ON THE CONNECTED AND WEAKLY CONVEX DOMINATION NUMBERS

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ABSTRACT. In this paper we study relations between connected and weakly convex domination numbers. We show that in general the difference between these numbers can be arbitrarily large and we focus on the graphs for which a weakly convex domination number equals a connected domination number. We also study the influence of the edge removing on the weakly convex domination number, in particular we prove that the weakly convex domination number is an interpolating function.

1. INTRODUCTION

All graphs considered in this paper are finite, undirected, simple and connected. Let $G = (V, E)$ be a connected graph of order $\mid V \mid = n$, where $V = V(G)$ is the set of the vertices of G and $E = E(G)$ denotes the set of edges of G. For a vertex $v \in V$, the *open neighbourhood* $N_G(v)$ is the set of all vertices adjacent to v and the *closed neighbourhood* $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v is $d_G(v) = |N_G(v)|$. We say that a vertex v is a *simplicial vertex* if $N_G[v]$ is a complete graph. A vertex v is an *end-vertex* (or a *leaf*) of G if v has exactly one neighbour in G. The set of all end-vertices in G is denoted by V_L and $n_L = |V_L|$. A vertex v is called a *support* if it is adjacent to an end-vertex. The set of all supports of G is denoted by V_S. A *cut-vertex* in G is a vertex $x \in V(G)$ such that the number of components of $G - \{x\}$ is bigger than the number of components of G. The set of all cut-vertices of G is denoted by V_C. Let $g(G)$ denotes the girth of G that is, the length of the shortest cycle in G.

A subset D of V is *dominating* in G if every vertex of $V - D$ has at least one neighbour in D. The set D is *connected dominating* in G if it is dominating and the subgraph $G[D]$ induced by D is connected. The minimum cardinality of a connected dominating set of G is a *connected domination number* of G and is denoted by $\gamma_c(G)$.

Dominating sets have been intensively studied since the fifties and the main interest is due to their relevance on both theoretical and practical side. Several variants of the classical concept of domination were obtained where additional

conditions on the subgraph induced by the dominating set were added. Connected dominating sets, introduced in [8], are one of these variants, and have useful applications in the wireless networks context (see [2]). In addition, the design of communication networks involves considering short distances between nodes in order to get quick transmition.

The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $(u - v)$-path in G. A $(u - v)$-path of length $d_G(u, v)$ is called $(u - v)$-*geodesic*. The *diameter* of a graph G, denoted as $\text{diam}(G)$, is defined as the maximum of distance over all pair of vertices. A set X is *weakly convex* in G if for any two vertices $a, b \in X$ there exists an $(a - b)$–geodesic such that all of its vertices belong to X. A set $X \subseteq V$ is a *weakly convex dominating set* if X is weakly convex and dominating.

The *weakly convex domination number of a graph* G denoted by $\gamma_{\text{wcon}}(G)$ equals to the minimum cardinality of a weakly convex dominating set in G. It was first introduced by Jerzy Topp in 2002 and formally defined and studied in [6]. This concept improves the applications of connected domination in the design of communication networks, by guarantee that the connections trough the nodes of the dominating set are the shortest.

In this work we investigate the relationship between the weakly convex domination and the connected domination. Moreover, we study edge removing and its effect on the weakly convex number for some graphs. Related to this, we have the idea of interpolation. In the 80s of the last century, the study of the interpolation properties started. It is often thought that the origin was the problem whether a graph G containing spanning trees having k and l end-vertices, respectively, also must contain a spanning tree with r end-vertices for every integer r such that $k < r < l$. Several authors published some results of interpolation theorems on various kinds of graph parameters with respect to the set of all spanning trees and some classes of spanning subgraphs of a given graph. In this paper, we conclude that for each connected graph G, the image of the function γ_{wcon} over the set of all spanning trees is an interval, what means that γ_{wcon} is an interpolating function.

The paper is organized as follows. In Section 2, we present relations between the weakly convex domination and the connected domination. In particular, we look for conditions on the graph G under which we get $\gamma_{\text{wcon}}(G) = \gamma_c(G)$. First, we prove that there are graphs for which the difference between $\gamma_{\text{wcon}}(G)$ and $\gamma_c(G)$ can be arbitrarily large. In the next section, we show some examples of families of graphs G with equality $\gamma_{\text{wcon}}(G) = \gamma_c(G)$, e.g. distance-hereditary graphs (in particular block graphs) and cacti graphs (in particular unicyclic graphs). Moreover we study the chordal graphs. This kind of graphs have an extensive literature and applications (see for example [1], [7] [10]). We complete Section 3 focussing our attention on induced subgraphs; more specifically, we give conditions for the weakly convex domination number to be equal to the connected domination number for a graph and every its induced subgraph. Section 4 is devoted to study the influence of the edge removing on the weakly convex domination number, in particular we show that a weakly convex domination number is an interpolating function.

2. Connected and weakly convex domination numbers

This section we start with proving that there are graphs for which the difference between γ_{wcon} and γ_c can be arbitrarily large.

Theorem 2.1. *For any $k \in \mathbb{N}$ and $k \geq 6$, there exists a graph G such that*

$$\gamma_{\mathrm{wcon}}(G) - \gamma_c(G) = k.$$

Proof. We begin with a cycle $C_{2k+6} = (x_1, v_1, v_2, \ldots, v_{k+2}, x_2, u_{k+2}, u_{k+1}, \ldots u_1, x_1)$. Next we add to this cycle: new vertices $x_1', v_1', v_2', \ldots, v_{k+2}', x_2'$ and edges $x_1 x_1'$, $x_2 x_2'$ and $v_i v_i'$ where $1 \leq i \leq k+2$, and we also add edges $v_j u_j$ for $1 \leq j \leq k+2$, and the edge $v_1 u_3$. The final graph is illustrated in the Figure 1. Let D be a minimum weakly convex dominating set of G. All supports of G belong to D. Moreover, the distance between v_1 and x_2 equals to $k+1$ and $u_3, u_4, u_5, \ldots, u_{k+2}$ belong to the shortest $(v_1 - x_2)$-path, so they also belong to D. Hence $\gamma_{\mathrm{wcon}}(G) = 2k+4$.

It is easy to observe that $\{x_1, v_1, v_2, \ldots, v_{k+2}, x_2\}$ is the minimum connected dominating set of G. Thus $\gamma_c(G) = k+4$ and $\gamma_{\mathrm{wcon}}(G) - \gamma_c(G) = k$. □
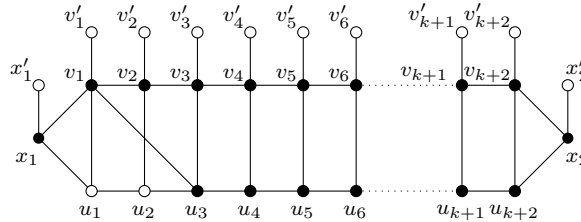

FIGURE 1. Construction of a graph G such that $\gamma_{\mathrm{wcon}}(G) - \gamma_c(G) = k$. Black vertices denote the minimum weakly convex dominating set of G.

In [8] the following result was proved for a connected domination number of any connected graph G.

Theorem 2.2. [8] *For any connected graph G with $n \geq 3$ vertices and m edges is $\gamma_c(G) \leq 2m - n$ with equality if and only if G is a path.*

Similar result we can prove for weakly convex domination number.

Theorem 2.3. *For any connected graph G with $n \geq 3$ vertices and m edges is $\gamma_{\mathrm{wcon}}(G) \leq 2m - n$ with equality if and only if G is a path or a cycle C_p with $p \geq 7$.*

Proof. Assume first that G is a tree. Then $\gamma_{\mathrm{wcon}}(G) = n - n_L = 2(n-1) - n + 2 - n_L = 2m - n + 2 - n_L \leq 2m - n$ with equality $\gamma_{\mathrm{wcon}}(G) = 2m - n$ when $n_L = 2$, what means that G is a path.

Assume now G is not a tree; thus $m \geq n$. Let first $\gamma_{\mathrm{wcon}}(G) \leq n - 1$. Then we have $\gamma_{\mathrm{wcon}}(G) \leq n - 1 = 2n - n - 1 < 2m - n$. Let now $\gamma_{\mathrm{wcon}}(G) = n$. Then we have $\gamma_{\mathrm{wcon}}(G) = n = 2n - n \leq 2m - n$ with equality for $m = n$. The only case where $\gamma_{\mathrm{wcon}}(G) = n$ and $m = n$ happens if $G = C_p, p \geq 7$. □

We continue with showing that we can get $\gamma_c(G) = \gamma_{\mathrm{wcon}}(G)$ if we consider graphs for which the subgraph induced by a minimum connected dominating set D has a small diameter. A connected perfect dominating set D is a connected dominating set where each vertex in G is dominated by exactly one vertex of D.

Lemma 2.4. *If G is a connected graph with a minimum connected dominating set D such that either $\mathrm{diam}(G[D]) \leq 2$ or D is a perfect connected dominating set with $\mathrm{diam}(G[D]) = 3$, then $\gamma_c(G) = \gamma_{\mathrm{wcon}}(G)$.*

Proof. Suppose there exist $x, y \in D$ such that $d_{G[D]}(x, y) > d_G(x, y)$. Of course $d_G(x, y) > 1$; otherwise $1 = d_G(x, y) = d_{G[D]}(x, y)$. If $\mathrm{diam}(G[D]) \leq 2$, also $d_G(x, y) \leq 2$ for all pair of different vertices $x, y \in D$, a contradiction. Now, assume that D is perfect dominating set with $\mathrm{diam}(G[D]) = 3$. Then $d_{G[D]}(x, y) \leq 3$ for all pair of different vertices $x, y \in D$. If $d_G(x, y) = 2 < d_{G[D]}(x, y) = 3$, then there is a vertex $z \notin D$ such that $z \in N(x) \cap N(y)$, a contradiction with the fact that D is perfect dominating set. $\square$

Note that there are no graphs G for which $\gamma_c(G) = n$ or $\gamma_c(G) = n - 1$. In fact, in [8] the following result was proved.

Theorem 2.5. [8] *For any connected graph G with at least three vertices is $\gamma_c(G) \leq n - 2$ with equality $\gamma_c(G) = n - 2$ if and only if $G = P_n$ or $G = C_n$.*

However, there are some graphs G for which $\gamma_{wcon}(G) = n$. Some of these graphs were characterized in [4] in the following theorem.

Theorem 2.6. [4] *If G is a connected graph with no end-vertex such that $g(G) \geq 7$, then $\gamma_{wcon}(G) = n$.*

By definition we have that in general, $\gamma_{\mathrm{wcon}}(G) \geq \gamma_c(G)$ for any graph G. As consequence of Theorems above we get necessary condition for $\gamma_{wcon}(G) = \gamma_c(G)$.

Corollary 2.7. *If $\gamma_{wcon}(G) = \gamma_c(G)$, then either G has end-vertices or $g(G) < 7$.*

We have a more general result for connected graph with $g(G) \geq 7$. We begin with the following observation.

Observation 2.8. *Let $G \neq K_n$ be a graph of order $n \geq 3$. If D is a minimum connected or weakly convex dominating set of G, then every cut-vertex belongs to D and no simplicial vertex belongs to D.*

Corollary 2.9. *If G contains only cut-vertices and simplicial vertices, then V_C is a minimum connected (and also weakly convex) dominating set of G and $\gamma_c(G) = \gamma_{\mathrm{wcon}}(G)$.*

Now we prove the following result.

Theorem 2.10. *Let G be a connected graph with $g(G) \geq 7$. Then,*
  (1) *$\gamma_{\mathrm{wcon}}(G) = n - n_L$;*
  (2) *$\gamma_{\mathrm{wcon}}(G) = \gamma_c(G)$ if and only if for every $u \in V$, u is either an end-vertex or a cut-vertex.*

Proof. (1). Every connected graph G with $g(G) \geq 7$ can be obtained from some graph G', which fulfills conditions from the Theorem 2.6, by adding only cut and end-vertices. As $\gamma_{\mathrm{wcon}}(G') = |V(G')|$ and every end-vertex is simplicial, we conclude from the Observation 2.8 that $\gamma_{\mathrm{wcon}}(G) = n - n_L$.

(2). Assume $\gamma_{\mathrm{wcon}}(G) = \gamma_c(G)$. From (1) we have $\gamma_{\mathrm{wcon}}(G) = \gamma_c(G) = n - n_L$. Suppose there is a vertex u such that u is neither an end-vertex nor cut-vertex. Then $d_G(u) \geq 2$ and for any $x, y \in N_G(u)$ there exists $(x - y)$-path not containing u. Then $V - (V_L \cup \{u\})$ is a connected dominating set of G and $\gamma_c(G) \leq |V - (V_L \cup \{u\})| = n - (n_L + 1) < n - n_L$, a contradiction.

Now assume for every $u \in V$, u is either an end-vertex or a cut-vertex. From Observation 2.8, the result holds. $\square$

3. Some graphs G with equality $\gamma_{\mathrm{wcon}}(G) = \gamma_c(G)$

In this section we provide conditions under which we have equality $\gamma_{\mathrm{wcon}}(G) = \gamma_c(G)$ in particular families of graphs G. We begin with cacti.

3.1. **Cacti.** A *cactus* is a connected graph in which any two simple cycles have at most one common vertex. Equivalently, it is a connected graph in which every edge belongs to at most one simple cycle. Notice that a unicyclic graph is a cactus with only one cycle and a trees is a cactus having no cycle.

Theorem 3.1. *Let G be a cactus. Then $\gamma_c(G) = \gamma_{\mathrm{wcon}}(G)$ if and only if:*
   (1) *for every cycle in G, $C = C_i$, $i \in \{5, 6\}$ we have $d(v) \geq 3$ for every $v \in V(C_i)$ or C_i has two adjacent vertices of degree 2 and*
   (2) *for every $C = C_i$, $i \geq 7$ in G, $d(v) \geq 3$ for every $v \in V(C)$.*

Proof. Assume (1) and (2) hold. Let D be a minimum connected dominating set of G. We show that D is also weakly convex. Suppose it is not true. Then there exist vertices x and y such that $\{x, y\} \subset D$ and $d_{G[D]}(x, y) > d_G(x, y)$. This implies that x and y belong to a cycle C of the length at least 5. Since (2) holds, C can have the length 5 or 6. In fact, if C is longer than 6, then from (2) every vertex on C is a cut-vertex, hence from Observation 2.8, $V(C) \subseteq V_C \subseteq D$, which contradicts $d_{G[D]}(x, y) > d_G(x, y)$. Hence, only one vertex from C does not belong to D, let us say $z \notin D$. Notice that $d_G(z) = 2$ and z is a common neighbour of x and y. The vertices from $V(C) - \{z\}$ form an $(x - y)$-path P such that $V(P) \subseteq D$. Since D is minimum, at most one vertex of P has degree 2 (if not we could find a smaller connected dominating set of G). It gives a contradiction with (1) and finally $\gamma_c(G) = \gamma_{\mathrm{wcon}}(G)$.

Conversely, let D be a minimum weakly convex dominating set of G. Since $\gamma_c(G) = \gamma_{\mathrm{wcon}}(G)$, D is also a minimum connected dominating set of G. Suppose first that (1) does not hold. Thus there exists a cycle C_i, $i \in \{5, 6\}$, such that C_i has at least one vertex of degree 2 and the set of the vertices of degree 2 of C_i is independent. Since vertices of degree 3 of C_i (as cut-vertices, by Observation 2.8) belong to D and since D is weakly convex and $i \in \{5, 6\}$, we obtain that $V(C_i) \subseteq D$. We get a contradiction becase $D - \{v\}$, where $d_G(v) = 2$ and $v \in V(C_i)$, is a connected dominating set of G. Next, suppose that (2) does not hold. Thus there exists a vertex v of degree 2 on a cycle C_i, $i \geq 7$. Since D

is weakly convex, $V(C_i) \subset D$ and then $D - \{v\}$ is a connected dominating set of G, a contradiction. □

3.2. **Distance-hereditary graphs.** A *distance-hereditary graph* is a connected graph in which every connected induced subgraph is isometric (that is, the distance of any two vertices in any connected induced subgraph equals their distance in the graph). From the definition we get the following

Lemma 3.2. *If G is a distance-hereditary graph, then $\gamma_{\mathrm{wcon}}(G) = \gamma_c(G)$.*

Proof. Let D be a connected dominating set of G. If $|D| = 1$, then $\gamma_{\mathrm{wcon}}(G) = \gamma_c(G)$. Now let x, y be two different vertices in D. Then there is an induced $(x - y)$-path with all its vertices in D. Using the fact that G is distance-hereditary we conclude that $d_{G[D]}(x, y) = d_G(x, y)$. Then D is weakly convex and we can get $\gamma_{\mathrm{wcon}}(G) = \gamma_c(G)$. □

Notice that the converse is not true, i.e. there exist graphs G with equality $\gamma_{\mathrm{wcon}}(G) = \gamma_c(G)$ which are not distance-hereditary; the example of such a graph can be a corona of a cycle C_7, i.e. $G = C_7 \circ K_1$.

A block in a graph G is a maximal connected subgraph H of G such that H does not contain any cut-vertex of H. A *block graph* is a connected graph G such that every block in G is a complete graph). Since a block graph is a distance-hereditary graph, we conclude the following

Corollary 3.3. *If G is a connected block graph, then $\gamma_c(G) = \gamma_{\mathrm{wcon}}(G)$.*

We say that a graph G is H–free if G does not contain H as an induced subgraph. In particular, P_4–free graph is called a *cograph*. Since a cograph is a distance-hereditary graph, we conclude the following

Corollary 3.4. *If G is a connected cograph, then $\gamma_c(G) = \gamma_{\mathrm{wcon}}(G)$.*

3.3. **Chordal graphs.** Now we foccus our attention on chordal graphs. Recall that a graph is chordal if every cycle of length at least 4 has a chord.
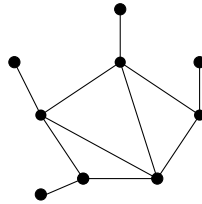

FIGURE 2. Graph H^*.

Theorem 3.5. *Let $G = (V, E)$ be a connected chordal graph of order n. If G is H^*–free (where H^* is a graph from Figure 2), then $\gamma_c(G) = \gamma_{\mathrm{wcon}}(G)$.*

Proof. Suppose that the statement is not true and let G be a minimal counterexample, i.e., G is a connected chordal graph without induced H^*, but $\gamma_c(G) < \gamma_{\mathrm{wcon}}(G)$. Let D be any minimum connected dominating set of G (obviously D is not weakly convex).

We consider cases depending on $\mathrm{diam}(G[D]) = d$. If $d \leq 2$, then $\gamma_c(G) = \gamma_{\mathrm{wcon}}(G)$, a contradiction. If $d = 3$, then there exist $x, y \in D$ such that $d_{G[D]}(x, y) > d_G(x, y)$. Observe that $d_{G[D]}(x, y) = 3 = d$ otherwise $d_G(x, y) = 1$, a contradiction.
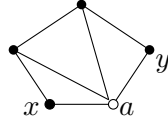

FIGURE 3. Graph H_a'.

Thus $d_G(x, y) = 2$ and there is $a \notin D$ such that $a \in N_G(x) \cap N_G(y)$. In this case we find a cycle C_5 in G such that four vertices belong to D and the other, namely a, does not belong to D.

Using the fact that G is chordal and $d = 3$ we get that the induced subgraph $G[C_5]$ contains a subgraph isomorphic to H_a' shown in the Figure 3 where $a \notin D$ and the rest black nodes belong to D. Since D is not weakly convex, for every vertex $v \neq a$ of C_5 exists $v' \in N_G(v)$ such that $v' \notin N_G(V(C_5) - \{v\})$, (otherwise $(D - \{v\}) \cup \{a\}$ would be a weakly convex dominating set of G of cardinality $\gamma_c(G)$).

From above and using that G is chordal and $d_{G[D]}(x, y) = 3$, we find an induced subgraph isomorphic to H^*, a contradiction.

Now we assume $d > 3$.

There are $x, y \in D$ such that $d_{G[D]}(x, y) = d > d_G(x, y)$. Because $t = d_G(x, y) > 1$ there exist a $(x - y)$-geodesic in $G[D]$, say

$$P = (x = z_0, z_1, z_2, \cdots, z_{d-1}, z_d = y)$$

and another $(x - y)$-geodesic in G, namely $P' = (x, w_1, w_2, \cdots, w_{t-1}, y)$ such that $w_i \notin D$ for $i \in \{1, \cdots, t-1\}$. Then we have a cycle C_{d+t}, where $d + t \geq 6$.

Using again the chordal condition and the fact that both P and P' are geodesic, we get that the possible chords are edges of the form zw with $w \in V(P')$ and $z \in V(P)$. Note that every two consecutive vertices z_i, z_{i+1} of P must share at least one neighbour in P'.

Case 1. If there exists a vertex a of $P' - \{x, y\}$ such that four consecutive vertices can be found in $N[a] \cap P$, then G has a subgraph isomorphic to H_a' and, with similar arguments as before we conclude that G has an induced subgraph isomorphic to H^*, a contradiction.

Case 2. Otherwise, since G is chordal, there are two consecutive vertices w_i, w_{i+1} and a sequence of five vertices in P namely $z_j, \cdots, z_{j+4}$ such that $N_G[w_i] \cap P = \{z_j, z_{j+1}, z_{j+2}\}$ and $N_G[w_{i+1}] \cap P = \{z_{j+2}, z_{j+3}, z_{j+4}\}$.

Denote by $H' = G[\{w_i, w_{i+1}, z_j, z_{j+1}, z_{j+2}, z_{j+3}, z_{j+4}\}]$ the induced subgraph of G.

Note that with an argument similar to the used above we get that: If the edge $w_i z_{j+4}$ is in H' then also the edge $w_i z_{j+3}$ is in H' and then we have the Case 1. The same occurs if the edge $w_{i+1} z_j$ is in H'.
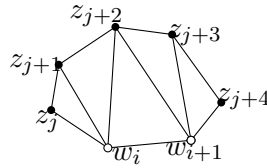
FIGURE 4. Graph H'.

Using similar arguments as before we get that H' is an induced subgraph of G and also $H'_{w_i,w_{i+1}}$.
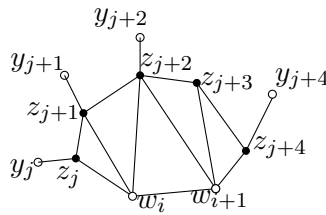
FIGURE 5. Graph $H'_{w_i,w_{i+1}}$.

If we delete y_{j+4} and z_{j+3} from $H'_{w_i,w_{i+1}}$, then we have an induced subgraph isomorphic to H^*. From the fact that G is H^*-free we conclude the result. □

3.4. **Perfect graphs.** We say that a graph G is $(\gamma_c - \gamma_{\text{wcon}})$-*perfect* if $\gamma_c(H) = \gamma_{\text{wcon}}(H)$ for every connected induced subgraph H of G.

The following result shows that chordal H^*-free graphs (where H^* is a graph from Figure 2) are $(\gamma_c - \gamma_{\text{wcon}})$-perfect.

Theorem 3.6. *Let $G = (V, E)$ be a connected chordal graph of order n. Then G is $(\gamma_c - \gamma_{\text{wcon}})$-perfect if and only if G is an H^*-free graph.*

Proof. Suppose $\gamma_c(H) = \gamma_{\text{wcon}}(H)$ for every induced subgraph H of G. Then $H \neq H^*$ since $\gamma_c(H^*) < \gamma_{\text{wcon}}(H^*)$.

Now, suppose G is H^*-free. Let H be an induced subgraph of G. Hence, H is chordal and H^*-free. By Theorem 3.5, $\gamma_c(H) = \gamma_{\text{wcon}}(H)$ and thus, G is $(\gamma_c - \gamma_{\text{wcon}})$-perfect. □

Now we generalize the necessary condition for any $(\gamma_c - \gamma_{\text{wcon}})$-perfect graph.

For a cycle C of G, denote by H a subgraph $G[N[V(C)]]$ induced by the closed neighbourhood of the vertices from the cycle C. In general, we have the following result.

Lemma 3.7. *If a graph G is a perfect graph, then G does not contain induced cycle greater than six and for every cycle (not necessarily induced) C of length five or six one of the two following conditions hold:*

(1) *there are two consecutive vertices of C such that they are not cut-vertices of H;*
(2) *for every vertex v of C such that v is a cut-vertex of H, their neighbours on C are connected by an edge or have a common neighbour on C different then v.*

Proof. Assume G to be a perfect graph. If there is an induced cycle C_p in G such that $p \geq 7$, then, since $\gamma_c(C_p) < \gamma_{\text{wcon}}(C_p)$, G is not perfect; so we conclude that for a perfect graph G there is no induced cycle of length greater than six.

Denote by A the set of vertices of C such that they are cut-vertices of H and let $B = V(C) - A$.

Assume C is a cycle of length five in G. Suppose that (1) does not hold; then $|A| \geq 3$, $|B| \leq 2$ and B is an independent set. Denote $C = (a, b, c, d, e, a)$; without loss of generality let $a, b, d \in A$. Since $\{a, b, d\} \subseteq A$, they have their private neighbours with respect to $V(C)$, a', b', d', respectively and $\{a'b', b'd', a'd'\} \notin E(G)$. Suppose that also (2) does not hold - then C is an induced cycle of G. In this case $F = G[\{a, a', b, b', c, d, d', e\}]$ is an induced subgraph of G such that $\gamma_c(F) < \gamma_{\text{wcon}}(F)$, a contradiction.

Now let C be a cycle of length six in G. If (1) does not hold, then $|A| \geq 3$, $|B| \leq 3$ and B is an independent set. Denote $C = (a, b, c, d, e, f, a)$; without loss of generality let $\{a, c, e\} \subseteq A$. Similarly like in the previous case, these vertices have their private neighbours with respect to $V(C)$, a', c', e', respectively such that $\{a'c', c'e', a'e'\} \notin E(G)$. If also (2) does not hold, then C is an induced cycle of G or there is one chord between vertices of A in C. In both cases $F' = G[\{a, a', b, c, c', d, e, e', f\}]$ is an induced subgraph of G such that $\gamma_c(F') < \gamma_{\text{wcon}}(F')$, a contradiction. $\qquad \square$

Notice that there are graphs which satisfy conditions (1) and (2), but are not perfect. Example of such a graph is graph G from Figure 6 for which there exists a cycle C of length five such that there are two consecutive non-cut-vertices a, b of G in C; but G is not perfect (observe that the set of all support vertices V_S is a minimum connected dominating set of G and $V_S \cup \{a, b\}$ is a minimum weakly convex set of G).
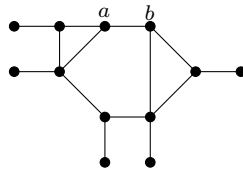

FIGURE 6. Example of a graph, which satisfies two conditions from Lemma 3.7, but is not perfect.

4. Edge removing

In this section we will analyze the influence of the edge removing on the domination numbers for graphs such that every vertex is simplicial or cut-vertex and also for unicycle graphs. As consequence we will conclude that weakly convex domination number is an interpolating function.

First, we see that in general, deleting an edge can arbitrarily increase and arbitrarily decrease the weakly convex domination number.

We say that edge $e \in E(G)$ is a *cut edge* if $G - e$ is not connected.

Theorem 4.1. *For every integer k there is a connected graph G and an edge not cut $e \in E(G)$ such that*

$$\gamma_{\mathrm{wcon}}(G - e) - \gamma_{\mathrm{wcon}}(G) = k.$$

Proof. If $k = 0$, then $G = C_3$. For $k > 0$ we show the construction of a graph G such that $\gamma_{\mathrm{wcon}}(G - e) - \gamma_{\mathrm{wcon}}(G) = k$ for some edge $e \in E(G)$. We begin with a cycle $C_{2k+2} = (x_1, v_1, v_2, \ldots, v_k, x_2, u_k, u_{k-1}, \ldots u_1, x_1)$. Next we add to this cycle: new vertices x_1', x_2', v_i' and edges $x_1 x_1', x_2 x_2', v_i v_i', v_i u_i$ where $1 \leq i \leq k$.

The final graph is illustrated in the Figure 7. It is easy to observe that all supports of G form a minimum weakly convex dominating set of G, so $\gamma_{\mathrm{wcon}}(G) = k + 2$. Let $G' = G - x_1 v_1$. Hence, all supports again belong to the minimum weakly convex dominating set of G'. But the distance between x_1 and x_2 equals to $k + 1$ and the only shortest $(x_1 - x_2)$-path consists of vertices $u_1, u_2, \ldots, u_k$. It implies that $\gamma_{\mathrm{wcon}}(G) \geq 2k + 2$. On the other hand, all but leaves vertices of G form a weakly convex dominating set of G', i.e. $\gamma_{\mathrm{wcon}}(G) \leq 2k + 2$. Hence, $\gamma_{\mathrm{wcon}}(G') = 2k + 2$. Summing up, we obtained that $\gamma_{\mathrm{wcon}}(G') - \gamma_{\mathrm{wcon}}(G) = k$.

Next, for $k < 0$ we show the construction of a graph such that $\gamma_{\mathrm{wcon}}(G - e) - \gamma_{\mathrm{wcon}}(G) = k$ for some edge e of G. We begin with a cycle $C_{2|k|+6} = (x_1, v_1, v_2, \ldots, v_{|k|+2}, x_2, u_{|k|+2}, u_{|k|+1}, \ldots u_1, x_1)$. Next we add to this cycle: new vertices x_1', x_2', v_i' and edges $x_1 x_1', x_2 x_2', v_i v_i', v_i u_i$ where $1 \leq i \leq |k| + 2$, and edge $v_1 u_3$. The final graph is illustrated in the Figure 8. Let D be a minimum weakly convex dominating set of G. All supports of G belong to D. Moreover, the distance between v_1 and x_2 equals to $|k| + 1$ and $u_3, u_4, \ldots, u_{|k|+2}$ belong to the shortest $(v_1 - x_2)$-path, so they also belongs to D. On the other hand, $V - (V_L \cup \{u_1, u_2\})$ is a weakly convex dominating set of G. Hence, $\gamma_{\mathrm{wcon}}(G) = 2|k| + 4$. Let us consider a graph $G' = G - u_{|k|+2} x_2$ It is easy to check, that the set of all supports of G' is a minimum weakly convex dominating set of G', and so $\gamma_{\mathrm{wcon}}(G') = |k| + 4$. Finally, $\gamma_{\mathrm{wcon}}(G') - \gamma_{\mathrm{wcon}}(G) = k$. □

In [5] is shown that removing an edge can not decrease the connected domination number, but it can increase it by at most two:

Theorem 4.2. [5] *If e is an edge of G and if G and $G - e$ are connected, then*

$$\gamma_c(G) \leq \gamma_c(G - e) \leq \gamma_c(G) + 2.$$
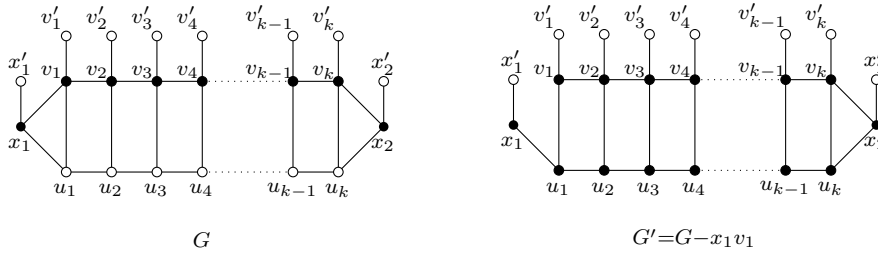
Hence, $\gamma_c(G - e) - \gamma_c(G) \in \{0, 1, 2\}$.

FIGURE 7. Construction of a graph G such that $\gamma_{\mathrm{wcon}}(G-e) - \gamma_{\mathrm{wcon}}(G) = k$ for some edge e, where k is a positive integer.
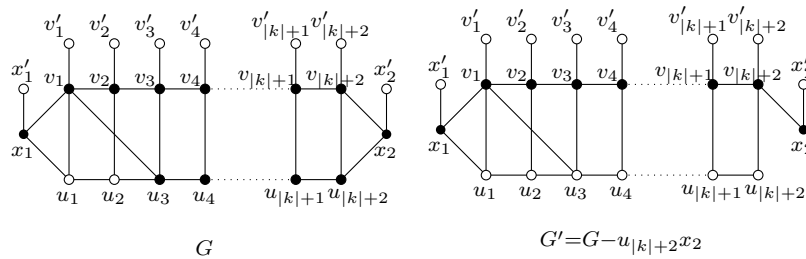

FIGURE 8. Construction of a graph G such that $\gamma_{\mathrm{wcon}}(G-e) - \gamma_{\mathrm{wcon}}(G) = k$ for some edge e, where k is a negative integer.

We are going to study the influence of removing an edge over the weakly convex domination number when we consider unicycle graphs or graphs such that every vertex is simplicial or cut-vertex.

4.1. **Graphs such that every vertex is simplicial or cut-vertex.** In this part, we consider graphs with at least three vertices such that every vertex is simplicial or cut-vertex. Examples of such graphs are block graphs. We get another example if we consider graphs G with $g(G) \geq 7$ and $\gamma_{\mathrm{wcon}}(G) = \gamma_c(G)$ (see Theorem 2.10).

Observe that if each vertex is a simplicial or a cut-vertex in G then by Corollary 2.9 the unique minimum connected, and also weakly convex, dominating set of G is the set of all cut-vertices of G.

Lemma 4.3. *Let G be a graph such that each vertex is either a cut-vertex or a simplicial vertex. If e is not a cut edge of G, then*

$$\gamma_c(G) \leq \gamma_c(G-e) \leq \gamma_c(G) + 1.$$

Proof. Let D_0 be a minimum connected dominating set of $G - e$. Then D_0 is also a connected dominating set of G and $\gamma_c(G) \leq |D_0| = \gamma_c(G-e)$.

Now let D be a minimum connected dominating set of G. Then $D = V_C$. We consider two cases for $e = uv$:

Case 1. If $u, v \in V - D$, then u, v are simplicial vertices. Then $D = V_C$ is a

connected dominating set of $G - e$ and $\gamma_c(G - e) \leq |D| = \gamma_c(G) \leq \gamma_c(G) + 1$.
Case 2. If $|\{u, v\} \cap D| \geq 1$, then since uv is not a cut edge, there is a vertex $v' \in N_G(v) \cap N_G(u)$. Then $D \cup \{v'\}$ is a connected dominating set of $G - e$ and $\gamma_c(G - e) \leq |D| + 1 = \gamma_c(G) + 1$.

$\square$

Lemma 4.4. *Let G be a graph such that each vertex is either a cut-vertex or a simplicial vertex. If e is not a cut edge of G, then $\gamma_c(G - e) = \gamma_{\mathrm{wcon}}(G - e)$.*

Proof. We consider two cases:
Case 1. $e = uv$, where both u, v are simplicial vertices. Then V_C is a minimum connected and weakly convex dominating set of $G - e$ and $|V_C| = \gamma_c(G - e) = \gamma_{\mathrm{wcon}}(G - e)$.
Case 2. $|\{u, v\} \cap V_C| \geq 1$. Then, since uv is not a cut edge, $|N_G(u) \cap N_G(v)| \geq 1$. If $(N_G(u) \cap N_G(v)) \cap V_C \neq \emptyset$, then V_C is a minimum connected and weakly convex dominating set of $G - e$ and $\gamma_c(G) = \gamma_{\mathrm{wcon}}(G) = |V_C|$. Otherwise, $V_C \cup \{v'\}$ is a minimum connected and weakly convex dominating set of $G - e$ and $\gamma_c(G) = \gamma_{\mathrm{wcon}}(G) = |V_C| + 1$ for any $v' \in N_G(u) \cap N_G(v)$.

$\square$

From Corollary 2.9, Lemma 4.3 and Lemma 4.4, we have the following corollary.

Corollary 4.5. *Let G be a graph such that each vertex is either a cut-vertex or a simplicial vertex. If e is not a cut edge of G, then*

$$\gamma_{\mathrm{wcon}}(G) \leq \gamma_{\mathrm{wcon}}(G - e) \leq \gamma_{\mathrm{wcon}}(G) + 1.$$

4.2. Unicyclic graphs.

Observation 4.6. *Let G be an unicyclic graph with the only cycle C_p and let uv be any cycle edge. Thus*

(1) $G - uv$ *is a spanning tree T of G;*
(2) $V_L(G) \subseteq V_L(T)$;
(3) $v \in V_L(T) - V_L(G)$ *if and only if $d_G(v) = 2$.*

Theorem 4.7. *Let G be a connected unicyclic graph with the only cycle C_p. If $e = uv$ is a cycle edge then*

$$\gamma_{\mathrm{wcon}}(G) - 2 \leq \gamma_{\mathrm{wcon}}(G - e) \leq \gamma_{\mathrm{wcon}}(G) + 2.$$

Proof. Note that $G - e$ is a spanning tree T of G. We have that $D_0 = V - V_L(T)$ is a minimum weakly convex dominating set of T (Observation 2.8). Denote $a = \gamma_{\mathrm{wcon}}(G)$. We analyze the following three cases:

Case 1. If $u, v \in D_0$, then, since $u, v \notin V_L(T)$, $d_G(u) \geq 3$ and $d_G(v) \geq 3$. Thus $V_L(T) = V_L(G)$ and $D_0 = V - V_L(G)$.

If $C = C_p$ with $p \geq 7$ then $V - V_L(G)$ is a minimum weakly convex dominating set of G and $\gamma_{\mathrm{wcon}}(G) = |D_0| = \gamma_{\mathrm{wcon}}(G - e)$.

If $p = 4, 5, 6$ and there are x, y two consecutive vertices in C with degree two then $V - (V_L(G) \cup \{x, y\})$ is a minimum weakly convex dominating set of G, and $\gamma_{\mathrm{wcon}}(G) = |D_0| - 2 = \gamma_{wcon}(T) - 2$. In other case, for $p = 5, 6$, D_0 is a minimum weakly convex dominating set of G, so $\gamma_{\mathrm{wcon}}(G) = |D_0| = \gamma_{\mathrm{wcon}}(T)$. For $p = 4$ if

there exists a vertex in C of degree 2, then $\gamma_{\text{wcon}}(G) = |D_0| - 1 = \gamma_{\text{wcon}}(T) - 1$, otherwise $\gamma_{\text{wcon}}(G) = |D_0| = \gamma_{\text{wcon}}(T)$.

If $p = 3$, then denote the third vertex on the C_3 by w. Depending on $w \in V_C(G)$ or $w \notin V_C(G)$, D_0 or $D_0 - \{w\}$ is a minimum weakly convex dominating set of G.

Thus, in this case, $\gamma_{\text{wcon}}(T) \in \{a, a+1, a+2\}$.

Case 2. Assume $|D_0 \cap \{u, v\}| = 1$, without loss of generality let $u \in D_0, v \in V - D_0$. Then $d_G(u) \geq 3$ and $d_G(v) = 2$. Note that $v \in V_L(T) - V_L(G)$. We have that $D_0 = V - (V_L(G) \cup \{v\})$ is a minimum weakly convex dominating set of T.

If $C = C_p$ with $p \geq 7$ then $V - V_L(G)$ is a minimum weakly convex dominating set of G and $\gamma_{\text{wcon}}(G) - 1 = |D_0| = \gamma_{\text{wcon}}(T)$.

If $p = 4, 5, 6$ and there are x, y two consecutive vertices in C with degree two then $V - (V_L(G) \cup \{x, y\})$ is a minimum weakly convex dominating set of G and $\gamma_{\text{wcon}}(G) + 1 = |D_0| = \gamma_{\text{wcon}}(T)$. In other case, for $p = 5, 6$, $V - V_L(G)$ is a minimum weakly convex dominating set of G, i.e., $\gamma_{\text{wcon}}(G) - 1 = \gamma_{\text{wcon}}(T)$, and for $p = 4$, D_0 is a minimum weakly convex dominating set of G, i.e. $\gamma_{\text{wcon}}(G) = \gamma_{\text{wcon}}(T)$.

If $p = 3$, then denote the third vertex on the C_3 by w. Depending on $w \in V_C(G)$ or $w \notin V_C(G)$, D_0 or $D_0 - \{w\}$ is a minimum weakly convex dominating set of G.

Thus, in this case, $\gamma_{\text{wcon}}(T) \in \{a-1, a, a+1\}$.

Case 3. Now let $u, v \in V - D_0$. Then $d_G(u) = 2 = d_G(v)$. Note that $u, v \in V_L(T) - V_L(G)$. We have that $D_0 = V - (V_L(G) \cup \{u, v\})$ is a minimum weakly convex dominating set of T.

If $C = C_p$ with $p \geq 7$, then $V - V_L(G)$ is a minimum weakly convex dominating set of G and $\gamma_{\text{wcon}}(G) - 2 = |D_0| = \gamma_{\text{wcon}}(T)$. If $p = 4, 5, 6$, then, since there are two consecutive vertices u, v in C with degree two, $V - (V_L(G) \cup \{u, v\})$ is also a minimum weakly convex dominating set of G and $\gamma_{\text{wcon}}(G) = \gamma_{\text{wcon}}(T)$.

Thus, in this case, $\gamma_{\text{wcon}}(T) \in \{a, a-2\}$.

We conclude that, $\gamma_{\text{wcon}}(T) \in \{a-2, a-1, a, a+1, a+2\}$ for every spanning tree T. $\square$

Let $\mathcal{T}$ be the set of all spanning trees of G. We say that function Π with integer values *interpolates over graph G* if and only if $\Pi(\mathcal{T}(G))$ is an interval; i.e the set $\Pi(\mathcal{T}(G)) = \{\Pi(T) : T \in \mathcal{T}(G)\}$ consists of consecutive integers. Function Π is an *interpolating function* if Π interpolates over every connected graph. The interpolation properties of some domination parameters were presented in [3] and in particular, it was shown that γ_c is an interpolating function.

In [9] Topp and Vestergaard proved the following theorem.

Theorem 4.8. [9] *Function Π with integer values is an interpolating function if and only if Π interpolates over every unicyclic graph.*

Now, we are in condition to show that weakly convex domination number is also an interpolating function.

Corollary 4.9. γ_{wcon} *is an interpolating function.*

Proof. Using Theorem 4.7 we have that $\gamma_{\mathrm{wcon}}(T) \in \{a-2, a-1, a, a+1, a+2\}$ for every spanning tree T, so by Theorem 4.8 we have that γ_{wcon} is an interpolation function. $\qquad\square$

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