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On the partition dimension of trees

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ABSTRACT

Given an ordered partition $\Pi = \{P_1, P_2, \ldots, P_t\}$ of the vertex set V of a connected graph $G = (V, E)$, the *partition representation* of a vertex $v \in V$ with respect to the partition Π is the vector $r(v|\Pi) = (d(v, P_1), d(v, P_2), \ldots, d(v, P_t))$, where $d(v, P_i)$ represents the distance between the vertex v and the set P_i. A partition Π of V is a *resolving partition* of G if different vertices of G have different partition representations, i.e., for every pair of vertices $u, v \in V$, $r(u|\Pi) \neq r(v|\Pi)$. The *partition dimension* of G is the minimum number of sets in any resolving partition of G. In this paper we obtain several tight bounds on the partition dimension of trees.

1. Introduction

The concepts of resolvability and location in graphs were described independently by Harary and Melter [9] and Slater [17]. After these papers were published several authors developed diverse theoretical works about this topic [3,2,4–10,14,19]. Slater described the usefulness of these ideas into long range aids to navigation [17]. Also, these concepts have some applications in chemistry for representing chemical compounds [12,13] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [15]. Other applications of this concept to navigation of robots in networks and other areas appear in [5,11,14]. Some variations on resolvability or location have been appearing in the literature, like those about conditional resolvability [16], locating domination [10], resolving domination [1] and resolving partitions [4,7,8,19].

Given a graph $G = (V, E)$ and an ordered set of vertices $S = \{v_1, v_2, \ldots, v_k\}$ of G, the *metric representation* of a vertex $v \in V$ with respect to S is the vector $r(v|S) = (d(v, v_1), d(v, v_2), \ldots, d(v, v_k))$, where $d(v, v_i)$ denotes the distance between the vertices v and v_i, $1 \leq i \leq k$. We say that S is a *resolving set* of G if different vertices of G have different metric representations, i.e., for every pair of distinct vertices $u, v \in V$, $r(u|S) \neq r(v|S)$. The *metric dimension*[1] of G is the minimum cardinality of any resolving set of G, and it is denoted by $\dim(G)$. The metric dimension of graphs is studied in [3,2,4–6,18].

Given an ordered partition $\Pi = \{P_1, P_2, \ldots, P_t\}$ of the vertices of G, the *partition representation* of a vertex $v \in V$ with respect to the partition Π is the vector $r(v|\Pi) = (d(v, P_1), d(v, P_2), \ldots, d(v, P_t))$, where $d(v, P_i)$, with $1 \leq i \leq t$, represents the distance between the vertex v and the set P_i, i.e., $d(v, P_i) = \min_{u \in P_i}\{d(v, u)\}$. We say that Π is a *resolving partition* of G if different vertices of G have different partition representations, i.e., for every pair of distinct vertices $u, v \in V$, $r(u|\Pi) \neq$

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[1] Also called the locating number.

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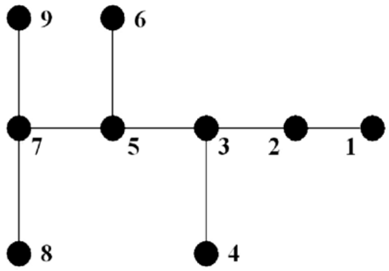

Fig. 1. In this tree the vertex 3 is an exterior major vertex of terminal degree two: 1 and 4 are terminal vertices of 3.

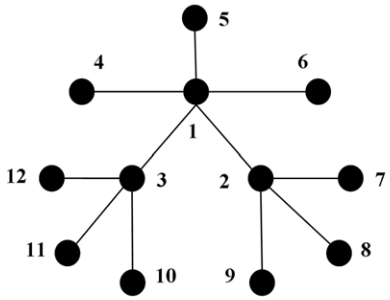

Fig. 2. $\Pi = \{\{1, 4, 9, 12\}, \{3, 5, 8, 11\}, \{2, 6, 7, 10\}\}$ is a resolving partition.

$r(v|\Pi)$. The *partition dimension* of G is the minimum number of sets in any resolving partition of G and it is denoted by $pd(G)$. The partition dimension of graphs is studied in [4,7,8,18].

2. The partition dimension of trees

It is natural to think that the partition dimension and metric dimension are related; in [7] it was shown that for any nontrivial connected graph G we have

$$pd(G) \leq \dim(G) + 1. \tag{1}$$

We know that the partition dimension of any path is two. That is, for any path graph P, it follows $pd(P) = \dim(P) + 1 = 2$. A formula for the dimension of trees that are not paths has been established in [5,9,17]. In order to present this formula, we need additional definitions. A vertex of degree at least 3 in a tree T will be called a *major vertex* of T. Any leaf u of T is said to be a *terminal vertex* of a major vertex v of T if $d(u, v) < d(u, w)$ for every other major vertex w of T. The *terminal degree* of a major vertex v is the number of terminal vertices of v. A major vertex v of T is an *exterior major vertex* of T if it has positive terminal degree.

Let $n_1(T)$ denote the number of leaves of T, and let $\mathrm{ex}(T)$ denote the number of exterior major vertices of T. We can now state the formula for the dimension of a tree [5,9,17]: if T is a tree that is not a path, then

$$\dim(T) = n_1(T) - \mathrm{ex}(T). \tag{2}$$

As a consequence, if T is a tree that is not a path, then

$$pd(T) \leq n_1(T) - \mathrm{ex}(T) + 1. \tag{3}$$

The above bound is tight, it is achieved for the graph in Fig. 1 where $\Pi = \{\{8\}, \{4, 9\}, \{1, 2, 3, 5, 6, 7\}\}$ is a resolving partition and $pd(T) = 3$. However, there are graphs for which the following bound gives better result than bound (3), for instance, the graph in Fig. 2.

Let $S = \{s_1, s_2, \ldots, s_\kappa\}$ be the set of exterior major vertices of $T = (V, E)$ with terminal degree greater than one; let $\{s_{i1}, s_{i2}, \ldots, s_{il_i}\}$ be the set of terminal vertices of s_i and let $\tau = \max_{1 \leq i \leq \kappa}\{l_i\}$. With the above notation we have the following result.

Theorem 1. *For any tree T which is not a path,*

$$pd(T) \leq \kappa + \tau - 1.$$

Proof. For a terminal vertex s_{ij} of a major vertex $s_i \in S$ we denote by S_{ij} the set of vertices of T, different from s_i, belonging to the $s_i - s_{ij}$ path. If $l_i < \tau - 1$, we assume $S_{ij} = \emptyset$ for every $j \in \{l_i + 1, \ldots, \tau - 1\}$. Now for every $j \in \{2, \ldots, \tau - 1\}$, let

$B_j = \cup_{i=1}^{\kappa} S_{ij}$ and, for every $i \in \{1, \ldots, \kappa\}$, let $A_i = S_{i1}$. Let us show that $\Pi = \{A, A_1, A_2, \ldots, A_\kappa, B_2, \ldots, B_{\tau-1}\}$ is a resolving partition of T, where $A = V - \left(\left(\cup_{i=1}^{\kappa} A_i\right) \cup \left(\cup_{j=2}^{\tau-1} B_j\right)\right)$. We consider two different vertices $x, y \in V$. Note that if x and y belong to different sets of Π, we have $r(x|\Pi) \neq r(y|\Pi)$.

Case 1: $x, y \in S_{ij}$. If $j = \tau$, then we have that $x, y \in A$ and it follows that $d(x, A_i) \neq d(y, A_i)$. Otherwise, we obtain that $d(x, A) = d(x, s_i) \neq d(y, s_i) = d(y, A)$.

Case 2: $x \in S_{ij}$ and $y \in S_{kl},\ i \neq k$. If $j = 1$ or $l = 1$, then x and y belong to different sets of Π. So we suppose $j \neq 1$ and $l \neq 1$. Hence, if $d(x, A_i) = d(y, A_i)$, then

$$\begin{aligned} d(x, A_k) &= d(x, s_i) + d(s_i, s_k) + 1 \\ &= d(x, A_i) + d(s_i, s_k) \\ &= d(y, A_i) + d(s_i, s_k) \\ &= d(y, s_k) + 2d(s_k, s_i) + 1 \\ &= d(y, A_k) + 2d(s_k, s_i) \\ &> d(y, A_k). \end{aligned}$$

Case 3: $x \in S_{i\tau}$ and $y \in A - \cup_{l=1}^{\kappa} S_{l\tau}$. If $d(x, A_i) = d(y, A_i)$, then $d(x, s_i) = d(y, s_i)$. Since $y \notin S_{l\tau}, l \in \{1, \ldots, \kappa\}$, there exists $A_j \in \Pi, j \neq i$, such that s_i does not belong to the $y - s_j$ path. Now let Y be the set of vertices belonging to the $y - s_j$ path, and let $v \in Y$ such that $d(s_i, v) = \min_{u \in Y}\{d(s_i, u)\}$. Hence,

$$\begin{aligned} d(x, A_j) &= d(x, s_i) + d(s_i, v) + d(v, s_j) + 1 \\ &= d(y, s_i) + d(s_i, v) + d(v, s_j) + 1 \\ &= d(y, v) + 2d(v, s_i) + d(v, s_j) + 1 \\ &= d(y, A_j) + 2d(v, s_i) \\ &> d(y, A_j). \end{aligned}$$

Case 4: $x, y \in A' = A - \cup_{l=1}^{\kappa} S_{l\tau}$. If for some exterior major vertex $s_i \in S$, the vertex x belongs to the $y - s_i$ path or the vertex y belongs to the $x - s_i$ path, then $d(x, A_i) \neq d(y, A_i)$. Otherwise, there exist at least two exterior major vertices s_i, s_j such that the $x - y$ path and the $s_i - s_j$ path share more than one vertex (if not, then $x, y \notin A'$). Let W be the set of vertices belonging to the $s_i - s_j$ path. Let $u, v \in W$ such that $d(x, u) = \min_{z \in W}\{d(x, z)\}$ and $d(y, v) = \min_{z \in W}\{d(y, z)\}$. We suppose, without loss of generality, that $d(s_i, u) > d(v, s_i)$. Hence, if $d(x, v) = d(y, v)$, then $d(x, u) \neq d(y, u)$, and if $d(x, u) = d(y, u)$, then $d(x, v) \neq d(y, v)$. We have

$$\begin{aligned} d(x, A_j) &= d(x, u) + d(u, s_j) + 1 \\ &\neq d(y, u) + d(u, s_j) + 1 \\ &= d(y, A_j) \end{aligned}$$

or

$$\begin{aligned} d(x, A_i) &= d(x, v) + d(v, s_i) + 1 \\ &\neq d(y, v) + d(v, s_i) + 1 \\ &= d(y, A_i). \end{aligned}$$

Therefore, for different vertices $x, y \in V$, we have $r(x|\Pi) \neq r(y|\Pi)$. □

One example where $pd(T) = \kappa + \tau - 1$ is the tree in Fig. 1.

Any vertex adjacent to a leaf of a tree T is called a *support* vertex. In the following result ξ denotes the number of support vertices of T and θ denotes the maximum number of leaves adjacent to a support vertex of T.

Corollary 2. *For any tree T of order $n \geq 2$, $pd(T) \leq \xi + \theta - 1$.*

Proof. If T is a path, then $\xi = 2$ and $\theta = 1$, so the result follows. Now we suppose T is not a path. Let v be an exterior major vertex of terminal degree τ. Let x be the number of leaves adjacent to v and let $y = \tau - x$. Since $\kappa + y \leq \xi$ and $x \leq \theta$, we deduce $\kappa + \tau \leq \xi + \theta$. □

The above bound is achieved, for instance, for the graph of order six composed of two support vertices a and b, where a is adjacent to b and four leaves; two of them are adjacent to a and the other two leaves are adjacent to b. One example of a graph for which Theorem 1 gives a better result than Corollary 2 is the graph in Fig. 1.

Since the number of leaves, $n_1(T)$, of a tree T is bounded below by $\xi + \theta - 1$, Corollary 2 leads to the following bound.

Remark 3. For any tree T of order $n \geq 2$, $pd(T) \leq n_1(T)$.

Now we are going to characterize all the trees for which $pd(T) = n_1(T)$. It was shown in [7] that $pd(G) = 2$ if and only if the graph G is a path. So by the above remark we obtain the following result.

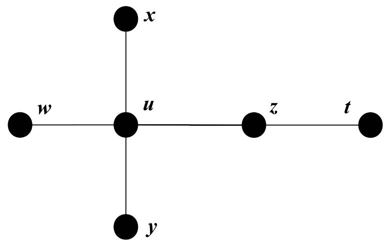

Fig. 3. A comet graph where $3 = \theta = pd(T) < n_1(T)$.

Remark 4. Let T be a tree of order $n \geq 4$. If $n_1(T) = 3$, then $pd(T) = 3$.

Theorem 5. *Let T be a tree with $n_1(T) \geq 4$. Then $pd(T) = n_1(T)$ if and only if T is the star graph.*

Proof. If $T = S_n$ is a star graph, it is clear that $pd(T) = n_1(T)$. Now, let $T = (V, E) \neq S_n$, such that $pd(T) = n_1(T) \geq 4$. Note that by (3) we have $\mathrm{ex}(T) = 1$. Let $t = n_1(T)$ and let $\Omega = \{u_1, u_2, \ldots, u_t\}$ be the set of leaves of T. Let $u \in V$ be the unique exterior major vertex of T. Let us suppose, without loss of generality, u_t is a leaf of T such that $d(u_t, u) = \max_{u_i \in \Omega}\{d(u_i, u)\}$.

For the leaves $u_1, u_2, u_t \in \Omega$ let the paths $P = uu_{t1}u_{t2}, \ldots, u_{tr_t}u_t$, $Q = uu_{11}u_{12}, \ldots, u_{1r_1}u_1$ and $R = uu_{21}u_{22}, \ldots, u_{2r_2}u_2$. Now, let us form the partition $\Pi = \{A_1, A_2, \ldots, A_{t-2}, A\}$, such that $A_1 = \{u_{11}, u_{12}, \ldots, u_{1r_1}, u_1, u_{t2}, u_{t3}, \ldots, u_{tr_t}, u_t\}$, $A_2 = \{u_{21}, u_{22}, \ldots, u_{2r_2}, u_2, u_{t1}\}$, $A_i = \{u_i\}$, $i \in \{3, \ldots, t-2\}$ and $A = V - \cup_{i=1}^{t-2} A_i$. Let us consider two different vertices $x, y \in V$. Hence, we have the following cases.

Case 1: $x, y \in A_1$. Let us suppose $x \in P$ and $y \in Q$. If $d(x, A_2) = d(y, A_2)$, then we have

$$\begin{aligned} d(x, A) &= d(x, u_{t1}) + 1 \\ &= d(x, A_2) + 1 \\ &= d(y, A_2) + 1 \\ &= d(y, A) + 2 \\ &> d(y, A). \end{aligned}$$

Now, if $x, y \in P$ or $x, y \in Q$, then $d(x, A) \neq d(y, A)$.

Case 2: $x, y \in A_2$. If $x = u_{t1}$ or $y = u_{t1}$, then let us suppose for instance, $x = u_{t1}$, so we have $d(x, A_1) = 1 < 2 \leq d(y, A_1)$. On the contrary, if $x, y \in R$, then $d(x, A) \neq d(y, A)$.

Case 3: $x, y \in A$. If $d(x, A_1) = d(y, A_1)$, then $t \geq 5$ and there exists a leaf u_i, $i \neq 1, 2, t-1, t$, such that $d(x, A_i) = d(x, u_i) \neq d(y, u_i) = d(y, A_i)$.

Therefore, for different vertices $x, y \in V$ we have $r(x|\Pi) \neq r(y|\Pi)$ and Π is a resolving partition in T, a contradiction. □

Let T be the comet graph shown in Fig. 3. A resolving partition for T is $\Pi = \{A_1, A_2, A_3\}$, where $A_1 = \{x, t\}$, $A_2 = \{y, z\}$ and $A_3 = \{u, w\}$. In this case, $\theta = pd(T) = 3 < 4 = n_1(T)$.

Remark 6. For any tree T of order $n \geq 2$, $pd(T) \geq \theta$.

Proof. Since different leaves adjacent to the same support vertex must belong to different sets of a resolving partition, the result follows. □

Other examples where $pd(T) = \theta$ are the star graphs and the graph in Fig. 2.

Theorem 7. *Let T be a tree which is not a path. If every vertex belonging to the path between two exterior major vertices of terminal degree greater than one is an exterior major vertex of terminal degree greater than one, then*

$$pd(T) \leq \max\{\kappa, \tau + 1\}.$$

Proof. We suppose $T = (V, E)$ is not a path. Let $S = \{s_1, s_2, \ldots, s_\kappa\}$ be the set of exterior major vertices of T with terminal degree greater than one and let $B_i = \{s_i\}$, $i = 1, \ldots, \kappa$. If $\kappa < \tau + 1$, then for $i \in \{\kappa + 1, \ldots, \tau + 1\}$ we assume $B_i = \emptyset$. Let l_i be the terminal degree of s_i, $i \in \{1, \ldots, \kappa\}$. If $l_i < i$, then we denote by $\{s_{i1}, \ldots, s_{il_i}\}$ the set of terminal vertices of s_i. On the contrary, if $l_i \geq i$, then the set of terminal vertices of s_i is denoted by $\{s_{i1}, \ldots, s_{ii-1}, s_{ii+1}, \ldots, s_{il_i+1}\}$. Also, for a terminal vertex s_{ij} of a major vertex s_i we denote by S_{ij} the set of vertices of T, different from s_i, belonging to the $s_i - s_{ij}$ path. Moreover, we assume $S_{ij} = \emptyset$ for the following three cases: (1) $i = j$, (2) $i \leq l_i < \tau$ and $j \in \{l_i + 2, \ldots, \tau + 1\}$, and (3) $i > l_i$ and $j \in \{l_i + 1, \ldots, \tau + 1\}$. Now, let $t = \max\{\kappa, \tau + 1\}$ and let $\Pi = \{A_1, A_2, \ldots, A_t\}$ be composed of the sets $A_i = B_i \cup \left(\cup_{j=1}^{\kappa} S_{ji}\right)$, $i = 1, \ldots, t$. Since every vertex belonging to the path between two exterior major vertices of terminal degree greater than one is an exterior major vertex of terminal degree greater than one, then Π is a partition of V.

Let us show that Π is a resolving partition. Let $x, y \in V$ be different vertices of T. If $x, y \in A_i$, we have the following three cases.

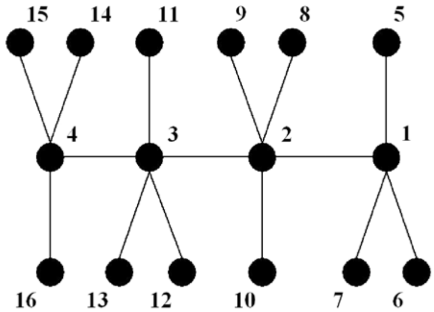

Fig. 4. $\Pi = \{\{1, 8, 11, 14\}, \{2, 5, 12, 15\}, \{3, 6, 9, 16\}, \{4, 7, 10, 13\}\}$ is a resolving partition.

Case 1: $x, y \in S_{ji}$. In this case $d(x, A_j) = d(x, s_j) \neq d(y, s_j) = d(y, A_j)$.
Case 2: $x \in S_{ji}$ and $y \in S_{ki}$, $j \neq k$. If $d(x, A_k) = d(y, A_k)$ we have $d(y, A_j) > d(y, s_k) = d(y, A_k) = d(x, A_k) > d(x, s_j) = d(x, A_j)$.
Case 3: $x = s_i$ and $y \in S_{ji}$. As s_i has at least two terminal vertices, there exists a terminal vertex s_{il} of s_i, $l \neq j$, such that $d(x, A_l) = d(x, S_{il}) = 1$. Hence, $d(y, A_l) > d(y, s_j) \geq 1 = d(x, A_l)$. Therefore, for different vertices $x, y \in V$, we have $r(x|\Pi) \neq r(y|\Pi)$. □

The above bound is achieved, for instance, for the graph in Fig. 4.

3. On the partition dimension of generalized trees

A *cut vertex* in a graph is a vertex whose removal increases the number of components of the graph and an *extreme vertex* is a vertex such that its closed neighborhood forms a complete graph. Also, a *block* is a maximal biconnected subgraph of the graph. Now, let $\mathfrak{F}$ be the family of sequences of connected graphs $G_1, G_2, \ldots, G_k$, $k \geq 2$, such that G_1 is a complete graph K_{n_1}, $n_1 \geq 2$, and G_i, $i \geq 2$, is obtained recursively from G_{i-1} by adding a complete graph K_{n_i}, $n_i \geq 2$, and identifying a vertex of G_{i-1} with a vertex in K_{n_i}.

From this point we will say that a connected graph G is a *generalized tree* if and only if there exists a sequence $\{G_1, G_2, \ldots, G_k\} \in \mathfrak{F}$ such that $G_k = G$ for some $k \geq 2$. Notice that in these generalized trees every vertex is either a cut vertex or an extreme vertex. Also, every complete graph used to obtain the generalized tree is a block of the graph. Note that if every G_i is isomorphic to K_2, then G_k is a tree, thus justifying the terminology used. In this section we will be centered in the study of partition dimension of generalized trees.

Let $G = (V, E)$ be a generalized tree and let $R_1, R_2, \ldots, R_k$ be the blocks of G. A cut vertex $v \in V$ is a *support cut vertex* if there is at least one block R_i of G, in which v is the unique cut vertex belonging to the block R_i. An extreme vertex is an *exterior extreme vertex* if it is adjacent to only one cut vertex. Let $S = \{s_1, s_2, \ldots, s_\zeta\}$ be the set of support cut vertices of G and let $\{s_{i1}, s_{i2}, \ldots, s_{il_i}\}$ be the set of exterior extreme vertices adjacent to $s_i \in S$. Also, let $Q = \{Q_1, Q_2, \ldots, Q_\vartheta\}$ be the set of blocks of G which contain more than one cut vertex and more than one extreme vertex and let $\{q_{i1}, q_{i2}, \ldots, q_{it_i}\}$ be the set of extreme vertices belonging to $Q_i \in Q$. Now, let $\phi = \max_{1\leq i\leq\zeta, 1\leq j\leq\vartheta}\{l_i, t_j\}$. With the above notation we have the following result.

Theorem 8. *For any generalized tree G,*

$$pd(G) \leq \begin{cases} \zeta + \vartheta + \phi - 1, & \text{if } \phi \geq 3; \\ \zeta + \vartheta + 1, & \text{if } \phi \leq 2. \end{cases}$$

Proof. For each support cut vertex $s_i \in S$, let $A_i = \{s_{i1}\}$ and for each block $Q_j \in Q$, let $B_j = \{q_{j1}\}$. Let us suppose $\phi \geq 3$. For every $j \in \{2, \ldots, l_i\}$ we take $M_{ij} = \{s_{ij}\}$ and, if $l_i < \phi - 1$, then for every $j \in \{l_{i+1}, \ldots, \phi - 1\}$ we consider $M_{ij} = \emptyset$. Analogously, for every $j \in \{2, \ldots, t_i\}$ we take $N_{ij} = \{q_{ij}\}$ and, if $t_i < \phi - 1$, then for every $j \in \{t_{i+1}, \ldots, \phi - 1\}$ we consider $N_{ij} = \emptyset$. Now, let $C_j = \bigcup_{i=1}^{\max\{\zeta,\vartheta\}}(M_{ij} \cup N_{ij})$, with $j \in \{2, \ldots, \phi - 1\}$.

Let us prove that $\Pi = \{A, A_1, A_2, \ldots, A_\zeta, B_1, B_2, \ldots, B_\vartheta, C_2, C_3, \ldots, C_{\phi-1}\}$ is a resolving partition of G, where $A = V - \cup_{i=1}^{\zeta} A_i - \cup_{i=1}^{\vartheta} B_i - \cup_{i=2}^{\phi-1} C_i$. To begin with, let x, y be two different vertices of G. We have the following cases.

Case 1: x is a cut vertex or y is a cut vertex. Let us suppose, for instance, x is a cut vertex. So there exists an extreme vertex s_{i1} such that x belongs to a shortest $y - s_{i1}$ path or y belongs to a shortest $x - s_{i1}$ path. Hence, we have $d(x, A_i) = d(x, s_{i1}) \neq d(y, s_{i1}) = d(y, A_i)$.

Case 2: x, y are extreme vertices. If x, y belong to the same block of G, then x, y belong to different sets of Π. On the contrary, if x, y belong to different blocks in G, then let us suppose that there exists an extreme vertex c such that $d(x, c) \leq 1$ or $d(y, c) \leq 1$. We can suppose $c \in A_i$, for some $i \in \{1, \ldots, \zeta\}$, or $c \in B_j$, for some $j \in \{1, \ldots, \vartheta\}$. Without the loss of generality, we suppose that $d(x, c) \leq 1$. Since x and y belong to different blocks of G, we have $d(y, c) > 1$. So we obtain either $d(x, A_i) = d(x, c) \leq 1 < d(y, c) = d(y, A_i)$ or $d(x, B_j) = d(x, c) \leq 1 < d(y, c) = d(y, B_j)$.

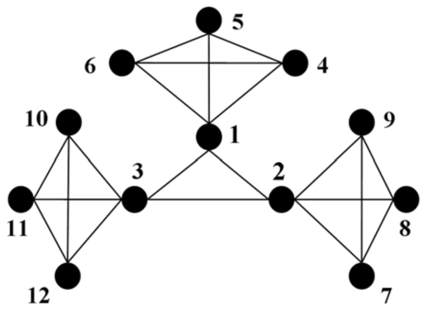

Fig. 5. $\Pi = \{\{4\}, \{7\}, \{10\}, \{5, 8, 11\}, \{1, 2, 3, 6, 9, 12\}\}$ is a resolving partition for the generalized tree.

Now, if there exists no such a vertex c, then there exist two blocks $H, K \notin Q$ with $x \in H$ and $y \in K$, which contain more than one cut vertex and only one extreme vertex. So $x, y \in A$. Let $u \in H$ be a cut vertex such that $d(y, u) = \max_{v \in H} d(y, v)$. Hence, there exists an extreme vertex s_{i1} such that u belongs to a shortest $x - s_{i1}$ path and $d(y, s_{i1}) = d(y, u) + d(u, s_{i1})$. As x, y belong to different blocks and $d(y, u) = \max_{v \in H} d(y, v)$ we have $d(y, u) \geq 2$. Thus,

$$\begin{aligned} d(y, A_i) &= d(y, s_{i1}) \\ &= d(y, u) + d(u, s_{i1}) \\ &\geq 2 + d(u, s_{i1}) \\ &> 1 + d(u, s_{i1}) \\ &= d(x, u) + d(u, s_{i1}) \\ &= d(x, A_i). \end{aligned}$$

Hence, we conclude that if $\phi \geq 3$, then for every $x, y \in V$, $r(x|\Pi) \neq r(y|\Pi)$. Therefore, Π is a resolving partition.

On the other hand, if $\phi \leq 2$, then $\Pi' = \{A, A_1, A_2, \ldots, A_\zeta, B_1, B_2, \ldots, B_\vartheta\}$ is a partition of V. Proceeding as above we obtain that Π' is a resolving partition. □

The above bound is achieved, for instance, for the graph in Fig. 5, where $\zeta = 3$, $\vartheta = 0$ and $\phi = 3$. Also, notice that for the particular case of trees we have $\zeta = \xi$, $\phi = \theta$ and $\vartheta = 0$. So the above result leads to Corollary 2.

References

[1] R.C. Brigham, G. Chartrand, R.D. Dutton, P. Zhang, Resolving domination in graphs, Mathematica Bohemica 128 (1) (2003) 25–36.
[2] J. Cáceres, C. Hernando, M. Mora, I.M. Pelayo, M.L. Puertas, C. Seara, On the metric dimension of some families of graphs, Electronic Notes in Discrete Mathematics 22 (2005) 129–133.
[3] J. Cáceres, C. Hernando, M. Mora, I.M. Pelayo, M.L. Puertas, C. Seara, D.R. Wood, On the metric dimension of Cartesian product of graphs, SIAM Journal of Discrete Mathematics 21 (2) (2007) 273–302.
[4] G. Chappell, J. Gimbel, C. Hartman, Bounds on the metric and partition dimensions of a graph, Ars Combinatoria 88 (2008) 349–366.
[5] G. Chartrand, L. Eroh, M.A. Johnson, O.R. Oellermann, Resolvability in graphs and the metric dimension of a graph, Discrete Applied Mathematics 105 (2000) 99–113.
[6] G. Chartrand, C. Poisson, P. Zhang, Resolvability and the upper dimension of graphs, Computers and Mathematics with Applications 39 (2000) 19–28.
[7] G. Chartrand, E. Salehi, P. Zhang, The partition dimension of a graph, Aequationes Mathematicae 59 (1–2) (2000) 45–54.
[8] M. Fehr, S. Gosselin, O.R. Oellermann, The partition dimension of Cayley digraphs, Aequationes Mathematicae 71 (2006) 1–18.
[9] F. Harary, R.A. Melter, On the metric dimension of a graph, Ars Combinatoria 2 (1976) 191–195.
[10] T.W. Haynes, M. Henning, J. Howard, Locating and total dominating sets in trees, Discrete Applied Mathematics 154 (2006) 1293–1300.
[11] B.L. Hulme, A.W. Shiver, P.J. Slater, A Boolean algebraic analysis of fire protection, Algebraic and Combinatorial Methods in Operations Research 95 (1984) 215–227.
[12] M.A. Johnson, Structure–activity maps for visualizing the graph variables arising in drug design, Journal of Biopharmaceutical Statistics 3 (1993) 203–236.
[13] M.A. Johnson, Browsable structure–activity datasets, in: R. Carbó-Dorca, P. Mezey (Eds.), Advances in Molecular Similarity, JAI Press Connecticut, 1998, pp. 153–170.
[14] S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, Discrete Applied Mathematics 70 (1996) 217–229.
[15] R.A. Melter, I. Tomescu, Metric bases in digital geometry, Computer Vision, Graphics, and Image Processing 25 (1984) 113–121.
[16] V. Saenpholphat, P. Zhang, Conditional resolvability in graphs: a survey, International Journal of Mathematics and Mathematical Sciences 38 (2004) 1997–2017.
[17] P.J. Slater, Leaves of trees, Congressus Numerantium 14 (1975) 549–559. Proc. 6th Southeastern Conference on Combinatorics, Graph Theory, and Computing.
[18] I. Tomescu, Discrepancies between metric and partition dimension of a connected graph, Discrete Mathematics 308 (2008) 5026–5031.
[19] I.G. Yero, J.A. Rodríguez-Velázquez, A note on the partition dimension of Cartesian product graphs, Applied Mathematics and Computation 217 (7) (2010) 3571–3574.