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On Applications of Fractional Derivatives in Electromagnetic Theory

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***Abstract*—In this paper, concepts of fractional-order (FO) derivatives are analysed from the point of view of applications in the electromagnetic theory. The mathematical problems related to the FO generalization of Maxwell's equations are investigated. The most popular formulations of the fractional derivatives, i.e., Riemann-Liouville, Caputo, Grünwald-Letnikov and Marchaud definitions, are considered. Properties of these derivatives are evaluated. It is demonstrated that some of formulations of the FO derivatives have limited applicability in the electromagnetic theory. That is, the Riemann-Liouville and Caputo derivatives with finite base point have a limited applicability whereas the Grünwald-Letnikov and Marchaud derivatives lead to reasonable generalizations of Maxwell's equations.**

***Index Terms*—Computational electromagnetics, electromagnetic modelling, Maxwell's equations, fractional calculus.**

I. Introduction

For many years, fractional-order (FO) mathematical models are applied in the electromagnetic theory based on various formulations of derivative and integral operators [1]–[7]. Such models are useful for characterization of dielectric media [8] and conductors [9], [10]. Hence, the FO modelling is applied in microwave engineering [11]–[13].

The FO derivative operator D^α ($\alpha \in \mathbb{R}$, $\alpha > 0$) is a generalization of the standard concept of the n-fold operator of differentiation D^n where n is a natural number ($n \in \mathbb{N}$). In classical monographs [14]–[16], one may find review of formulations of fractional derivatives. Some of definitions are well established and widely applied in the electromagnetic theory, whereas some ideas appeared quite recently and are not too much popular. In this contribution, the applicability of four important derivative definitions, i.e., Riemann-Liouville, Caputo, Grünwald-Letnikov and Marchaud, is discussed. The motivation for our research stems from an ambiguity of definitions of the FO derivative whose properties sometimes exclude them from applications in the electromagnetic theory. Therefore, opinions appear that questionize applicability of FO derivatives and models in electrical sciences and engineering [17], [18]. Recent discussion in literature suggests that the proposed analysis of properties of FO derivatives from the electromagnetic theory point of view is necessary.

In the sequel, it is only assumed that functions are smooth enough for the FO derivative operator D^α ($\alpha > 0$) to be applied. Quite recently, several attempts have been made to specify the conditions that constitute fractional derivatives [19]–[21]. Among them, one may find the linearity of the fractional derivative operator

$$D^\alpha(af(t) + bg(t)) = aD^\alpha f(t) + bD^\alpha g(t) \tag{1}$$

and the semigroup property (also called *the index law*)

$$D^\alpha D^\beta f(t) = D^\beta D^\alpha f(t) = D^{\alpha+\beta} f(t), \qquad \alpha, \beta \in \mathbb{R}. \tag{2}$$

The semigroup property is sometimes validated under additional assumptions that, e.g., either $\alpha, \beta < 0$ or $0 < \alpha, \beta, \alpha + \beta < 1$. These assumptions appear to be crucial when applying operator D^α in the FO generalizations of Maxwell's equations. However, as it is noticed in [18], the condition (2) may not be satisfied for widely applied definitions of FO derivative. Furthermore, for the phasor representation of signals in the electromagnetic theory, the *trigonometric functions invariance* property is required

$$D^\alpha e^{j\omega t} = (j\omega)^\alpha e^{j\omega t} \tag{3}$$

where $j = \sqrt{-1}$. The last property is a generalization of the obvious formula taken from the integer-order (IO) calculus.

We demonstrate that properties (1)–(3) are natural when using the classical methods of the electromagnetic theory and refer to the wave-type equations formulated either in time or frequency domain. Then, a short review of different definitions of fractional derivatives is given from this perspective. Finally, conclusion is drawn at the end of the paper.

II. Problem Statement

The electromagnetic problem can be solved in either time or frequency domain. However, solutions obtained in both domains should be equivalent, i.e, they should be related by the Fourier transformation. In this section, we demonstrate that it requires satisfaction of the semigroup condition (2) by the definition of FO derivative applied in the electromagnetic analysis.

A. Maxwell's Equations

Let us formulate Maxwell's equations in isotropic and homogeneous media

$$\nabla \cdot \mathbf{D} = \rho \tag{4}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{6}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{7}$$

where $\mathbf{E}$ and $\mathbf{H}$ denote respectively the electric- and magnetic-field intensities, $\mathbf{D}$ and $\mathbf{B}$ denote respectively the displacement- and magnetic-flux densities, $\mathbf{J}$ denotes the current density, ρ denotes the charge density. Constitutive relations for a linear medium described by IO model (IOM) are defined as follows:

$$\mathbf{J} = \sigma \mathbf{E} \tag{8}$$

$$\mathbf{D} = \epsilon \mathbf{E} \tag{9}$$

$$\mathbf{B} = \mu \mathbf{H}. \tag{10}$$

For the sake of brevity, let us consider a free space without losses ($\sigma = 0$) and sources. Using (4)–(10) and the semigroup property (2) for IO derivatives (where $\alpha = \beta = 1$), one obtains the following wave equations in the time domain:

$$\nabla^2 \begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix} - \mu\epsilon \frac{\partial^2}{\partial t^2} \begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix} = 0. \tag{11}$$

Let us consider the phasor representation of the electromagnetic field, i.e.

$$\mathbf{E} = \Re(\tilde{\mathbf{E}} e^{j\omega t}) \tag{12}$$

$$\mathbf{H} = \Re(\tilde{\mathbf{H}} e^{j\omega t}) \tag{13}$$

where $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ are electric and magnetic field phasors that are functions of the spatial variables only and ω denotes the angular frequency. From (11)–(13), one obtains the wave equations in the frequency domain

$$\nabla^2 \begin{Bmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{Bmatrix} + \mu\epsilon\omega^2 \begin{Bmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{Bmatrix} = 0. \tag{14}$$

Let us consider the function $f(t)$ defined in the entire real line for which the bilateral Laplace transform can be defined

$$\hat{f} = \mathcal{L}\{f(t)\} = \int_{-\infty}^{+\infty} f(t) e^{-st} dt. \tag{15}$$

It is identical to the unilateral Laplace transform given by

$$\hat{f} = \mathcal{L}\{f(t)\} = \int_{0}^{+\infty} f(t) e^{-st} dt \tag{16}$$

for causal functions (i.e., $f(t) = 0$ for $t \in (-\infty, 0)$). Then, (16) is applied to the wave equation (11). Hence, one obtains

$$\nabla^2 \begin{Bmatrix} \hat{\mathbf{E}} \\ \hat{\mathbf{H}} \end{Bmatrix} - \mu\epsilon s^2 \begin{Bmatrix} \hat{\mathbf{E}} \\ \hat{\mathbf{H}} \end{Bmatrix} = 0. \tag{17}$$

Substituting $s = j\omega$ into (17), one obtains the Fourier transformed wave equations which are equivalent to (14).

B. FO Maxwell's Equations

Let us focus now on time-fractional generalizations of Maxwell's equations. Let us treat the fractional derivative of the order $\alpha \in (0, 1)$ with respect to t as an abstract operator D_t^{α} defined on some functional domain. Let us then introduce constitutive relations for a medium described by FO model (FOM) [22]–[25]

$$\mathbf{J} = \sigma_\alpha D_t^{1-\alpha} \mathbf{E}, \quad 0 < \alpha \le 1 \tag{18}$$

$$\epsilon_\beta \mathbf{E} = D_t^{1-\beta} \mathbf{D}, \quad 0 < \beta \le 1 \tag{19}$$

$$\mu_\gamma \mathbf{H} = D_t^{1-\gamma} \mathbf{B}, \quad 0 < \gamma \le 1. \tag{20}$$

Time-fractional derivatives appear in the constitutive relations when hereditary mechanisms of power-law type exist in electromagnetic media. For $\alpha = 1$, $\beta = 1$ and $\gamma = 1$, one obtains the classical constitutive relations (8)–(10).

Let us consider Maxwell's equations (4)–(7) with FO constitutive relations (18)–(20) in a free space without losses ($\sigma_\alpha = 0$) and sources. Then, one obtains

$$\nabla \cdot \mathbf{E} = 0 \tag{21}$$

$$\nabla \times \mathbf{E} = -\mu_\gamma D_t^{\gamma} \mathbf{H} \tag{22}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{23}$$

$$\nabla \times \mathbf{H} = \epsilon_\beta D_t^{\beta} \mathbf{E}. \tag{24}$$

From (21)–(24), one can derive the following diffusion-wave equations:

$$\nabla^2 \mathbf{E} - \mu_\gamma \epsilon_\beta D_t^{\gamma} D_t^{\beta} \mathbf{E} = 0 \tag{25}$$

$$\nabla^2 \mathbf{H} - \mu_\gamma \epsilon_\beta D_t^{\beta} D_t^{\gamma} \mathbf{H} = 0. \tag{26}$$

As can be noticed, the electric- and magnetic-field intensities satisfy the same diffusion-wave equation if and only if $\alpha = \beta$ or the FO operators are commutative (i.e., $D_t^{\gamma} D_t^{\beta} = D_t^{\beta} D_t^{\gamma}$).

Let us apply the phasor representation (12)–(13) to FO Maxwell's equations (21)–(24)

$$\nabla \cdot \tilde{\mathbf{E}} = 0 \tag{27}$$

$$\nabla \times \tilde{\mathbf{E}} = -\mu_\gamma (j\omega)^{\gamma} \tilde{\mathbf{H}} \tag{28}$$

$$\nabla \cdot \tilde{\mathbf{H}} = 0 \tag{29}$$

$$\nabla \times \tilde{\mathbf{H}} = \epsilon_\beta (j\omega)^{\beta} \tilde{\mathbf{E}}. \tag{30}$$

The following diffusion-wave equations can be obtained in the frequency domain from (27)–(30):

$$\nabla^2 \begin{Bmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{Bmatrix} - \mu_\gamma \epsilon_\beta (j\omega)^{\beta+\gamma} \begin{Bmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{Bmatrix} = 0. \tag{31}$$

As can be noticed, inconsistency is obtained because if one applies the phasor representation (12)–(13) to (21)–(24) then the same equation is obtained for the electric- and magnetic-field intensities. On the other hand, the electric- and magnetic-field intensities satisfy different diffusion-wave equations in the time domain, refer to (25)–(26). Furthermore, it cannot be assumed *a priori* that the set of common solutions to (25) and (26) exists as demonstrated in [26]. In this paper, one can see the relatively simple example demonstrating that the equation $D^{\alpha+\beta}y = y$ is not equivalent to $D^\alpha B^\beta y = y$ for the Riemann-Liouville derivative.

C. Solution to the Problem

For definitions of the fractional derivative that satisfy both (2) and (3), the time-domain diffusion-wave equations (25)–(26) are consistent with the frequency-domain formulation (31). Hence, the diffusion-wave equations (25)–(26) take the form

$$\nabla^2 \begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix} - \mu_\gamma \epsilon_\beta D_t^{\beta+\gamma} \begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix} = 0 \tag{32}$$

which is consistent with (31) originating from the phasor representation (12)–(13).

III. FO Derivatives

Let us now consider a few definitions of the fractional derivative. Let us assume that the function $f\colon \mathbb{R} \to \mathbb{R}$ is defined on the real line. Each of the presented definitions has its *left-sided* and *right-sided* version. We do not refer to both of them but only to the *left-sided* version. These approaches are symmetric and the *left-sided* version looks at the past times, so it is closer to the concept of causality. One should note that both Riemann-Liouville and Caputo derivatives require a value of the *base point*. The classical approach assumes that the base point $a \in \mathbb{R}$ is a number, but the definitions may also be extended to $a = -\infty$. Usually, due to the causality assumption, $a = 0$ is the natural selection. If this is the case, one may restrict the domain of the function f to the interval $[0, +\infty)$. One should note that all of the definitions discussed in the paper satisfy the linearity property (1).

A. Riemann-Liouville

The Riemann-Liouville integral of the function $f : \mathbb{R} \mapsto \mathbb{R}$ is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau \tag{33}$$

where $\alpha > 0$ is an order of integration, Γ is the Gamma function and a is the fixed base point. Based on this definition, the Riemann-Liouville derivative of order $\alpha \in (n-1, n)$ is introduced as

$$D^\alpha f(t) = D^n J^{n-\alpha} f(t). \tag{34}$$

One should note that the Riemann-Lioville derivative does not satisfy neither (2) nor (3). A valuable discussion on the semigroup property for the Riemann-Liouville intergral may be found in Section 3.2. of [19]. The interesting counterexamples (actually related to fractional differential equations) are given in [26] (see Example 6.2 and Remark 5). For instance, the base point $a = 0$ leads to the derivatives of the sine and cosine functions being nonelementary functions (see e.g. [27, Propositions 11, 12])

$$D^\alpha \sin(\omega t) = \frac{t^{-\alpha}}{2} \left(E_{1,1-\alpha}(j\omega t) + E_{1,1-\alpha}(-j\omega t)\right) \tag{35}$$

$$D^\alpha \cos(\omega t) = \frac{t^{-\alpha}}{2} \left(E_{1,1-\alpha}(j\omega t) - E_{1,1-\alpha}(-j\omega t)\right) \tag{36}$$

where $E_{\alpha,\beta}(z)$ denotes the generalized Mittag-Leffler function. On the other hand, if $a = -\infty$ then the Riemann-Liouville integral of neither sine nor cosine exists due to divergent integral in the unbounded domain for any $\alpha > 0$.

The Laplace transform of the order α derivative ($\alpha \in [n-1, n)$ where $n \in \mathbb{N}$) of the function $f\colon [0, +\infty) \to \mathbb{R}$ is given by

$$\mathcal{L}\{D^\alpha f(t)\} = s^\alpha \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} s^k [D^{\alpha-k-1} f(t)]_{t=0}. \tag{37}$$

It is the formula (1.85) from [16].

B. Caputo

The Caputo derivative may be defined in a similar way as the Riemann-Liouville derivative (34) but with the reverse order of operators (i.e., order of fractional integration and IO derivative is interchanged)

$$D^\alpha f(t) = J^{n-\alpha} D^n f(t) \tag{38}$$

where $\alpha \in (n-1, n)$. However, one may also refer to another definition based on the Riemann-Liovuille integral (see [16, Formula (1.12)] or [15, Formula (2.4.1)])

$${}_C D^\alpha f(t) = {}_{RL} D^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right] \tag{39}$$

where ${}_C D^\alpha$ denotes the Caputo derivative and ${}_{RL} D^\alpha$ denotes the Riemann-Liouville derivative with the same base point $a = 0$. These two definitions agree for a function f of the class C^n, although the definition (39) formally requires only the existence of $(n-1)$-th derivative in the neighbourhood of the base point $a = 0$. The formula (39) applied to a function $f(t) = \sin t$ returns for $\alpha \in (0, 1)$ that (35) is also valid for the Caputo derivative with the base point $a = 0$. Hence, the Caputo derivative does not satisfy (3) – for exact formulas see [27, Propositions 11, 12]. Furthermore, the semigroup property (2) fails in general for this definition, refer again to [26] and especially to Example 6.1 therein.

The Laplace transform of the derivative of the order $\alpha \in (n-1, n]$ (where $n \in \mathbb{N}$) of the function $f\colon [0, +\infty) \to \mathbb{R}$ is given by

$$\mathcal{L}\{D^\alpha f(t)\} = s^\alpha \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0). \tag{40}$$

It is the formula (1.88) from [16].

C. Grünwald-Letnikov

The (left-sided) Grünwald-Letnikov derivative of the order $\alpha > 0$ of the function f defined on the real line is given by the discrete formula (see [14, Formula (20.7)])

$$D^\alpha f(x) = \lim_{h\to 0^+} \frac{1}{h^\alpha} \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} f(x - mh) \tag{41}$$

where $\binom{\alpha}{m} = \frac{\alpha(\alpha-1)\ldots(\alpha-m+1)}{m!}$. It is well know that this definition satisfies both (2) (see [28, Section 2.6.1]) and (3) (see [28, Formula (2.65)]).

The bilateral Laplace transform of the derivative of the order $\alpha \in (n-1, n)$ (where $n \in \mathbb{N}$) of the function $f\colon \mathbb{R} \to \mathbb{R}$ is given by

$$\mathcal{L}\{D^\alpha f(t)\} = s^\alpha \mathcal{L}\{f(t)\}, \qquad \Re s > 0. \tag{42}$$

The details of this result and an interesting discussion may be found in Sections 2.7.3 and 2.8 of [28].

D. Marchaud

The definition of the Marchaud derivative is similar to the Riemann-Liouville derivative with a base point $a = -\infty$ and for a broad class of functions these two definitions are equivalent (i.e., for a class of sufficiently smooth functions with an appropriate behaviour at $-\infty$, as discussed in [14, Section 5.4]). However, there are some very important differences between both definitions. The most important difference is that the Marchaud derivative can be calculated for a broader class of functions than the Riemann-Liouville derivative – including the sine and cosine functions. Furthermore, one should notice that the Marchaud defintion is equivalent to the Grünwald-Letnikov definition (see Theorems 20.2 and 20.4 in [14]). We would like to refer here to the recent survey paper [29] which discusses both approaches in detail.

In general, the Marchaud derivative is defined for $\alpha > 0$ (refer to Section 5.5 in [14], Section 1.3.1 in [16], and [30] for a historical perspective). When $\alpha \in (n-1, n)$ and $n \in \mathbb{N}$ then

$$D^\alpha f(t) = \frac{\alpha - n + 1}{\Gamma(n-\alpha)} \int_0^{+\infty} \frac{f^{(n-1)}(t) - f^{(n-1)}(t-\tau)}{\tau^{2+\alpha-n}} \mathrm{d}\tau \tag{43}$$

where f is assumed to be smooth enough, e.g., $f \in C^{n-1}(\mathbb{R})$ with $f^{(n-1)}$ bounded. As mentioned above, it is equivalent to the Grünwald-Letnikov derivative which immediately confirms that it satisfies both (2) and (3). Still, one may verify (3) directly from the Formula 5 in Table 9.2 in [14]. The formulas

$$I_+^{(1-\alpha)}(\sin \omega t) = \omega^{\alpha-1} \sin(\omega t - (1-\alpha)\frac{\pi}{2}) \tag{44}$$

$$I_+^{(1-\alpha)}(\cos \omega t) = \omega^{(\alpha-1} \cos(\omega t - (1-\alpha)\frac{\pi}{2}) \tag{45}$$

are satisfied for $\omega > 0$ and $\alpha \in (0, 1)$. Then, (44)–(45) should be differentiated to get the fractional Marchaud derivative of the order α. Hence, one obtains

$$D^\alpha(\cos \omega t + j \sin \omega t) = \tag{46}$$

$$\omega^\alpha(-\sin(\omega t - (1-\alpha)\frac{\pi}{2}) + j\cos(\omega t - (1-\alpha)\frac{\pi}{2}) =$$

$$\omega^\alpha(-\sin(\omega t + \alpha\frac{\pi}{2} - \frac{\pi}{2}) + j\cos(\omega t + \alpha\frac{\pi}{2} - \frac{\pi}{2}) =$$

$$\omega^\alpha e^{j\omega t} e^{j\alpha\frac{\pi}{2}} = (j\omega)^\alpha e^{j\omega t}$$

for $\alpha \in (0, 1)$ and $\omega > 0$. Similar derivations are obtained for $\omega < 0$. For $\alpha > 1$, it is sufficient to refer to the semigroup property (2) with $\alpha = n - 1 + \{\alpha\}$ where $\{\alpha\} \in (0, 1)$.

IV. Conclusion

The time- and frequency-domain methods of analysis should return equivalent results when applied to FO generalizations of Maxwell's equations. It is demonstrated that in order to obtain the equivalence between results in the time and frequency domains, the FO differential operator should satisfy the linearity and semigroup conditions as well as be representable in the phasor domain. Out of four of the most popular approaches considered in this paper, only two of them are looking at the entire time-history of an input function and are appropriate choices for the electromagnetic theory. The Riemann-Liouville and Caputo derivatives with finite base point have a limited applicability, whereas the Grünwald-Letnikov and Marchaud definitions (which are actually equivalent) lead to reasonable generalizations of Maxwell's equations.

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