ON STATISTICAL LINEAR INVERSE PROBLEMS IN FISHERY ACOUSTICS

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The paper summarizes several aspects of solving statistical linear inverse problems that occur in the processes related to fish abundance estimation performed on the basis of acoustically acquired data. As the difficulty in solving such problems inherently comes from uncertainty in determination of so called inversion kernel, the paper discusses the key approaches that lead to formulation of statistical relations between acoustic echoes observed from swimming fish in their natural environment and mostly required fish parameter, namely physical length of the fish.

INTRODUCTION

In many areas of practical sciences where observation has statistical nature the reconstruction of unknown features leads very often to so called statistical linear inverse problems (SLIP). In paticular in fishery acoustics they occur in two problems related to fish abundance estimation. Firstly, in the problem of fish target strength estimation from single beam observation where unknown fish position in the beam may be removed by statistical processing of fish echo. Secondly, when transforming acoustic measure of fish scattering namely target strength into physical measure namely fish fork length where the effect of fish random orientation needs to be removed.

In case of fish target strength estimation from single beam data the so-called single beam integral equation needs to be solved for target strength probability distribution functions (PDFs). The integral equation may be formulated in so-called absolute (linear) domain where unknown PDF is represented by fish backscattering length or fish backscattering cross section, or in logarithmic domain where unknown function is target strength PDF. However, it was shown in [1] that in logarithmic domain the problem is less ill conditioned than in absolute domain, although it may lead to oversmoothing. In all this formulations the kernel of inverse problem is represented by the term, which in fishery acoustics is called beam pattern PDF. Actually, it is used to remove beam pattern effect from fish echo data. Although beam pattern is deterministic function, it is used in statistical sense as a function, which transforms random fish position into new random variable combining random fish position observed by beam pattern. Theoretically, unknown fish target strength does not depend on its position in the beam, so both distributions can be treated as random variables. Nevertheless, as the observed data are limited due to hydracoustic system threshold, they both became dependent.

In case of fish length estimation the inversion techniques may be used to convert fish target strength PDF into maximum fish target strength PDF by removal of fish directivity pattern effect. The term maximum fish target strength refers to the situation when the fish is insonified by the wave from the direction perpendicular to its maximum acoustic aperture, which for different species may not be perpendicular to its fish body or swimming direction. In this case, as the maximum fish target strength and fish directivity pattern both depend on fish length the problem requires processing PDFs of dependant random variables. As it will be shown in this chapter, the solution can be obtained by introducing conditional PDFs. Moreover, when the relation between maximum fish target strength and actual fish length (depends rather only on morphological parameters of the fish) is established, the PDF of fish length can be obtained.

In this paper, several aspects related to determination of PDFs of transducer beam pattern and fish directivity pattern are considered, as both constitute the kernel used in inversion scheme.

1. FISH ANGULAR POSITION IN THE TRANSDUCER BEAM

Widely used assumption of uniform spatial distribution of fish in a water column, leads to *sine*-law distribution of the angular position of the fish in the beam [2],[3],[4]. This assumption is valid only for the case of the single (non-multiple) echoes received from individual fish in consecutive pings. However, when acquiring actual data from acoustic surveys, the multiple or correlated echoes may be collected from the same fish forming the fish traces.

Typically, calculation of PDF of fish angular position $p_\theta()$ is based on the assumption of uniform distribution of fish in a water column (cartesian coordinates). Let us then assume that a distribution of variable z representing depth on which fish appears in the conical area defined by observation angle $\theta_{\max}$ is uniform, i.e.

$$p_z(z) = \frac{1}{z_{\max}} \tag{1}$$

where $z_{\max}$ represents maximum depth. Due to linear relation between depth and radius of a circular slice $\underline{z} = \underline{R}\tan\theta_{\max}$, the distribution of random variable $\underline{R}$ become also uniform, i.e.

$$p_R(R) = \frac{1}{R_{\max}} \tag{2}$$

where $R_{\max} = z_{\max} / \sin\theta_{\max}$ is maximum possible radius of the slice. Substituting $\underline{\theta} = \mathrm{acos}(\underline{z}/\underline{r})$ and making simple calculations as presented in [5] the distribution of angular position θ has *sine* like law:

$$p_\theta(\theta) = \frac{1}{1-\cos\theta_{\max}}\sin\theta \tag{3}$$

where $\theta_{\max}$ is maximum angle of beam pattern involved in calculation.

However, in most cases in acoustics surveys the same fish is observed in several contiguous pings so the set of echoes does not represent independent statistics. That is why it is more proper to include the effect of multiple echoes in the set. Theoretically, it is possible to consider two models of fish traces statistics: 1) assuming the vessel movement with stationary fish, and 2) assuming stationary vessel and moving fish. Both models were described by Moszynski in [6]. In the first model the uniform vessel movement with stationary fish is assumed. The second model assumes fish movement along arbitrary path in the transducer beam pattern cross-section. Actually, both models differ in a distribution of the crossing angle (the angle in which a fish enters the beam), which has *sine*-law PDF in the first model and uniform distribution in the second one. Although the behaviour of fish crossing the transducer beam is unpredictable [7], the distrubution of number of echoes in the trace obtained theoretically using the same assumptions, are in agreement with those observed in the data acquired during acoustic surveys [8].

Let us now consider the distribution of angular position of fish $\underline{\theta}$ that is necessary for calculation of beam pattern PDF. From the geometrical relations we have:

$$\underline{\theta} = a\cos\frac{\underline{z}}{\underline{r}} = a\cos\frac{1}{\sqrt{1+\left(\underline{\rho}/z\right)^2}} \tag{4}$$

where random variables $\underline{r}$, $\underline{\rho}$, $\underline{z}$ represent coordinates of fish position related by equation: $\underline{r}^2 = \underline{\rho}^2 + \underline{z}^2$. Let us also consider the random variable $\underline{t}$ called trace distance, which represents distance of the fish from the crossing point of circular slice. If fish swims on the chord and is "sampled" uniformly in the consecutive pings, we may treat its distribution as uniform in a range $(0, 2\,\underline{r}\sin\underline{\alpha})$. Thus, the trace distance random variable may be expressed as $\underline{t} = 2\underline{r}\sin\underline{\alpha}\,\underline{u}$, where $\underline{u}$ has again normalised uniform distribution. Taking into account the cosine law in the non-right angled triangle we obtain:

$$\underline{\rho}^2 = \underline{r}^2 + \underline{t}^2 - 2\underline{r}\,\underline{t}\sin\underline{\alpha} = \underline{r}^2\left(1-(2\sin\underline{\alpha})^2(\underline{u}-\underline{u}^2)\right) \tag{5}$$

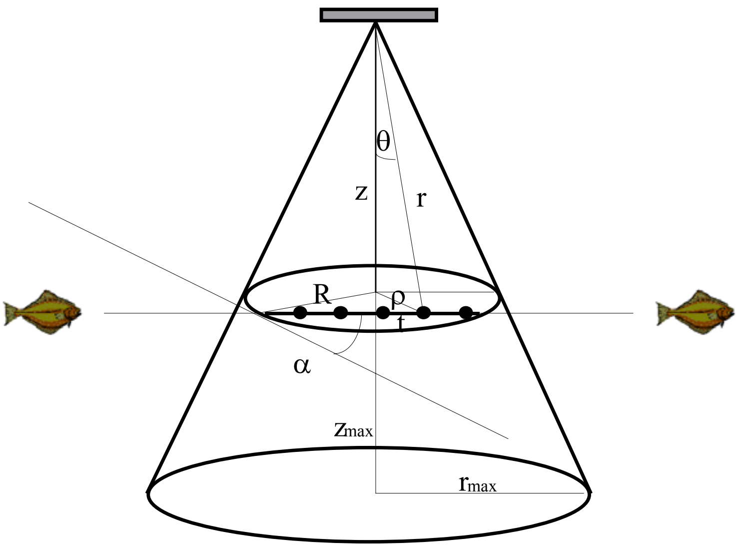

Fig.1 Geometry of multiple echoes from a fish trace

Using the formulae for PDF of the product of random variables and substituting $\underline{z} = \underline{r} \tan \theta_{\max}$ which removes $\underline{z}$ dependence from Eq.(4) and $\underline{r}$ dependence in Eq.(5), we receive the PDF of angular position for the moving vessel model in the following form:

$$p_\theta(\theta) = \frac{1}{\tan^2 \theta_{\max}} K\left(\frac{\tan\theta}{\tan\theta_{\max}}\right) \frac{\sin\theta}{\cos^3\theta} \tag{6}$$

where $K(x) = \int_0^{\pi/2} (1 - x^2 \sin^2 \varphi)^{-1/2} d\varphi$ represents here complete elliptic integral of the first kind. It is worth to note that as one could expect there are more echoes received from off-axis angles, which results in increased distribution at this range of angles as compared to *sine*-like distribution known for the case of non-multiple echoes statistics.

Practically, the statistics of actual angular position of fish in the beam may be calculated using echo trace algorithm, which for split-beam system observations allows determing fish trace for each fish.

2. TRANSDUCER BEAM PATTERN PDF

In general, transducer beam pattern is deterministic quantity but in the context of unknown position of fish in the beam it can acquire nondeterministic meaning. For beam pattern PDF calculation let us consider first the ideal circular piston in a infinite baffle. Its one-way intensity domain beam pattern function $b(\cdot)$ (or two-way pressure domain) is:

$$b(\theta) = \left(\frac{2J_1(x)}{x}\right)^2 \tag{7}$$

where x is defined by $x = x(\theta) = ka \ \sin\theta$ (k – wave number, a – transducer radius) and $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)d\theta$ is the Bessel function of first kind order n. It is worth to note that in the fishery acoustics and oceanography literature there are different definitions of beam pattern function. Defined here $b(\cdot)$ is equivalent to b^2 of MacLennan and Simmonds [9] and D^2 of Medwin and Clay [10], as they define its beam pattern function in pressure domain. The different definitions come from two facts: 1) the transmitting b_t (or D_t) and the receiving b_r (or D_r) patterns are theoretically the same due to reciprocity of a transducer $D_t = D_r$, 2) due to two-way transmission the net effect is multiplication of both patterns $b = D_r \cdot D_t = D^2$ (as in [11]), hence the square in Eq. (7) for the beam pattern of circular transducer, when interpreting this function in pressure domain.

The logarithmic version of two-way pattern is derived by transform:

$$B(\theta) = 10\log b^2(\theta) = 20\log b(\theta) \tag{8}$$

and is presented in Fig.2.

The beam pattern PDF is defined using theory of function of random variable for b($\underline{\theta}$) [12]:

$$p_b(b) = \frac{p_\theta(\theta)}{\left|\frac{db}{d\theta}\right|} \tag{9}$$

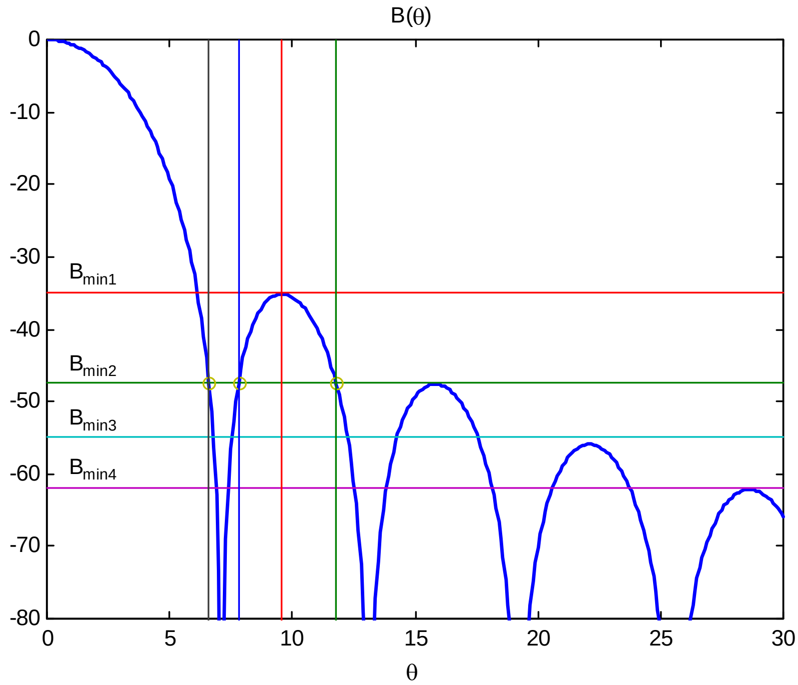

Fig.2 Two-way beam pattern function for ideal circular piston in an infinite baffle

This function for circular transducer may be expressed as a parametric function $p_b(b) = (b(\theta), p_b(\theta))$ with angle θ as a parameter [13]:

$$p_b(b) = \left(\left(\frac{2J_1(x)}{x} \right)^2, \frac{p_\theta(\theta)\,\mathrm{tg}\,\theta}{\left| \frac{8J_1(x)J_2(x)}{x} \right|} \right) \tag{10}$$

where $p_\theta()$ is a probablity density function of random angular position of fish. Using logarithmic transform of variables $B(b) = 20\log b$ its PDF relation may be written as:

$$p_B(B) = \frac{\ln 10}{20} \left| 10^{\frac{B}{20}} \right| p_b\left(10^{\frac{B}{20}} \right) \tag{11}$$

Practically, the beam pattern in commercially manufactured echo sounders may differ from theretically obtained for circular piston. The approximation used by Moszynski [14] is particularly useful for PDF calculation. As he showed the following approximation:

$$b(\theta) = \left(1 - (1 - 2^{-\gamma}) \frac{1 - \cos\theta}{1 - \cos\theta_{3dB}} \right)^{\frac{2}{\gamma}} \tag{12}$$

allows fitting the slope to the actual pattern by the usage of the exponential coefficient γ. In case of angular distribution of fish position described by Eq.(3), it leads to simple formulae for beam pattern PDF:

$$p_b(b) = \frac{c_b}{b^{1-2/\gamma}} \tag{13}$$

with c_b as normalization constant. The logarithmic transform gives following equation:

$$p_B(B) = \frac{\ln 10}{20}\frac{\gamma}{1-2^{-\gamma}}\frac{1-\cos\theta_{3dB}}{1-\cos\theta_{\max}}10^{\frac{\gamma B}{20}} \tag{14}$$

where $\theta_{\max}$ is maximum angle considered in analysis. Taking into account multiple echo statistics described by Eq.(6), it results in a parametric equation (with angular position θ as a parameter):

$$p_B(B) = \left(20\log b(\theta)\;\;,\;\; \frac{\ln 10}{20}\frac{\gamma}{1-2^{-\gamma}}\frac{1-\cos\theta_{3dB}}{\tan^2\theta_{\max}}K\left(\frac{\tan\theta}{\tan\theta_{\max}}\right)\frac{b(\theta)^{\gamma}}{\cos^3\theta}\right) \tag{15}$$

Another practical note is worth to be mentioned. When only main lobe of beam pattern is used, the inverse of beam pattern function needed for $p_b()$ computation is single-valued. However, when side-lobes are involved the inverse function is non single-valued Fig.2 and then its PDF calculation must be made for all angles θ having the same value B. This makes the equations for beam pattern PDF $p_B(B)$ still valid but inverse function $b^{-1}(\cdot)$ needs to be computed to find all occureances of θ for every b being now independent variable. The procedure must be followed by the sumation of PDF values for those angles. The inverse of beam pattern function now called $\theta(b)$ may be calculated iteratively using Newton iteration as showed in [15]. The final result of this approach is presented in Fig.3, showing theoretical beam pattern PDF $p_B(\cdot)$ with inclusion of three side-lobes. The components from sidelobes are presented as dotted lines. Note the theoretical left-sided infinities in PDF where sidelobes reach its local maxima. Practically, it is recommended to operate without side-lobe, where beam pattern PDF has monotonic representation described by an approximation i.e. Eq.(14) or Eq.(15).

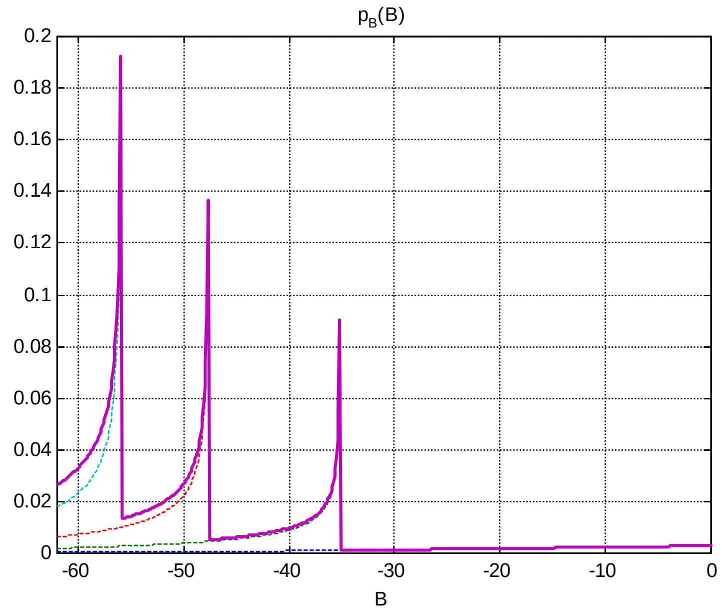

Fig.3 Theoretical beam pattern probability density function (PDF) for ideal circular piston when three sidelobes are included

To construct the kernel of inverse problem, the beam pattern PDF needs to be discretized. In most cases this leads to simple sampling of theoretical PDF, however in this case due to infinities and histogram nature of observed echo level PDF $p_E()$ it must be done carefully. To check the behaviour of discretized version of PDF another method of its determination is suggested. The proposed method is based on random generation of assumed angular position. The distribution described by PDF function can be generated using inverse of cumulative distribution function (CDF) method. Note that the discrete approximation depends not only on bin size but also on bins positions, especially when the bin contains theoretical infinity and low values alltogether. Additionally, the values in border bins may be inaccurate. Sample simulations and the analysis of actual data from the acoustic survey assuming side-lobe acquisition were presented in [15].

3. INFLUENCE OF THRESHOLD ON THE BEAM PATTERN KERNEL

In the acoustic surveying of fish stocks and subsequent estimation of biomass, the sampling volume of the acoustic instruments is of great importance, as for randomly distributed targets the received echo energy is linearly related to this volume. As a signal threshold is applied in order to eliminate contribution from noise the effective sampling volume is always less than the full volume of the acoustic beam. The problem was treated in several papers. Kalikhman [16] combined the beam pattern of the transducer with scattering characteristics of the fish averaged over azimuth. Their conclusion is that, for a single fish, the effective equivalent beam angle depends not only on the threshold but also on the angle of insonification. Foote [17] developed an expression for the effective equivalent beam angle in terms of the directivity of the transducer, the backscattering cross-section of the fish as function of tilt angle and the signal threshold. Detailed literature survey on the problem can be found in papers of Reynisson [18] and Fleischman [19]. The problem of bias introduced by threshold will be described here by statistical analysis of the target strength and beam pattern distributions.

In simplified and often used case, the random variables $\underline{TS}$ and $\underline{B}$ are treated as independent random variables, which allows expressing their probability density functions (PDFs) in the form of convolution integral equation. However, although from dual-beam systems we have exact TS for each detected fish echo $\underline{E}$, when applying threshold to obtained echoes it restricts not only the dynamic range of the data but also introduces dependence on $\underline{TS}$ and $\underline{B}$ random variables. Thus, statistically, we can write the equivalent equation for dual beam case:

$$\underline{E'} = \underline{TS'} + \underline{B'} \qquad (16)$$

where primes denote that we operate on conditinal variables. As the consequence, the PDFs of these variables should be expresses by a conditional PDF as follows:

$$p_{E'}(E') = p_E(E \mid E > E_{\min}) \qquad (17)$$

$$p_{TS'}(TS') = p_{TS}(TS \mid E > E_{\min}) \qquad (18)$$

$$p_{TS'}(TS') = p_{TS}(TS \mid E > E_{\min}) \qquad (19)$$

where $E_{\min}$ is the echo level threshold value.

The net effect is that the mean value of backscattering cross-section $\overline{\sigma}'$ evaluated from transformed distribution of conditional random variable $\underline{TS}$' is biased as compared to actual mean value $\overline{\sigma}$ evaluated from variable $\underline{TS}$. It is noteworthy that in the single beam case the fact of introducing the threshold modifies only the range of integration in convolution-like integral:

$$p_E(E \mid E > E_{\min}) = \int_{E_{\min}-B}^{0} p_{TS}(E-B)\, p_B(B)\, dB \tag{20}$$

and the reconstructed $p_{TS}(TS)$ is unconditional if properly assumed $p_B(B)$ is used.

Statistical removal of the bias introduced by the threshold in dual beam processing requires calculation of $p_{TS}(TS)$ from conditional $p_{TS}(TS|E > E_{\min})$. The latter distribution, which is *de facto* observed, can be expressed as:

$$p_{TS}(TS \mid E > E_{\min}) = p_{TS}(TS \mid TS + B > E_{\min}) \tag{21}$$

which may be evaluated by integration of joint distribution of independent random variables $\underline{TS}$ and $\underline{B}$:

$$p_{TS}(TS \mid E > E_{\min}) = \frac{\int_{E_{\min}-TS}^{\infty} p_{TS,B}(TS,B)\, dB}{\int_{-\infty}^{+\infty} \int_{E_{\min}-TS}^{\infty} p_{TS,B}(TS,B)\, dB\, dTS} \tag{22}$$

As the denominator evaluate to the constant value (normalization constant) and independency of variables $\underline{TS}$ and $\underline{B}$ allows replacing joint PDF by multiplication of its PDFs, it results in:

$$p_{TS}(TS \mid E > E_{\min}) = c_1 p_{TS}(TS) \int_{E_{\min}-TS}^{\infty} p_B(B)\, dB \tag{23}$$

The integral in above equation can be expressed using cumulative distribution function (CDF) $F_B(\,)$ of beam pattern random variable $\underline{B}$, which finally gives:

$$p_{TS}(TS \mid E > E_{\min}) = c_1\ p_{TS}(TS)\,[1 - F_B(E_{\min} - TS)] \tag{24}$$

which describes the connection between conditional distribution of observed target strength and required unconditional distribution. Note that, it requires the knowledge of unconditional distribution of beam pattern CDF.

The same approach applied to conditional distribution of beam pattern function $p_B(B|E > E_{\min})$ gives the following equation:

$$p_B(B \mid E > E_{\min}) = c_2\ p_B(B)\,[1 - F_{TS}(E_{\min} - B)] \tag{25}$$

In both cases, the expression in brackets represents CDF of the second function scaled to domain of first function. Thus, dependence introduced by threshold is observed as a multiplication of unconditional PDF of one function by scaled CDF of the second one. The constants c_1 and c_2 normalize equivalent distributions. The removal of threshold effect on measured distribution of target strength requires solution of equations (24) and (25), which represents a set of integral equations.

4. FISH DIRECTIVITY PATTERN PDF

Statistical processing does not require using high resolution fish backscattering model, so it is not necessary to reflect precisely fish scattering properties. For that pupose it is reasonable to use a simple tilted cylinder model that allows fish target strength to be rewritten in the logarithmic form [20]. It more clearly shows dependence on angular position of the fish expressed by fish body tilt angle χ and swimbladder tilt angle χ_0:

$$TS = TS_0(l_{ecb}, a_{ecb}) + B_f(\chi, \chi_0, l_{ecb}) \tag{26}$$

where $TS_0 = 20\log l_{bs0}$ is maximum target strength of the fish and B_f is the fish angular directivity pattern in dorsal aspect expressed in logarithmic domain:

$$B_f(\chi,\chi_0,l_{ecb}) = 20\log\left(\frac{\sin\left(kl_{ecb}\sin(\chi+\chi_0)\right)}{kl_{ecb}\sin(\chi+\chi_0)}\sqrt{\cos(\chi+\chi_0)}\right) \qquad (27)$$

When the echoes are acquired from fish population having random orientation the Eq. (26) can be interpreted as the sum of random variables. However, in this case the variables are not independent due to dependence of both variables $\underline{TS}_0$ and $\underline{B}_f$ on equivalent cylinder length l_{ecb}.

In general case, the PDF of sum of two random variables $\underline{z} = \underline{x} + \underline{y}$ can be expressed by integral [12]:

$$p_z(z) = \int p_{x,y}(x, z-x)dx \qquad (28)$$

where $p_{x,y}(x,y)$ is joint PDF of two random variables $\underline{x}$ and $\underline{y}$. If $\underline{x}$ and $\underline{y}$ are dependent variables then using Bayesian theory the joint PDF can be replaced by the product of PDF of one variable $p_x(x)$ and appropriate conditional PDF of the second variable:

$$p_z(z) = \int p_x(x)\, p_{y|x}(z-x, x)dx \qquad (29)$$

For considered TS_0 and B_f random variables it means:

$$p_{TS}(TS) = \int p_{TS_0}(TS_0)\, p_{B_f|TS_0}(TS-TS_0, TS_0)dTS_0 \qquad (30)$$

Eq. (30) represents integral equation for reconstructing the distribution of maximum target strength of the fish TS_0 from the observation of fish with random orientation. Conditional fish pattern PDF $p_{B_f|TS_0}(B_f, TS_0)$ represents kernel of this equation. To calculate this function it is required to establish the value of maximum target strength TS_0 of the fish and then calculate its directivity pattern B_f. When fish is observed from dorsal aspect, its directivity pattern depends mainly on orientation expressed by the sum of fish body tilt angle χ and swimbladder tilt angle χ_0. Thus, additionally, the distribution of both random variables influencing the angle in which fish is observed should be known. Finally, it is recomended to construct fish directivity pattern PDF by random generation of such distribution and transforming obtained PDF by fish directivity pattern function.

Sample randomly generated fish conditional directivity pattern PDF at 120 kHz is presented in Fig.4. During PDF calculation, it was assumed that the fish body tilt angle has normal distribution with a mean value of 8° and the variance 3°. The mean value can be interpreted as a swimbladder tilt angle χ_0 and variance is a measure of changes in fish body orientation. Note, however, that the swimbladder tilt angle may be treated only as a mean value for particular species as it may change during fish grow history. In other words, it means that the different size fish may have different swimbladder tilt angle. Additiononally, the changes in fish orientation observed by the acoustic system are also related to fish movement in the beam and are limited due to transducer beam pattern.

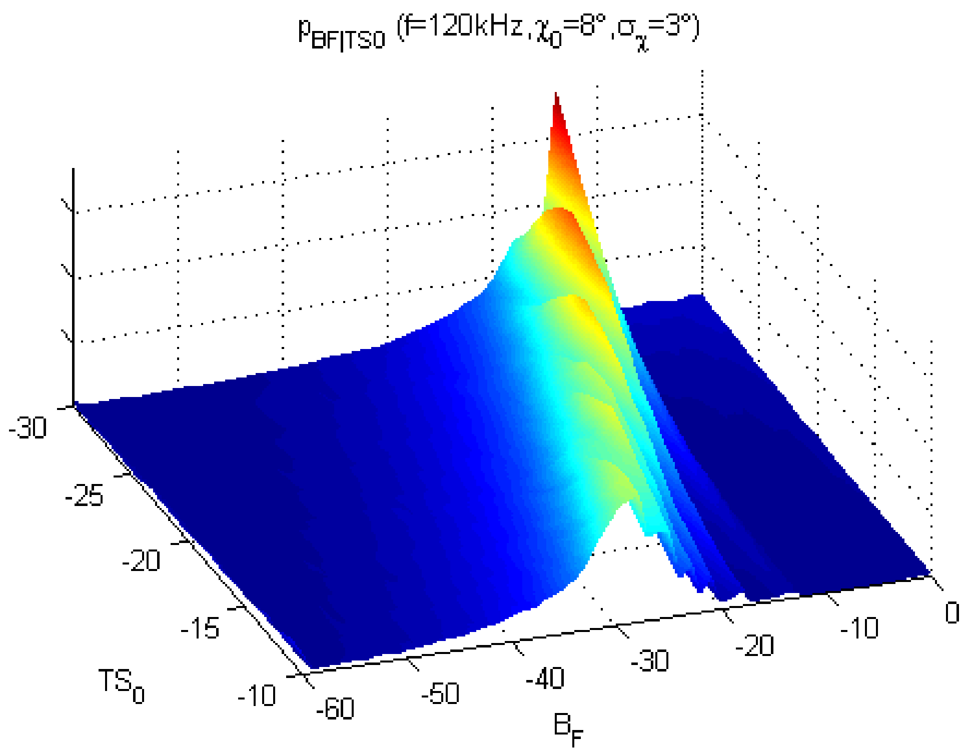

Fig.4 Sample randomly generated fish conditional directivity pattern PDF at 120 kHz assuming normal distribution of tilt angle (χ_0 = 8°, σ_χ = 3°)

Theoretically, when only fish straight movement in the beam is considered, the tilt angle statistics would be the same as fish angular position in the beam presented before. Hence, to calculate conditional fish directivity pattern it is possible to use fish angle statistics from echo tracing algorithm. However, true tilt angle statistics needs to reflect fish behaviour, which may be registered only by video observations or additional knowledgde on its migrations.

As the fish scattering model is used to calculate the fish directivity pattern PDF, it is obvious that not only distribution of maximum target strength could be calculated from inverse processing but also fish length distribution. This is true, if for observed fish its equivalent parameters for scattering model are known, what is possible if fish species are identified. Species identification is well-recognized problem in fishery acoustics and for some cases of commercial fishes it is well worked out [21].

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