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## On existence and uniqueness of weak solutions for linear pantographic beam lattices models

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**Abstract** In this paper, we discuss well-posedness of the boundary-value problems arising in some “gradient-incomplete” strain-gradient elasticity models, which appear in the study of homogenized models for a large class of metamaterials whose microstructures can be regarded as beam lattices constrained with internal pivots. We use the attribute “gradient-incomplete” strain-gradient elasticity for a model in which the considered strain energy density depends on displacements and only on some specific partial derivatives among those constituting displacements first and second gradients. So, unlike to the models of strain-gradient elasticity considered up-to-now, the strain energy density which we consider here is in a sense degenerated, since it does not contain the full set of second derivatives of the displacement field. Such mathematical problem was

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motivated by a recently introduced new class of metamaterials (whose microstructure is constituted by the so-called pantographic beam lattices) and by woven fabrics. Indeed, as from the physical point of view such materials are strongly anisotropic, it is not surprising that the mathematical models to be introduced must reflect such property also by considering an expression for deformation energy involving only some among the higher partial derivatives of displacement fields. As a consequence, the differential operators considered here, in the framework of introduced models, are neither elliptic nor strong elliptic as, in general, they belong to the class so-called hypoelliptic operators. Following (Eremeyev et al. in *J Elast* 132:175–196, 2018. <https://doi.org/10.1007/s10659-017-9660-3>) we present well-posedness results in the case of the boundary-value problems for small (linearized) spatial deformations of pantographic sheets, i.e., 2D continua, when deforming in 3D space. In order to prove the existence and uniqueness of weak solutions, we introduce a class of subsets of anisotropic Sobolev's space defined as the energy space  $E$  relative to specifically assigned boundary conditions. As introduced by Sergey M. Nikolskii, an anisotropic Sobolev space consists of functions having different differential properties in different coordinate directions.

**Keywords** Strain-gradient elasticity · Weak solutions · Beam lattice · Pantographic sheets · Anisotropic Sobolev's spaces

## 1 Introduction

The strain-gradient theory of elasticity has its origin in the early works of some giants of continuum mechanics: see [1–6] for historical developments in the mechanics of generalized continua, and it was developed further in the original works by Toupin [7] and Mindlin [8,9]. The main conceptual tool for formulating these theories is given by the principle of virtual work and/or the principle of least action: indeed also continuum mechanics finds its more effective formulation when one bases its postulation on variational principles. This opinion was also shared by Hellinger, see [10–12] who, in his masterpiece “Fundamentals of the mechanics of continua”, showed, already with the knowledge available in 1913, that the unifying vision given by variational principles could allow for an effective presentation of all field theories.

The strain-gradient elasticity found recently various applications to the modelling of behavior of various materials with complex inner microstructure. A particular class of such microstructured materials are metamaterials, see, e.g., [13, 14]. Among them let us mention fiber-reinforced composites and woven fabrics [15–20], fiber-reinforced composites with debonded fibers [21] and beam lattices [22–25], see also the report of two recent conferences [26,27]. Example of beam lattice is given in Fig. 1, whereas a typical woven fabric is shown in Fig. 2. Unlike to the general framework of the strain-gradient elasticity given by Toupin–Mindlin, the model of pantographic beam lattices relates to a strain energy density which depends on functions having different differential properties in different spatial directions, see [22–24,28]. We say that such a model belongs to reduced strain-gradient elasticity [29] and we claim that the theory of anisotropic Sobolev spaces [30] finds an interesting application in the study of pantographic structures.

These structures have been introduced in order to give an example of metamaterial which can undergo very large deformations still remaining in an elastic regime. First preliminary experimental and theoretical results are presented in [16,22,33–35]. These papers show that the concept underlying the design of pantographic metamaterials deserved to be developed and therefore its mathematical modelling is needed, for getting detailed predictions via suitably developed numerical codes. Being said metamaterials constituted by lattices of beams, their numerical analysis may be inspired by discrete or semi-discrete models, see [36,37],

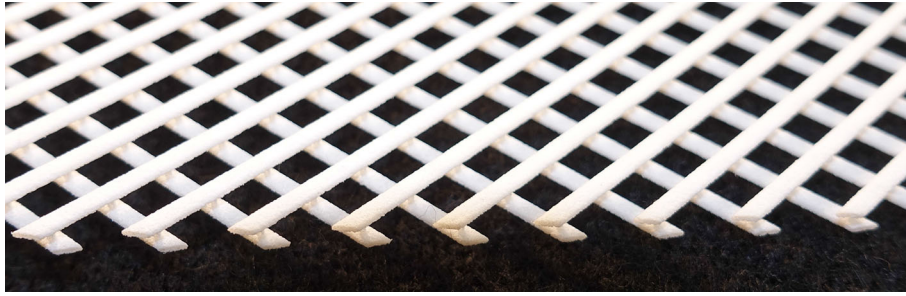
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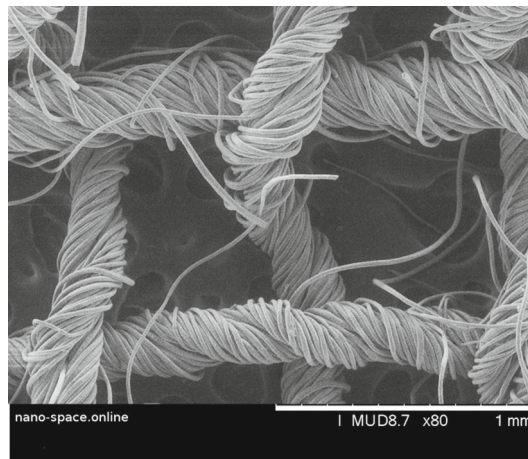
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**Fig. 1** 3D printed specimen of a pantographic sheet



**Fig. 2** Dress and suiting fabric, 100% wool. The image was provided by nano-space.online. Morphology of the samples was studied in native conditions in order to exclude surface effects from conductive film coating, see [31,32]

or by continuum models see, e.g., [38,39]. Also metamaterials with granular structures can be developed by considering heuristically homogenized continuum models as presented in [40–43]. In order to perform effectively the numerical analysis of complex beam lattices, the most efficient methods are desired, (as those presented, e.g., in [44–50] and the references therein). Let us note that for polymer and metal materials when certain level of deformations is reached it could be important to take into account also inelastic phenomena [51–55]. Another source of inelastic behavior is the contact of the beams in the lattice and the related adhesion interactions [56,57]. For current state of the pantographic metamaterials, we refer to [58–63]. Beam lattices can be used as “meso-models” of cellular solids or regular open-cell foams which are widely used in engineering and tissue engineering: we therefore believe that the homogenized models introduced in the present paper can have a wider range of application. Remark also that (see e.g., [63]) in pantographic metamaterials some “phase segregations” or “phase transitions” have been observed: therefore it seems natural to assume that the mathematical techniques used in [64–66] are applicable also in the present context. A generalized form of Pott model has been used to simulate static and kinetic phenomena in foams and the biological morphogenesis [67,68]. Pantographic sheets modelling is closely related to mechanics of networks, see [69–71] and the reference therein.

It has also to be investigated the whole damage mechanisms occurring in them, with methods which may be inspired to peridynamics, see, e.g., [72–76]. The relationship between peridynamics and higher gradient continuum theories, on the other hand, was already known to Piola himself [77,78], see also [79], and we believe that it deserves to be fully investigated.

The main object of this paper is to prove a result of well-posedness of the deformation problem of linear elastic pantographic sheets deforming in space: i.e., bidimensional continua generalizing standard plate models, as their deformation energy not only depend on the second gradient of out-of-plane displacement but also on second gradients of in-plane displacements. We believe that this is an essential intermediate step in the study of large deformation of pantographic metamaterials or of composite reinforcements, in particular when

wrinkling occurs. The experimental evidence about these phenomena, occurring in the forming of composite reinforcements, is discussed in [80–83].

We believe that the insight gained by the results which we present here can supply a useful tool for developing the analysis of the more general nonlinear case. To this end, one may need more advanced techniques as used in the case of nonlinear theories of plates and shallow shells, see, e.g., [84–87].

## 2 Derivation of a continuum model

Let us consider a rectangular pantographic beam lattice, which consists of  $m$  vertical fibers of length  $l_2$  and  $n$  horizontal fibers of length  $l_1$  connected by short pivots of length  $h$ , see Fig. 3. In what follows, we describe infinitesimal deformations of the lattice using the Euler–Bernoulli beam model. First, we introduce two kinematically independent fields of translations and rotations and then using the standard kinematic Euler–Bernoulli constraints we will replace rotations by derivatives of translations.

Assuming a hyperelastic behavior for the fibers' and pivots' material, we get that the stored energy functional has the form

$$\mathcal{E} = \sum_{j=1}^n \int_0^{l_1} \mathcal{U}_1(x_1) dx_1 + \sum_{i=1}^m \int_0^{l_2} \mathcal{U}_2(x_2) dx_2 + \sum_{i=1}^m \sum_{j=1}^n \int_0^h \mathcal{U}_3(x_3) dx_3. \quad (1)$$

Here,  $\mathcal{U}_1$  and  $\mathcal{U}_2$  are the strain energy densities of fibers, whereas  $\mathcal{U}_3$  is the strain energy of the pivots. With  $x_1$ ,  $x_2$ ,  $x_3$ , and  $\mathbf{i}_1$ ,  $\mathbf{i}_2$ ,  $\mathbf{i}_3$ , we denote the Cartesian coordinates and the corresponding base vectors, see Fig. 3.

Introducing the translations  $\mathbf{u}^{(\alpha)} = u_1^{(\alpha)} \mathbf{i}_1 + u_2^{(\alpha)} \mathbf{i}_2 + u_3^{(\alpha)} \mathbf{i}_3$  and rotations  $\boldsymbol{\phi}^{(\alpha)} = \phi_1^{(\alpha)} \mathbf{i}_1 + \phi_2^{(\alpha)} \mathbf{i}_2 + \phi_3^{(\alpha)} \mathbf{i}_3$  vectors for the  $\alpha$ th family of fibers,  $\alpha = 1, 2$ , we assume the general form of  $\mathcal{U}_\alpha$

$$\mathcal{U}_1 = \mathcal{U}_1(\mathbf{u}_{,1}^{(1)}, \boldsymbol{\phi}_{,1}^{(1)}), \quad \mathcal{U}_2 = \mathcal{U}_2(\mathbf{u}_{,2}^{(2)}, \boldsymbol{\phi}_{,2}^{(2)}). \quad (2)$$

Hereinafter, for brevity, we use the following notations for partial derivatives:  $(\dots)_{,i} = \frac{\partial}{\partial x_i}(\dots)$ , while  $\phi_1^{(1)}$  and  $\phi_2^{(2)}$  are the angles of torsion of first and second family of fibers, respectively, whereas other angles relate to the bending.

For simplicity, we consider a symmetric cross section of fibers (such as circular or square ones), so that the bending stiffness in the principal directions takes the same values. As a result, we get the following expressions for  $\mathcal{U}_1$  and  $\mathcal{U}_2$

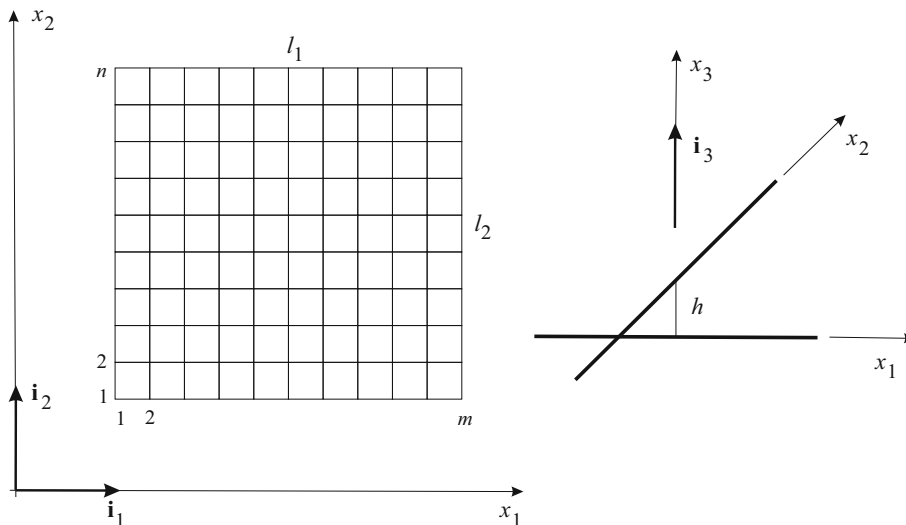


Fig. 3 Scheme of a pantographic lattice

$$\mathcal{U}_1 = \frac{1}{2}\mathbb{K}_e^{(1)} \left(u_{1,1}^{(1)}\right)^2 + \frac{1}{2}\mathbb{K}_b^{(1)} \left[\left(\phi_{2,1}^{(1)}\right)^2 + \left(\phi_{3,1}^{(1)}\right)^2\right] + \frac{1}{2}\mathbb{K}_t^{(1)} \left(\phi_{1,1}^{(1)}\right)^2, \quad (3)$$

$$\mathcal{U}_2 = \frac{1}{2}\mathbb{K}_e^{(2)} \left(u_{2,2}^{(2)}\right)^2 + \frac{1}{2}\mathbb{K}_b^{(2)} \left[\left(\phi_{1,2}^{(2)}\right)^2 + \left(\phi_{3,2}^{(2)}\right)^2\right] + \frac{1}{2}\mathbb{K}_t^{(2)} \left(\phi_{2,2}^{(2)}\right)^2. \quad (4)$$

Here,  $\mathbb{K}_e^{(\alpha)}$ ,  $\mathbb{K}_b^{(\alpha)}$ , and  $\mathbb{K}_t^{(\alpha)}$  are the tangent, bending, and torsional stiffness parameters of the  $\alpha$ th family of beams, respectively. Formulae (3) and (4) present a simple form of energy of a Timoshenko-type shear-deformable beam taking into account its bending, stretching, and torsion. Remark that more general form of the strain energy for a beam under infinitesimal deformations is given in the literature, see, e.g., [88].

In what follows, we utilize the Bernoulli hypothesis. This leads to the standard relations between  $\mathbf{u}^{(\alpha)}$  and  $\boldsymbol{\phi}^{(\alpha)}$ , see, e.g., [88, 89],

$$\begin{aligned} \boldsymbol{\phi}^{(\alpha)} &= \phi_1^{(\alpha)} \mathbf{i}_1 + \phi_2^{(\alpha)} \mathbf{i}_2 + \phi_3^{(\alpha)} \mathbf{i}_3, \\ \phi_2^{(1)} &= -u_{3,1}^{(1)}, \quad \phi_3^{(1)} = u_{2,1}^{(1)}, \quad \phi_1^{(2)} = u_{3,2}^{(2)}, \quad \phi_3^{(2)} = -u_{1,2}^{(2)}. \end{aligned}$$

As a result,  $\mathcal{U}_1$  and  $\mathcal{U}_2$  take the form

$$\mathcal{U}_1 = \frac{1}{2}\mathbb{K}_e^{(1)} \left(u_{1,1}^{(1)}\right)^2 + \frac{1}{2}\mathbb{K}_b^{(1)} \left[\left(u_{2,11}^{(1)}\right)^2 + \left(u_{3,11}^{(1)}\right)^2\right] + \frac{1}{2}\mathbb{K}_t^{(1)} \left(\phi_{1,1}^{(1)}\right)^2, \quad (5)$$

$$\mathcal{U}_2 = \frac{1}{2}\mathbb{K}_e^{(2)} \left(u_{2,2}^{(2)}\right)^2 + \frac{1}{2}\mathbb{K}_b^{(2)} \left[\left(u_{1,22}^{(2)}\right)^2 + \left(u_{3,22}^{(2)}\right)^2\right] + \frac{1}{2}\mathbb{K}_t^{(2)} \left(\phi_{2,2}^{(2)}\right)^2. \quad (6)$$

Note that here we have displacements  $u_k^{(\alpha)}$  and angles of torsion  $\phi_2^{(2)}$  and  $\phi_1^{(1)}$  as kinematical descriptors of the model.

Referring then to the deformation of the pivots interconnecting the fibers, or referring to the fiber interaction in the woven fabric, we must now describe the corresponding deformation energy. Using theoretical, numerical, and experimental results, it has been concluded that deformations of pantographic lattices cannot be described without considering pivots deformations [22, 90, 91]. It has to be remarked here that the deformation at microlevel of pivots induces a deformation of the pantographic sheet at macrolevel. Indeed, there are at least two length scales for considering pantographic sheets: a microlevel where each fiber and each pivot can be modelled as beams and a macrolevel where a homogenized continuum model is introduced generalizing Mindlin plate theory. The generalization consists in the introduction in the deformation strain energy of the second gradients of in-plane displacement components, see [18, 92]. Therefore, a correspondence relationship among microdeformations and macrodeformations needs to be specified: (i) micro-twist of a pivot results into an apparent macro-shear, (ii) micro-bending of a pivot results into relative twist of fibers and (iii) micro-elongation of a pivot results into relative displacement of fibers, which can be further distinguished into relative fibers slip and relative detachment of fibers. Thus, considering stretching, bending, and torsion of the pivots, we get the following efficient model by introducing the following strain energy density

$$\mathcal{U}_3 = \frac{1}{2}\mathbb{K}_e^{(3)} \left(u_{3,3}^{(3)}\right)^2 + \frac{1}{2}\mathbb{K}_b^{(3)} \left[\left(\phi_{1,3}^{(3)}\right)^2 + \left(\phi_{2,3}^{(3)}\right)^2\right] + \frac{1}{2}\mathbb{K}_t^{(3)} \left(\phi_{3,3}^{(3)}\right)^2, \quad (7)$$

where  $\mathbb{K}_e^{(3)}$ ,  $\mathbb{K}_b^{(3)}$ , and  $\mathbb{K}_t^{(3)}$  are the stiffness moduli of the pivots.

In order to present 2D model of the pantographic lattice deformations, let us replace the third term in Eq. (1) by an approximate value which depends only on the integrand values at  $x_3 = 0, h$ . This approximation can be applied only in the case of relatively short pivots, and its applicability limits are investigated in [91]. First, we replace  $u_{3,3}^{(3)}$  by the finite difference

$$u_{3,3}^{(3)} = \frac{u_3^{(3)}|_{x_3=h} - u_3^{(3)}|_{x_3=0}}{h} = \frac{u_3^{(2)} - u_3^{(1)}}{h}.$$



Hereinafter, we use the assumption on the continuity of displacements and rotations at the pivot fiber interface. In a similar way, we treat the derivatives of angles

$$\begin{aligned}\phi_{1,3}^{(3)} &= \frac{\phi_1^{(3)}|_{x_3=h} - \phi_1^{(3)}|_{x_3=0}}{h} = \frac{\phi_1^{(2)} - \phi_1^{(1)}}{h} = \frac{u_{3,2}^{(2)} - \phi_1^{(1)}}{h}, \\ \phi_{2,3}^{(3)} &= \frac{\phi_2^{(3)}|_{x_3=h} - \phi_2^{(3)}|_{x_3=0}}{h} = \frac{\phi_2^{(2)} - \phi_2^{(1)}}{h} = \frac{\phi_2^{(2)} + u_{3,1}^{(1)}}{h}, \\ \phi_{3,3}^{(3)} &= \frac{\phi_3^{(3)}|_{x_3=h} - \phi_3^{(3)}|_{x_3=0}}{h} = \frac{\phi_3^{(2)} - \phi_3^{(1)}}{h} = -\frac{u_{1,2}^{(2)} + u_{2,1}^{(1)}}{h},\end{aligned}$$

where we replace the angles by the correspondent derivatives of displacements.

As a result, we get the approximated expression

$$\begin{aligned}\int_0^h \mathcal{U}_3(x_3) dx_3 &= \frac{1}{2h} \mathbb{K}_e^{(3)} \left( u_3^{(2)} - u_3^{(1)} \right)^2 \\ &+ \frac{1}{2h} \mathbb{K}_b^{(3)} \left[ \left( u_{3,2}^{(2)} - \phi_1^{(1)} \right)^2 + \left( \phi_2^{(2)} + u_{3,1}^{(1)} \right)^2 \right] \\ &+ \frac{1}{2h} \mathbb{K}_t^{(3)} \left( u_{1,2}^{(2)} + u_{2,1}^{(1)} \right)^2.\end{aligned}\quad (8)$$

The first term in (8) corresponds to a spring model. Indeed, it describes the energy of a vertical spring connecting beams. Other terms in (8) can be introduced considering rotational springs like in [22] describing pivot bending and twisting.

Being motivated by Fig. 2, we extend the model introducing horizontal springs of stiffness  $\mathbb{K}_s^{(3)}/h$  with the energy

$$\frac{1}{2h} \mathbb{K}_s^{(3)} \left[ \left( u_{p1}^{(2)} - u_{p1}^{(1)} \right)^2 + \left( u_{p2}^{(2)} - u_{p2}^{(1)} \right)^2 \right],$$

where  $u_{p\beta}^{(2)}$  and  $u_{p\beta}^{(1)}$  are the in-plane displacements of the upper and lower ends of the pivot. For the latter, there are formulae

$$u_{p1}^{(2)} = u_1^{(2)} - r_2 \phi_2^{(2)}, \quad u_{p2}^{(2)} = u_2^{(2)}, \quad u_{p1}^{(1)} = u_1^{(1)}, \quad u_{p2}^{(1)} = u_2^{(1)} - r_1 \phi_1^{(1)},$$

where  $r_1$  and  $r_2$  are distances between the fiber cross section centers and ends of a pivot.

Thus, we can replace (1) with a semi-discrete energy given by the functional

$$\begin{aligned}\mathcal{E}[\mathbf{u}^{(1)}(x_1), \mathbf{u}^{(2)}(x_2), \phi_1^{(1)}(x_1), \phi_2^{(2)}(x_2)] &= \frac{1}{2} \sum_{j=1}^n \int_0^{l_1} \left\{ \mathbb{K}_e^{(1)} \left( u_{1,1}^{(1)} \right)^2 + \mathbb{K}_b^{(1)} \left[ \left( u_{2,11}^{(1)} \right)^2 + \left( u_{3,11}^{(1)} \right)^2 \right] + \frac{1}{2} \mathbb{K}_t^{(1)} \left( \phi_{1,1}^{(1)} \right)^2 \right\} dx_1 \\ &+ \frac{1}{2} \sum_{i=1}^m \int_0^{l_2} \left\{ \mathbb{K}_e^{(2)} \left( u_{2,2}^{(2)} \right)^2 + \mathbb{K}_b^{(2)} \left[ \left( u_{1,22}^{(2)} \right)^2 + \left( u_{3,22}^{(2)} \right)^2 \right] + \frac{1}{2} \mathbb{K}_t^{(2)} \left( \phi_{2,2}^{(2)} \right)^2 \right\} dx_2 \\ &+ \frac{1}{2h} \sum_{i=1}^m \sum_{j=1}^n \left\{ \mathbb{K}_e^{(3)} \left( u_3^{(2)} - u_3^{(1)} \right)^2 + \mathbb{K}_b^{(3)} \left[ \left( u_{3,2}^{(2)} - \phi_1^{(1)} \right)^2 + \left( \phi_2^{(2)} + u_{3,1}^{(1)} \right)^2 \right] \right. \\ &+ \mathbb{K}_s^{(3)} \left[ \left( u_1^{(2)} - u_1^{(1)} - r_2 \phi_2^{(2)} \right)^2 + \left( u_2^{(2)} - u_2^{(1)} - r_1 \phi_1^{(1)} \right)^2 \right] \\ &\left. + \mathbb{K}_t^{(3)} \left( u_{1,2}^{(2)} + u_{2,1}^{(1)} \right)^2 \right\} \Big|_{x_1=x_{1i}, x_2=x_{2j}},\end{aligned}\quad (9)$$

where  $(x_{1i}, x_{2j})$  are the coordinates of nodes, each one corresponding to a pivot. From the variational equation  $\delta\mathcal{E} = 0$ , it follows a system of linear ordinary differential equations with respect to  $\mathbf{u}^{(1)}(x_1)$  and  $\mathbf{u}^{(2)}(x_2)$ ,  $\phi_1^{(1)}(x_1)$ , and  $\phi_2^{(2)}(x_2)$ .

Instead of the discrete model with the energy functional (9), we introduce equivalent continuum model assuming that  $\mathbf{u}^{(1)}$  and  $\mathbf{u}^{(2)}$ ,  $\phi_1^{(1)}$ , and  $\phi_2^{(2)}$  are functions which depend on  $x_1$  and  $x_2$ :

$$\begin{aligned}\mathbf{u}^{(1)} &= \mathbf{u}^{(1)}(x_1, x_2), & \mathbf{u}^{(2)} &= \mathbf{u}^{(2)}(x_1, x_2), \\ \boldsymbol{\phi} &= \phi_1^{(1)} \mathbf{i}_1 + \phi_2^{(2)} \mathbf{i}_2, & \phi_1^{(1)} &= \phi_1(x_1, x_2), & \phi_2^{(2)} &= \phi_2(x_1, x_2).\end{aligned}$$

This assumption for averaging a la Piola was used in [22,23,77,78]. The continuous counterpart of (9) takes the form

$$\begin{aligned}\mathcal{E}[\mathbf{u}^{(1)}(x_1, x_2), \mathbf{u}^{(2)}(x_1, x_2), \boldsymbol{\phi}(x_1, x_2)] &= \frac{1}{2} \int_0^{l_1} \int_0^{l_2} \left\{ \mathbb{K}_e^{(1)} (u_{1,1}^{(1)})^2 + \mathbb{K}_b^{(1)} \left[ (u_{2,11}^{(1)})^2 + (u_{3,11}^{(1)})^2 \right] + \frac{1}{2} \mathbb{K}_t^{(1)} \phi_{1,1}^2 \right. \\ &+ \mathbb{K}_e^{(2)} (u_{2,2}^{(2)})^2 + \mathbb{K}_b^{(2)} \left[ (u_{1,22}^{(2)})^2 + (u_{3,22}^{(2)})^2 \right] + \frac{1}{2} \mathbb{K}_t^{(2)} \phi_{2,2}^2 \\ &+ \mathbb{K}_e^{(3)} (u_3^{(2)} - u_3^{(1)})^2 + \mathbb{K}_b^{(3)} \left[ (u_{3,2}^{(2)} - \phi_1)^2 + (\phi_2 + u_{3,1}^{(1)})^2 \right] \\ &+ \mathbb{K}_s^{(3)} \left[ (u_1^{(2)} - u_1^{(1)} - r_2 \phi_2)^2 + (u_2^{(2)} - u_2^{(1)} - r_1 \phi_1)^2 \right] \\ &\left. + \mathbb{K}_t^{(3)} (u_{1,2}^{(2)} + u_{2,1}^{(1)})^2 \right\} dx_1 dx_2, \tag{10}\end{aligned}$$

where with an abuse of notation we keep the same notations for the semi-discrete and continuum stiffness parameters. Remark that continuum stiffness parameters are related to continuum ones by using the ratios  $m/l_1$  and  $n/l_2$ .

In order to describe the behavior of the lattice as a whole and the detachment of fibers, we introduce mean and relative displacements as follows

$$\mathbf{u} = \frac{1}{2}(\mathbf{u}^{(1)} + \mathbf{u}^{(2)}), \quad \mathbf{v} = \frac{1}{2}(\mathbf{u}^{(2)} - \mathbf{u}^{(1)}).$$

Here,  $\mathbf{u}$  can be interpreted as the displacements of the midsurface of the lattice, whereas  $\mathbf{v}$  describes the relative deformations that is the difference between displacements of the fibers. As a result, we have  $\mathbf{u}^{(1)} = \mathbf{u} - \mathbf{v}$ ,  $\mathbf{u}^{(2)} = \mathbf{u} + \mathbf{v}$ ,  $\boldsymbol{\phi} = \phi_1 \mathbf{i}_1 + \phi_2 \mathbf{i}_2$ , and  $\mathcal{E}$  takes the form

$$\begin{aligned}\mathcal{E}[\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}] &= \int_0^{l_1} \int_0^{l_2} \mathcal{W} dx_1 dx_2, \tag{11} \\ \mathcal{W} &= \frac{1}{2} \left\{ \mathbb{K}_e^{(1)} (u_{1,1} - v_{1,1})^2 + \mathbb{K}_b^{(1)} \left[ (u_{2,11} - v_{2,11})^2 + (u_{3,11} - v_{3,11})^2 \right] \right. \\ &+ \mathbb{K}_t^{(1)} \phi_{1,1}^2 + \mathbb{K}_t^{(2)} \phi_{2,2}^2 \\ &+ \mathbb{K}_e^{(2)} (u_{2,2} + v_{2,2})^2 + \mathbb{K}_b^{(2)} \left[ (u_{1,22} + v_{1,22})^2 + (u_{3,22} + v_{3,22})^2 \right] \\ &+ \mathbb{K}_e^{(3)} v_3^2 \\ &+ \mathbb{K}_b^{(3)} \left[ (u_{3,2} + v_{3,2} - \phi_1)^2 + (\phi_2 + u_{3,1} - v_{3,1})^2 \right] \\ &+ \mathbb{K}_s^{(3)} \left[ (v_1 - r_2 \phi_2)^2 + (v_2 - r_1 \phi_1)^2 \right] \\ &\left. + \mathbb{K}_t^{(3)} (u_{1,2} + u_{2,1} + v_{1,2} - v_{2,1})^2 \right\}. \tag{12}\end{aligned}$$

In what follows, we analyze the properties of the differential equations which are the Euler–Lagrange stationarity conditions for given energy and their weak solutions.

### 3 Strain energy density and equilibrium conditions

An important element of the analysis of existence and uniqueness of weak solutions is the study of the energy null-space, that is, set of admissible functions for which the given strain energy density is vanishing. For standard linear elasticity, such null-space reduces to the so-called infinitesimal rigid body motions, see [93–95]. For linear pantographic sheets considered within reduced strain-gradient elasticity, there are other possible energy-free solutions [28].

#### 3.1 Energy-free deformations and rigid body motions: full model

Let us find all smooth solutions of  $\mathcal{W} = 0$ , that is, the set of energy-free deformations. As  $\mathbb{K}_t^{(1)}$  and  $\mathbb{K}_t^{(2)}$  are positive, we get that  $\phi_{1,1} = \phi_{2,2} = 0$  which result in

$$\phi_1 = \phi_1(x_2), \quad \phi_2 = \phi_2(x_1). \quad (13)$$

From  $\mathbb{K}_e^{(3)} > 0$ , we have  $v_3 = 0$ . From  $\mathbb{K}_s^{(3)} > 0$ , it follows that

$$v_1 = r_2 \phi_2(x_1), \quad v_2 = r_1 \phi_1(x_2), \quad (14)$$

and  $\mathcal{W}$  takes the form

$$\begin{aligned} \mathcal{W} = & \frac{1}{2} \left[ \mathbb{K}_e^{(1)} (u_{1,1} - v_{1,1})^2 + \mathbb{K}_b^{(1)} (u_{2,11}^2 + u_{3,11}^2) \right. \\ & + \mathbb{K}_e^{(2)} (u_{2,2} + v_{2,2})^2 + \mathbb{K}_b^{(2)} (u_{1,22}^2 + u_{3,22}^2) \\ & \left. + \mathbb{K}_b^{(3)} \left[ (u_{3,2} - \phi_1)^2 + (\phi_2 + u_{3,1})^2 \right] + \mathbb{K}_t^{(3)} (u_{1,2} + u_{2,1})^2 \right]. \end{aligned} \quad (15)$$

From  $\mathcal{W} = 0$ , we get the following system of differential equations

$$u_{3,11} = u_{3,22} = 0, \quad (16)$$

$$u_{3,2} = \phi_1, \quad u_{3,1} = -\phi_2, \quad (17)$$

$$u_{1,1} = v_{1,1}, \quad u_{2,2} = -v_{2,2}, \quad u_{2,11} = u_{1,22} = 0, \quad (18)$$

$$u_{1,2} + u_{2,1} = 0. \quad (19)$$

From (16), we get that  $u_3 = a_{11}x_1x_2 + a_{10}x_1 + a_{01}x_2 + u_3^{(0)}$ . This function corresponds to a hyperbolic paraboloid which is an example of doubly ruled surface. From (13) and (17), we have that  $a_{11} = 0$ ,  $\phi_1 = a_{01}$ , and  $\phi_2 = -a_{10}$ . From (14), it follows that  $v_1 = -r_2a_{10}$  and  $v_2 = r_1a_{01}$ . So, in the null-space displacements for deformation energy, the relative in-plane displacements are constants.

From (18), we have  $u_1 = f_1x_2 + u_1^{(0)}$ ,  $u_2 = f_2x_1 + u_2^{(0)}$ . Hereinafter,  $a_{11}$ ,  $a_{10}$ ,  $a_{01}$ ,  $f_1$ ,  $f_2$ , and  $u_i^{(0)}$  are constants. In addition, from (19) it follows that  $f_2 = -f_1$  and  $u_2 = -f_1x_1 + u_2^{(0)}$ . As a result, we get six linearly independent energy-free deformations given by

$$u_1^{(f)} = f_1x_2 + u_1^{(0)}, \quad (20)$$

$$u_2^{(f)} = -f_1x_1 + u_2^{(0)}, \quad (21)$$

$$u_3^{(f)} = a_{10}x_1 + a_{01}x_2 + u_3^{(0)}, \quad (22)$$

$$\phi_1^{(f)} = a_{01}, \quad \phi_2^{(f)} = -a_{10}, \quad (23)$$

$$v_1^{(f)} = -r_2a_{10}, \quad v_2^{(f)} = r_1a_{01}. \quad (24)$$

In standard linear elasticity, energy-free deformations coincide with the infinitesimal rigid body motions. In all 2D generalized plate considered here models the infinitesimal rigid motions must be included in the energy null-space even if, in general, they do not coincide with it. So in order to recognize this property starting from (20)–(24), let us recall that the infinitesimal deformations related to the rigid body motions take the form

$$\mathbf{u}^{(r)} = \mathbf{u}^{(0)} + \boldsymbol{\omega} \times \mathbf{x},$$



where  $\mathbf{u}^{(0)} = u_k^{(0)} \mathbf{i}_k$  and  $\boldsymbol{\omega} = \omega_k \mathbf{i}_k$  are constant vectors, and  $\mathbf{x} = x_k \mathbf{i}_k$ . In scalar form, these deformations can be written as follows

$$u_1^{(r)} = \omega_2 x_3 - \omega_3 x_2 + u_1^{(0)}, \quad (25)$$

$$u_2^{(r)} = -\omega_1 x_3 + \omega_3 x_1 + u_2^{(0)}, \quad (26)$$

$$u_3^{(r)} = \omega_1 x_2 - \omega_2 x_1 + u_3^{(0)}. \quad (27)$$

Obviously,  $\mathcal{W}$  is invariant under transformations  $\mathbf{u} \rightarrow \mathbf{u} + \mathbf{u}^{(r)}$ . The analysis of energy-free solutions within the linear Kirchhoff theory of plates [96, p. 271] brings the following solution

$$u_1^{(r)} = -\omega_3 x_2 + u_1^{(0)}, \quad u_2^{(r)} = \omega_3 x_1 + u_2^{(0)}, \quad u_3^{(r)} = a_{10} x_1 + a_{01} x_2 + u_3^{(0)}, \quad (28)$$

which coincides with (20)–(22). In other words, the displacement part of the elements of the null-space for strain energy of pantographic sheets is a finite dimensional vector space and it is the same as in the standard theory of plates.

Analyzing in-plane deformations of linear pantographic sheets, we demonstrated [28] the importance of shear energy of pivots to avoid additional energy-free shear deformations. To consider both in- and out-of-plane deformations, let us also underline the importance of the bending energies of pivots.

### 3.2 Energy-free deformations: pivot spring model

Let us remark that, when assuming

$$\mathbb{K}_b^{(3)} = 0, \quad \mathbb{K}_t^{(3)} = 0, \quad (29)$$

the elements of energy null-space have a non-vanishing  $u_3$  including the addend  $a_{11} x_1 x_2$  and rotations having the form (13). Indeed, if we consider a spring model for pivots which is considering only stretching, the corresponding strain energy takes the form

$$\begin{aligned} \mathcal{W}_0 = \frac{1}{2} \left\{ \mathbb{K}_e^{(1)} (u_{1,1} - v_{1,1})^2 + \mathbb{K}_b^{(1)} \left[ (u_{2,11} - v_{2,11})^2 + (u_{3,11} - v_{3,11})^2 \right] \right. \\ + \mathbb{K}_t^{(1)} \phi_{1,1}^2 + \mathbb{K}_t^{(2)} \phi_{2,2}^2 \\ + \mathbb{K}_e^{(2)} (u_{2,2} + v_{2,2})^2 + \mathbb{K}_b^{(2)} \left[ (u_{1,22} + v_{1,22})^2 + (u_{3,22} + v_{3,22})^2 \right] \\ \left. + \mathbb{K}_e^{(3)} v_3^2 + \mathbb{K}_s^{(3)} \left[ (v_1 - r_2 \phi_2)^2 + (v_2 - r_1 \phi_1)^2 \right] \right\}. \quad (30) \end{aligned}$$

Considering equation  $\mathcal{W}_0 = 0$ , we obtain the following energy-free solution

$$\begin{aligned} \phi_1 &= \phi_1(x_2), \quad \phi_2 = \phi_2(x_1), \\ v_1 &= r_2 \phi_2(x_1), \quad v_2 = r_1 \phi_1(x_2), \quad v_3 = 0, \\ u_1 &= f_1 x_2 + r_2 \phi_2(x_1) + u_1^{(0)}, \quad u_2 = f_2 x_1 - r_1 \phi_1(x_2) + u_2^{(0)}, \\ u_3 &= a_{11} x_1 x_2 + a_{10} x_1 + a_{01} x_2 + u_3^{(0)}. \end{aligned}$$

Thus, the null-space for the energy is even not finite dimensional and has the previously stated structure.

### 3.3 Energy-free deformations: fibers without torsional energy and pivots without bending energy

Assuming that torsional deformations of fibers and bending energy of pivots are vanishing, that is when

$$\mathbb{K}_t^{(1,2)} = \infty, \quad \mathbb{K}_b^{(3)} = 0, \quad (31)$$

we derive the following strain energy

$$\begin{aligned} \mathcal{W}_{00} = & \frac{1}{2} \left\{ \mathbb{K}_e^{(1)} (u_{1,1} - v_{1,1})^2 + \mathbb{K}_b^{(1)} \left[ (u_{2,11} - v_{2,11})^2 + (u_{3,11} - v_{3,11})^2 \right] \right. \\ & + \mathbb{K}_e^{(2)} (u_{2,2} + v_{2,2})^2 + \mathbb{K}_b^{(2)} \left[ (u_{1,22} + v_{1,22})^2 + (u_{3,22} + v_{3,22})^2 \right] \\ & \left. + \mathbb{K}_e^{(3)} v_3^2 + \mathbb{K}_s^{(3)} [v_1^2 + v_2^2] + \mathbb{K}_t^{(3)} (u_{1,2} + u_{2,1} + v_{1,2} - v_{2,1})^2 \right\} \end{aligned} \quad (32)$$

whose finite dimensional null-space consists of the functions

$$v_1 = v_2 = v_3 = 0, \quad (33)$$

$$u_1 = f_1 x_2 + u_1^{(0)}, \quad u_2 = -f_1 x_1 + u_2^{(0)}, \quad u_3 = a_{11} x_1 x_2 + a_{10} x_1 + a_{01} x_2 + u_3^{(0)}. \quad (34)$$

### 3.4 Energy-free deformations: perfectly connected fibers without torsion energy and pivots without bending energy

Finally, if one in addition to (31) assumes that

$$\mathbb{K}_s^{(3)} = \mathbb{K}_e^{(3)} = \infty, \quad (35)$$

the relative deformations are vanishing. In this case, the strain energy function has the simplest form

$$\begin{aligned} \mathcal{W}_{000} = & \frac{1}{2} \left[ \mathbb{K}_e^{(1)} u_{1,1}^2 + \mathbb{K}_e^{(2)} u_{2,2}^2 + \mathbb{K}_b^{(1)} (u_{2,11}^2 + u_{3,11}^2) \right. \\ & \left. + \mathbb{K}_b^{(2)} (u_{1,22}^2 + u_{3,22}^2) + \mathbb{K}_t^{(3)} (u_{1,2} + u_{2,1})^2 \right]. \end{aligned} \quad (36)$$

The null-space for  $\mathcal{W}_{000}$  consists of functions given by (34).

### 3.5 Equilibrium equations

In order to derive the corresponding equilibrium equations and to establish possible applicable external actions, we calculate the first variation of the functional

$$\mathcal{E} = \iint_{\omega} \mathcal{W} \, dx_1 \, dx_2, \quad (37)$$

where  $\omega \subset \mathbb{R}^2$  is a bounded area with smooth enough boundary. The first variation of  $\mathcal{E}$  reads

$$\delta \mathcal{E}[\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}; \delta \mathbf{u}, \delta \mathbf{v}, \delta \boldsymbol{\phi}] = \iint_{\omega} \delta \mathcal{W} \, dx_1 \, dx_2, \quad (38)$$

where

$$\begin{aligned} \delta \mathcal{W} = & \mathbb{K}_e^{(1)} (u_{1,1} - v_{1,1}) (\delta u_{1,1} - \delta v_{1,1}) \\ & + \mathbb{K}_b^{(1)} \left[ (u_{2,11} - v_{2,11}) (\delta u_{2,11} - \delta v_{2,11}) + (u_{3,11} - v_{3,11}) (\delta u_{3,11} - \delta v_{3,11}) \right] \\ & + \mathbb{K}_e^{(2)} (u_{2,2} + v_{2,2}) (\delta u_{2,2} + \delta v_{2,2}) \\ & + \mathbb{K}_b^{(2)} \left[ (u_{1,22} + v_{1,22}) (\delta u_{1,22} + \delta v_{1,22}) + (u_{3,22} + v_{3,22}) (\delta u_{3,22} + \delta v_{3,22}) \right] \\ & + \mathbb{K}_t^{(1)} \phi_{1,1} \delta \phi_{1,1} + \mathbb{K}_t^{(2)} \phi_{2,2} \delta \phi_{2,2} \\ & + \mathbb{K}_e^{(3)} v_3 \delta v_3 + \mathbb{K}_s^{(3)} [(v_1 - r_2 \phi_2) (\delta v_1 - r_2 \delta \phi_2) + (v_2 - r_1 \phi_1) (\delta v_2 - r_1 \delta \phi_1)] \\ & + \mathbb{K}_t^{(3)} (u_{1,2} + u_{2,1} + v_{1,2} - v_{2,1}) (\delta u_{1,2} + \delta u_{2,1} + \delta v_{1,2} - \delta v_{2,1}). \end{aligned} \quad (39)$$

The consistent form of the work of external surface loads acting on the pantographic sheet is given by

$$\delta\mathcal{A} = \iint_{\omega} (b_1\delta u_1 + b_2\delta u_2 + b_3\delta u_3 + g_1\delta v_1 + g_2\delta v_2 + g_3\delta v_3 + m_1\delta\phi_1 + m_2\delta\phi_2) dx_1 dx_2,$$

where  $b_i, i = 1, 2, 3$ , are the surface external forces,  $g_i$  are external double forces, see [97], and  $m_\alpha$  are external torques, respectively.

From the variational equation,

$$\delta\mathcal{E} - \delta\mathcal{A} = 0 \quad (40)$$

we get the following equilibrium equations

$$- \mathbb{K}_t^{(1)} \phi_{1,11} - r_1 \mathbb{K}_s^{(3)} (v_2 - r_1 \phi_1) = m_1, \quad (41)$$

$$- \mathbb{K}_t^{(2)} \phi_{2,22} - r_2 \mathbb{K}_s^{(3)} (v_1 - r_2 \phi_2) = m_2, \quad (42)$$

$$- \mathbb{K}_e^{(1)} (u_{1,11} - v_{1,11}) - \mathbb{K}_t^{(3)} (u_{1,22} + u_{2,12} + v_{1,22} - v_{2,12}) + \mathbb{K}_b^{(2)} (u_{1,2222} + v_{1,2222}) = b_1, \quad (43)$$

$$\mathbb{K}_e^{(1)} (u_{1,11} - v_{1,11}) + \mathbb{K}_s^{(3)} (v_1 - r_2 \phi_2) - \mathbb{K}_t^{(3)} (u_{1,22} + u_{2,12} + v_{1,22} - v_{2,12}) + \mathbb{K}_b^{(2)} (u_{1,2222} + v_{1,2222}) = g_1, \quad (44)$$

$$- \mathbb{K}_e^{(2)} (u_{2,22} + v_{2,22}) - \mathbb{K}_t^{(3)} (u_{1,12} + u_{2,11} + v_{1,12} - v_{2,11}) + \mathbb{K}_b^{(1)} (u_{2,1111} - v_{2,1111}) = b_2, \quad (45)$$

$$- \mathbb{K}_e^{(2)} (u_{2,22} + v_{2,22}) + \mathbb{K}_s^{(3)} (v_2 - r_1 \phi_1) + \mathbb{K}_t^{(3)} (u_{1,12} + u_{2,11} + v_{1,12} - v_{2,11}) - \mathbb{K}_b^{(1)} (u_{2,1111} - v_{2,1111}) = g_2, \quad (46)$$

$$\mathbb{K}_b^{(1)} (u_{3,1111} - v_{3,1111}) + \mathbb{K}_b^{(2)} (u_{3,2222} + v_{3,2222}) = b_3, \quad (47)$$

$$- \mathbb{K}_b^{(1)} (u_{3,1111} - v_{3,1111}) + \mathbb{K}_b^{(2)} (u_{3,2222} + v_{3,2222}) + \mathbb{K}_e^{(3)} v_3 = g_3. \quad (48)$$

This coupled system of PDEs contains differential operators of different order, so the unknown functions have partial derivatives of different order depending on the direction. This means that the system is not elliptic, in general, see [98–100]. To underline the peculiarities of corresponding differential operators, let us consider the simplest case (36). Now, the corresponding equilibrium equations have the following decoupled form

$$- \mathbb{K}_e^{(1)} u_{1,11} + \mathbb{K}_b^{(2)} u_{1,2222} - \mathbb{K}_t^{(3)} (u_{1,22} + u_{2,12}) = b_1, \quad (49)$$

$$- \mathbb{K}_e^{(2)} u_{2,22} + \mathbb{K}_b^{(1)} u_{2,1111} - \mathbb{K}_t^{(3)} (u_{1,12} + u_{2,11}) = b_2, \quad (50)$$

$$\mathbb{K}_b^{(1)} u_{3,1111} + \mathbb{K}_b^{(2)} u_{3,2222} = b_3. \quad (51)$$

For example, here  $u_1(x_1, x_2)$  possesses second derivatives with respect to  $x_1$  and fourth derivatives with respect to  $x_2$ . Introducing differential operators  $P_{ij}$  as follows

$$P_{11} = -\mathbb{K}_e^{(1)} \partial_1^2 - \mathbb{K}_t^{(3)} \partial_2^2 + \mathbb{K}_b^{(2)} \partial_2^4,$$

$$P_{12} = P_{21} = -\mathbb{K}_t^{(3)} \partial_1 \partial_2,$$

$$P_{22} = -\mathbb{K}_t^{(3)} \partial_1^2 - \mathbb{K}_e^{(2)} \partial_2^2 + \mathbb{K}_b^{(1)} \partial_1^4,$$

$$P_{33} = \mathbb{K}_b^{(1)} \partial_1^4 + \mathbb{K}_b^{(2)} \partial_2^4,$$

where  $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$ , we rewrite (49)–(51) in the symbolic form

$$P_{11}u_1 + P_{12}u_2 = b_1, \quad P_{21}u_1 + P_{22}u_2 = b_2, \quad P_{33}u_3 = b_3.$$

Operators  $P_{11}$  and  $P_{22}$  are neither elliptic nor strongly elliptic, but they belong to the class of hypoelliptic differential equations [101, 102]. The existence and uniqueness of weak solutions for (49)–(50) was analyzed in [28]. Operator  $P_{33}$  is strongly elliptic.

#### 4 Existence and uniqueness of weak solutions

For the proof of the existence and uniqueness of weak solutions, we use the same technique as in [28] which uses the anisotropic Sobolev's spaces as the corresponding energy space for considered functionals. These functional spaces were introduced in [30, 103–105], see also [106]. They are generalizations of the Sobolev's spaces [107, 108]. In what follows, we are restricted ourselves by the functionals which have a finite dimensional null-space and non-singular boundary conditions [28]. Our non-singular boundary conditions are nothing else as the Shapiro–Lopatinskii or complementary boundary conditions, see original works [109, 110] and [100, 108, 111, 112] for the mathematical definitions. For example, for strongly elliptic PDEs the Dirichlet- and von Neumann-type boundary conditions satisfy the Shapiro–Lopatinskii conditions.

Without loss of generality, we consider the problems in the dimensionless forms. We start from the simplest case.

##### 4.1 Weak solutions for $\mathcal{W}_{000}$

We introduce the following bilinear form

$$B_{000}(\mathbf{u}; \mathbf{w}) = \iint_{\omega} [u_{1,1}w_{1,1} + u_{2,2}w_{2,2} + u_{2,11}w_{2,11} + u_{3,11}w_{3,11} + u_{1,22}w_{1,22} + u_{3,22}w_{3,22} + (u_{1,2} + u_{2,1})(w_{1,2} + w_{2,1})] dx_1 dx_2, \quad (52)$$

and the linear functional

$$L_{000}\mathbf{w} = \iint_{\omega} (b_1w_1 + b_2w_2 + b_3w_3) dx_1 dx_2.$$

Obviously, if one uses the substitution  $\mathbf{w} = \delta\mathbf{u}$  the bilinear form  $B_{000}$  coincides with the first variation of  $\mathcal{E}$ , whereas  $L_{000}\mathbf{w}$  is the dimensionless work of external loads.

The quadratic functional  $B_{000}(\mathbf{u}; \mathbf{u})$  has all properties of a squared seminorm, but it is not a norm as the norm requirement

$$B_{000}(\mathbf{u}; \mathbf{u}) = 0 \quad \text{iff} \quad \mathbf{u} = \mathbf{0}$$

is violated. Indeed,  $B_{000}(\mathbf{u}; \mathbf{u}) = 0$  results in nontrivial solutions (34).

Let us consider the homogenous boundary conditions

$$\mathbf{u}|_{\partial\omega} = \mathbf{0}. \quad (53)$$

It is easy to show that in this case from  $B_{000}(\mathbf{u}; \mathbf{u}) = 0$  it follows that  $\mathbf{u} = \mathbf{0}$ . Thus,  $B_{000}(\mathbf{u}; \mathbf{u})$  becomes a norm on a set of functions satisfying (53). We assume the natural boundary conditions followed from (40) as other boundary conditions.

We introduce the energy space  $E_0^{000}$  as completion in the norm

$$\|\mathbf{u}\|_{E_0^{000}}^2 = B_{000}(\mathbf{u}; \mathbf{u})$$

of  $C_2(\omega)$  functions which verify (53). The energy space  $E_0^{000}$  can be characterized through the anisotropic Sobolev's spaces as follows

$$\mathbf{u} = (u_1, u_2, u_3) \in E_0^{000} \Leftrightarrow u_1 \in W_2^{(1,2)}(\omega), u_2 \in W_2^{(2,1)}(\omega), u_3 \in W_2^2(\omega).$$

Thus,  $E_0^{000} = \overset{\circ}{W}_2^{(1,2)}(\omega) \oplus \overset{\circ}{W}_2^{(2,1)}(\omega) \oplus \overset{\circ}{W}_2^2(\omega)$ . As  $B_{000}(\mathbf{u}; \mathbf{u})$  constitutes the squared norm in  $E_0^{000}$ , it is obvious that  $B_{000}$  is coercive.

Let us recall that the anisotropic Sobolev's spaces contain functions which have different differential properties. For example, for  $\omega \in \mathbb{R}^2$ , the norms in the spaces  $\overset{\circ}{W}_2^{(1,2)}(\omega)$ ,  $\overset{\circ}{W}_2^{(2,1)}(\omega)$ ,  $W_2^{(1,2)}(\omega)$ , and  $W_2^{(2,1)}(\omega)$  coincide each other and are given by the formulae

$$\|f\|_{W_2^{(1,2)}} = \|f\|_{L_2} + \|f_{,1}\|_{L_2} + \|f_{,22}\|_{L_2}, \quad (54)$$

$$\|f\|_{W_2^{(2,1)}} = \|f\|_{L_2} + \|f_{,11}\|_{L_2} + \|f_{,2}\|_{L_2}. \quad (55)$$



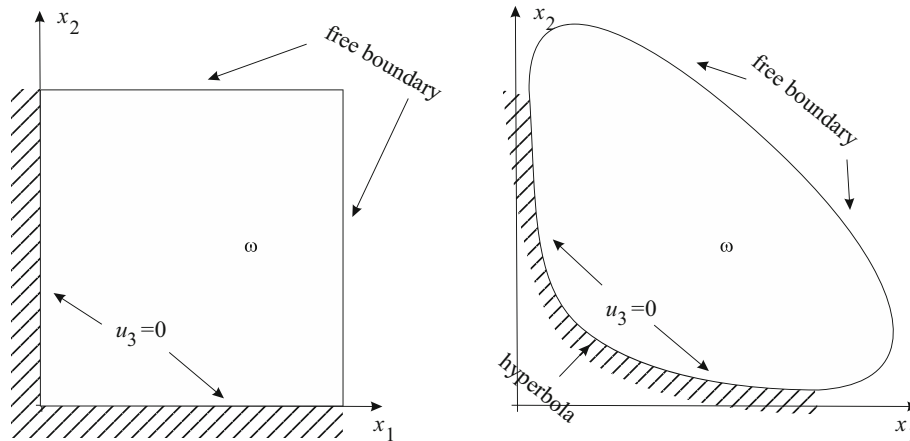


Fig. 4 Examples of singular boundary conditions for (51) with non-unique solutions

**Definition 1** We call  $\mathbf{u} \in E_0^{000}$  a weak solution of the equilibrium equations (49)–(51), if the equation

$$B_{000}(\mathbf{u}; \mathbf{w}) = L_{000}\mathbf{w} \quad (56)$$

is fulfilled for any test function  $\mathbf{w}$  from a dense set in  $E_0^{000}$ .

Using standard Riesz representation theorem and the Lax-Milgram theorem arguments [93–95, 113], we can prove the following

**Theorem 1** Let  $b_1, b_2,$  and  $b_3$  belong to the space  $L_2(\omega)$ . There exists a weak solution  $\mathbf{u}^* \in E_0^{000}$  to the corresponding equilibrium problem (49)–(51), which for any  $\mathbf{w} \in E_0^{000}$  satisfies (56)

Furthermore,  $\mathbf{u}^*$  is unique and it is a minimizer of the energy functional:

$$F(\mathbf{u}^*) = \inf_{\mathbf{u} \in E_0^{000}} F(\mathbf{u}), \quad F(\mathbf{u}) \equiv \mathcal{E}(\mathbf{u}) - L_{000}\mathbf{u}.$$

Let us note that since (49)–(51) are decoupled the problem can be splitted into two independent problems for (49), (50) and for (51). This gives the possibility to consider these problems independently, that is independently for  $u_1$  and  $u_2$  and for  $u_3$ . The boundary-value problems for (49), (50) are studied in [28] in details. Some solution of (51) is presented in [114, p. 1014]. Let us also note that for mixed boundary conditions we have non-unique solutions of (51). For example, let us consider a rectangle such shown in Fig. 4 (on the left) where a half of its boundary fixed, that is  $u_3 = 0$ , whereas other part is free. As the natural boundary conditions for  $\mathcal{W}_{000}$  includes second and third derivatives, it is clear that for  $b_3 = 0$  there are two solutions. They are

$$u_3 = 0, \quad \text{and} \quad u_3 = a_{11}x_1x_2$$

for any number  $a_{11}$ .

Another example with smooth boundary can be obtained as follows. As the equation

$$a_{11}x_1x_2 + a_{10}x_1 + a_{01}x_2 + u_3^{(0)} = 0$$

constitutes an equations of a hyperbola, let us consider an area which boundary includes a part of hyperbola, Fig. 4 (on the right). On this part, we again assume that  $u_3 = 0$  and other boundary conditions are natural ones. Here, we also have two solutions that are trivial one  $u_3 = 0$  and nonzero solution

$$u_3 = c \left( a_{11}x_1x_2 + a_{10}x_1 + a_{01}x_2 + u_3^{(0)} \right),$$

with any number  $c$ . This solution belongs to the energy null-space, see (34).

These examples show that the consideration of out-of-plane deformations may lead to non-unique solutions. Indeed, for in-plane deformations of pantographic sheets given in Fig. 4, the solution of (49) and (50) is unique [28], whereas for out-of-plane deformations we have many solutions.



#### 4.2 Weak solutions for $\mathcal{W}_{00}$

In a similar way, we introduce the following dimensionless bilinear form and the linear functional

$$\begin{aligned}
 B_{00}(\mathbf{u}, \mathbf{v}; \mathbf{w}, \mathbf{z}) = & \iint_{\omega} [(u_{1,1} - v_{1,1})(w_{1,1} - z_{1,1}) + (u_{2,11} - v_{2,11})(w_{2,11} - z_{2,11}) \\
 & + (u_{2,2} + v_{2,2})(w_{2,2} + z_{2,2}) + (u_{1,22} + v_{1,22})(w_{1,22} + z_{1,22}) \\
 & + (u_{3,11} - v_{3,11})(w_{3,11} - z_{3,11}) + (u_{3,22} + v_{3,22})(w_{3,22} + z_{3,22}) \\
 & + (u_{1,2} + u_{2,1} + v_{1,2} - v_{2,1})(w_{1,2} + w_{2,1} + z_{1,2} - z_{2,1}) \\
 & + v_3 z_3 + v_1 z_1 + v_2 z_2] dx_1 dx_2, \tag{57}
 \end{aligned}$$

$$L_{00}(\mathbf{w}, \mathbf{z}) = \iint_{\omega} (b_1 w_1 + b_2 w_2 + b_3 w_3 + g_1 z_1 + g_2 z_2 + g_3 z_3) dx_1 dx_2. \tag{58}$$

We again consider the clamped boundary that is with the following boundary conditions

$$\mathbf{u}|_{\partial\omega} = \mathbf{v}|_{\partial\omega} = \mathbf{0}. \tag{59}$$

For these boundary conditions,  $B_{00}(\mathbf{u}, \mathbf{v}; \mathbf{u}, \mathbf{v})$  is a squared norm in some energy space  $E_0^{00}$  which follows from the completion of continuously differentiable functions in this norm,

$$B_{00}(\mathbf{u}, \mathbf{v}; \mathbf{u}, \mathbf{v}) = \|\mathbf{u}, \mathbf{v}\|_{E_0^{00}}^2.$$

So,  $B_{00}$  becomes the coercive in  $E_0^{00}$  and similar theorems on the uniqueness and existence as above can be formulated.

#### 4.3 Remarks on other cases

Unlike two previous case, this technique cannot be applied straightforwardly to the pivot spring model introduced by (30). Indeed, for this model, the corresponding null-space is not finite dimensional. And vice versa, the full model with the strain energy density (12) can be analyzed similarly to these cases as its null-space is finite dimensional. So, energy-null solutions can be avoided choosing proper boundary conditions.

Let us also underline the crucial difference between the semi-discrete model (9) and its continual counterparts. The semi-discrete model corresponds to a system of linear ODEs. So its well-posedness is rather obvious. On the other hand, the continual ‘‘homogenized’’ models may lose this property. So, one should be aware of such situations.

### 5 Conclusions and future steps

In this paper, we formulate a linear elastic model for pantographic sheets which has the following features:

1. The placements of the two involved families of fibers are independent fields both defined in a bidimensional Lagrangian reference configuration and having images in the 3D Eulerian Euclidean affine space of positions;
2. Both families of fibers are modelled as beams which can store deformation energy due to elongation, bending and twisting;
3. The pivots between the fibers are assumed to be connected with two sections belonging to two different fibers (see Fig. 3), and these sections are assumed to behave as rigid bodies;
4. The strain energy of these pivots depends on the relative displacement and rotations of the aforementioned fibers' sections.

We are aware of the fact that the listed assumptions do limit the range of applicability of the proposed model. In particular, when the forming process of composite reinforcements has to be described the assumption of small displacements and deformations cannot be accepted. Also, the friction phenomena among fibers have to be included in the model. However, we consider the presented model as an improvement of those already considered in the literature, see [18,28,92] and the reference therein, as the process of pivot deformation or inter-fiber elastic interaction has not been taken suitably into account. Moreover, see [115], recently some attention has been paid to the vibration phenomena of pantographic specimens and the experimental evidence which has been obtained indicates that a linear elastic dynamical model can be very useful in applications. Therefore, the first development which we will address of the results presented in this paper will surely include the discussion of:

1. The most suitable inertial terms to be added in the Lagrangian for considered systems;
2. The properties of obtained dispersion formulas;
3. The properties of eigenfrequencies and modal forms of finite pantographic specimens suitably constrained and excited.

Another result which we present in this paper concerns the study of well-posedness of the equilibrium problem for the considered pantographic sheet. The mathematical problems to be faced are not as simple as one should have expected. The standard Sobolev space setting is not suitable to frame in a satisfactory way. Instead, anisotropic Sobolev spaces are needed and a careful study of the null-space of the postulated deformation energy plays a crucial role in some instances of pantographic sheets where some stiffnesses are vanishing. The null-space of some kinds of sheets may include finite or even infinite dimensional spaces of displacements which strictly include rigid motions. The elements of these null-space have been sometimes called "floppy modes." The boundary conditions to be imposed to assure well-posedness of equilibrium problems must, in these singular cases, assure that all possible floppy modes are not allowed.

Let us note that here we restricted ourselves to infinitesimal deformations. On the other hand, high flexibility of considered beam lattice structures results in necessity to analyze existence and uniqueness/non-uniqueness of the corresponding nonlinear boundary-value problems. Unlike classic plates and shallow shells [84–87], where deflections are larger than in-plane displacements, in general, for a beam lattice in-plane displacements may have the same order of magnitude as out-of-plane ones. So, one can expect an essential nonlinearity of the corresponding boundary-value problems.

We expect that the presented mathematical analysis, which shows that the treatment presented in [28] can be extended to sheets deforming in 3D space, will guide us to treat the more complicated problems to be faced when considering pantographic sheets undergoing large deformations, see, e.g., [18,22,92]. In other words, we believe that for any linear or nonlinear physical model its linear mathematical counterpart that is a linear boundary-value problem with suitable boundary conditions, such as fixed boundary conditions, should have unique solution. Otherwise, this results not only in some pathological mathematical properties, but also in essential difficulties in numerical calculations.

This conclusion may be also useful for the mathematical analysis of other enhanced models of continua and structures, as for example polar, dipolar, non-local media and continua with additional kinematical descriptors [95,116–120], where the enhanced kinematics may result in unusual floppy modes and corresponding constraints to external loading.

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