1}^{n} \operatorname{Var}(f_i(X_i)) \leq \operatorname{Var}\left(\sum_{i=1}^{n} f_i(X_i)\right),$$

where M is a positive constant, is not satisfied in general when  $C \ge 2$ .

Consider the following simple example: Let  $(Y_i, i = 1, 2, ...) \subset N(0, 1)$ be a sequence of independent Gaussian random variables. Let  $a, b \in \mathbb{R}$  be such that  $a^2 + b^2 = 1$  and define

$$X_{3k-2} = -Y_{2k}, \quad X_{3k-1} = a Y_{2k-1} - bY_{2k}, \quad X_{3k} = aY_{2k-1} + b Y_{2k}, \quad k \ge 1.$$

Moreover, we put

$$f_{3k-2}(t) = 2bt$$
,  $f_{3k-1}(t) = -t$ ,  $f_{3k}(t) = t$ ,  $t \in \mathbb{R}, k \ge 1$ .

It is easy to check that

$$C = \sup_{i \ge 1} \sum_{j \ge 1} |\rho_{ij}| = 1 + |b| + \max\{|b|, |1 - 2b^2|\} \ge 2$$

and

$$\operatorname{Var}\left(\sum_{i=1}^{3n} f_i(X_i)\right) = 0 \quad \text{and} \quad \sum_{i=1}^{3n} \operatorname{Var}(f_i(X_i) > 0, \quad n \ge 1.$$

2. Main result. In this section we are going to prove the inequality (1.5) under a slightly weaker condition than (1.4). First let us introduce some notations. For a given correlation matrix  $R = (\rho_{ij})_{i,j\geq 1}$ , we put

$$R_n^{(m)} = (\rho_{ij}^m)_{1 \le i,j \le n}, \quad m, n \ge 1,$$

and let  $\lambda_{n,1}^{(m)}$  and  $\lambda_{n,n}^{(m)}$  denote the least and the greatest of the eigenvalues of the matrix  $R_n^{(m)}$ . By the Schur lemma (see [B]) the matrix  $R_n^{(m)}$  is nonnegative definite. Hence, the eigenvalues  $\lambda_{n,1}^{(m)}$  are nonnegative. For the matrix  $R_n = R_n^{(1)}$  we use the well known decomposition

$$R_n = U_n D_n U_n^T,$$

where

$$D_n = \begin{pmatrix} \lambda_{n,1}^{(1)} & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \lambda_{n,n}^{(1)} \end{pmatrix}$$

is a diagonal matrix with eigenvalues  $\lambda_{n,i}^{(1)}$ ,  $i = 1, \ldots, n$ , of  $R_n$  on the main diagonal. The matrix  $U_n = (u_{n,ij})_{1 \le i,j \le n}$  is an orthogonal matrix and T stands for transposition. It follows that

(2.1) 
$$\rho_{ij} = \sum_{k=1}^{n} \lambda_{n,k}^{(1)} u_{n,ik} u_{n,jk}, \quad 1 \le i, j \le n.$$

Now we can state the following simple result.

LEMMA 2.1. Fix  $n \geq 1$ . Then the least and the greatest eigenvalues of the matrix  $R_n^{(m)}$  are monotonic with respect to m, i.e.

(2.2) 
$$\lambda_{n,1}^{(m+1)} \ge \lambda_{n,1}^{(m)} \quad and \quad \lambda_{n,n}^{(m+1)} \le \lambda_{n,n}^{(m)}, \quad for \ m = 1, 2, \dots$$

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*Proof.* Since the matrix  $R_n^{(m+1)}$  is symmetric, we have

(2.3) 
$$\lambda_{n,1}^{(m+1)} = \inf_{\|c\|=1} (R_n^{(m+1)}c, c) = \inf_{\|c\|=1} \sum_{i,j=1}^n \rho_{ij}^{m+1}c_i c_j,$$

where  $c = (c_1, \ldots, c_n) \in l_2^n$  and  $l_2^n$  is the *n*-dimensional Euclidean space with the scalar product denoted here by  $(\cdot, \cdot)$ . From (2.3) and (2.1) we conclude that for every  $c = (c_1, \ldots, c_n) \in l_2^n$  with ||c|| = 1 we have

$$(2.4) \qquad \sum_{i,j=1}^{n} \rho_{ij}^{m+1} c_i c_j = \sum_{i,j=1}^{n} \rho_{ij}^m \rho_{ij} c_i c_j = \sum_{i,j=1}^{n} \rho_{ij}^m \sum_{k=1}^{n} \lambda_{n,k}^{(1)} u_{n,ik} u_{n,jk} c_i c_j = \sum_{k=1}^{n} \lambda_{n,k}^{(1)} \Big( \sum_{i,j=1}^{n} \rho_{ij}^m c_i u_{n,ik} c_j u_{n,jk} \Big) \geq \sum_{k=1}^{n} \lambda_{n,k}^{(1)} \sum_{i=1}^{n} c_i^2 u_{n,ik}^2 \inf_{\|b\|=1} (R_n^{(m)} b, b) = \lambda_{n,1}^{(m)},$$

since

$$\sum_{k=1}^{n} \lambda_{n,k}^{(1)} \sum_{i=1}^{n} c_i^2 u_{n,ik}^2 = \sum_{i=1}^{n} c_i^2 \sum_{k=1}^{n} \lambda_{n,k}^{(1)} u_{n,ik}^2 = 1$$

by (2.1). Taking the infimum in (2.4) over all  $c = (c_1, \ldots, c_n) \in l_2^n$  with ||c|| = 1 we obtain the first inequality of (2.2). The proof of the second one runs similarly.

We can now formulate our main result.

THEOREM 2.1. Let  $(X_i, i = 1, 2, ...)$  be a Gaussian sequence with  $X_i \in N(0, 1)$  for each *i* and suppose its correlation matrix  $R = (\rho_{ij})_{i,j\geq 1}$  is the matrix of some operator  $R: l_2 \rightarrow l_2$ . Then for  $f_i \in L^2(\mu)$ , i = 1, 2, ..., and for every  $n \geq 1$  we have

(2.5) 
$$\lambda_{\min} \sum_{i=1}^{n} \operatorname{Var}(f_i(X_i)) \leq \operatorname{Var}\left(\sum_{i=1}^{n} f_i(X_i)\right) \leq \lambda_{\max} \sum_{i=1}^{n} \operatorname{Var}(f_i(X_i)),$$

where

$$\lambda_{\min} = \inf_{\|x\|=1} (Rx, x), \quad \lambda_{\max} = \sup_{\|x\|=1} (Rx, x).$$

REMARK. Let us point out that the assumption concerning the correlation matrix  $R = (\rho_{ij})_{i,j\geq 1}$  of the sequence  $(X_i, i = 1, 2, ...)$  is slightly weaker than the hypothesis (1.4). To see this, consider the following example: Let  $(Y_i, i = 1, 2, ...) \subset N(0, 1)$  be a sequence of independent Gaussian random variables and define

$$X_1 = aY_1 + \sum_{j=2}^{\infty} Y_j/j, \text{ where } a = \sqrt{2 - \pi^2/6},$$
$$X_i = Y_i \text{ for } i \ge 2.$$

It follows immediately that the correlation matrix  $R = (\rho_{ij})_{i,j\geq 1}$  of the sequence  $(X_i, i = 1, 2, ...)$  is the matrix of some linear operator  $R : l_2 \to l_2$  and the hypothesis (1.4) is not satisfied.

Proof of Theorem 1.1. First we prove the left inequality of (2.5). Without loss of generality we assume that  $E(f_i(X_i)) = 0, i = 1, 2, ...$  If  $\lambda_{\min} = 0$ then the inequality holds trivially. Assume that  $\lambda_{\min} \neq 0$ . Expanding each  $f_i, i \geq 1$ , with respect to the Hermite basis in  $L^2(\mu)$  we obtain

(2.6) 
$$f_i = \sum_{k=1}^{\infty} c_{ik} h_k, \quad \|f_i\|_{\mu}^2 = \sum_{k=1}^{\infty} c_{ik}^2, \quad i = 1, 2, \dots$$

From (1.1) and (1.2) and the orthonormality of Hermite polynomials  $\{h_n\}_{n\geq 1} \subset L^2(\mu)$  it follows that

(2.7) 
$$E[h_n(X_i)h_m(X_j)] = \rho_{ij}^n \delta_m^n, \quad n, m, i, j = 1, 2, \dots,$$

where  $\delta_m^n$  is the Kronecker delta. Combining (2.6) with (2.7) and using Lemma 2.1 we get

$$\operatorname{Var}\left(\sum_{i=1}^{n} f_{i}(X_{i})\right) = E\left(\sum_{i=1}^{n} f_{i}(X_{i})\right)^{2}$$
$$= \lim_{N \to \infty} E\left(\sum_{i=1}^{n} \sum_{k=1}^{N} c_{ik}h_{k}(X_{i})\right)^{2} = \lim_{N \to \infty} E\left(\sum_{k=1}^{N} \sum_{i=1}^{n} c_{ik}h_{k}(X_{i})\right)^{2}$$
$$= \lim_{N \to \infty} \sum_{k,l=1}^{N} E\left[\left(\sum_{i=1}^{n} c_{ik}h_{k}(X_{i})\right)\left(\sum_{j=1}^{n} c_{jl}h_{l}(X_{j})\right)\right]$$
$$= \lim_{N \to \infty} \sum_{k=1}^{N} E\left[\sum_{i=1}^{n} c_{ik}h_{k}(X_{i})\right]^{2} = \lim_{N \to \infty} \sum_{k=1}^{N} \sum_{i,j=1}^{n} \rho_{ij}^{k}c_{ik}c_{jk}$$
$$\geq \lim_{N \to \infty} \sum_{k=1}^{N} \lambda_{n,1}^{(k)} \sum_{i=1}^{n} c_{ik}^{2} \geq \lim_{N \to \infty} \sum_{k=1}^{N} \lambda_{n,1}^{(1)} \sum_{i=1}^{n} c_{ik}^{2}$$
$$\geq \lambda_{\min} \sum_{i=1}^{n} \sum_{k=1}^{\infty} c_{ik}^{2} = \lambda_{\min} \sum_{i=1}^{n} E\left[f_{i}(X_{i})\right]^{2} = \lambda_{\min} \sum_{i=1}^{n} \operatorname{Var}(f_{i}(X_{i}))$$

This proves the left inequality of (2.5). The proof of the right one is similar.

REMARK. Let us point out that under the assumptions of Theorem 2.1 the inequality (1.6) holds for all  $f_i \in L^2(\mu)$ , i = 1, 2, ..., with a positive constant M if and only if the operator  $R: l_2 \to l_2$  is invertible.

Adapting now the methods from [BC1] and [BC2] we can write the following two statements:

LEMMA 2.2 (Borel–Cantelli Lemma). Let  $(X_i, i = 1, 2, ...)$  be a Gaussian sequence with  $X_i \in N(0, 1)$  for  $i \ge 1$  and suppose its correlation matrix  $R = (\rho_{ij})_{i,j\ge 1}$  is the matrix of some linear operator  $R : l_2 \to l_2$ . Then for every sequence of Borel sets  $(A_i, i = 1, 2, ...)$  such that  $\sum_{i=1}^{\infty} P\{X_i \in A_i\} = \infty$  we have  $P\{X_i \in A_i \text{ i.o.}\} = 1$ .

THEOREM 2.2 (Strong Law of Large Numbers). Let  $(X_i, i = 1, 2, ...)$  be a Gaussian sequence with  $X_i \in N(0, 1)$  for  $i \ge 1$  and suppose its correlation matrix  $R = (\rho_{ij})_{i,j\ge 1}$  is the matrix of some linear operator  $R : l_2 \to l_2$ . Then for  $f \in L^1(\mu)$  we have

$$\frac{1}{n}\sum_{i=1}^{n}f(X_i)\xrightarrow[n\to\infty]{} Ef(X_1) \quad a.s. \bullet$$

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