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OPTIMUM CONTROL OF SELECTED VIBRATION FORMS IN MECHANICAL SYSTEMS

In the paper, the authors describe and solve the problem of optimum control of selected vibration forms in mechanical systems. Two illustrative examples have been used to present the procedure for determination of the optimum controller coefficients.

In the first example, a simplified mechanical system is considered, while in the the second one – a rotor with magnetic bearing. In both cases, the integral performance indices have been defined in order to minimize the vibration level at selected points of the structures.

The system with the magnetic bearing is structurally unstable. For this reason, the authors present the way of finding the weight coefficients of integral performance index for unstable, multi-degrees-of-freedom system. In that way, the selected modal forms attain the previously assumed dynamic properties and the performance index takes the minimum value. The results of numerical analysis show that the proposed way is efficient and makes it possible to control selected forms of vibration in the system.

1. Introduction

Due to the danger of high displacement level, the problem of transverse rotor vibrations control is very important and still requires investigations.

Vibrations, as a side effect of machine functioning, result from unbalance, asymmetry or the presence of non-axial elements placed on the rotor. Vibrations are unavoidable and cause that the rotor, working in a stable area, vibrates at a certain frequency and amplitude. Vibration level affects the magnitude of the dynamic reactions in bearing nodes and, in consequence,

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the durability of the toe-pan system. For the above reasons, in the paper one has made an effort to suppress the vibrations using an optimum control procedure.

Although the algorithms for choosing the optimum controller settings are known, some difficulties with their physical realization can appear. The level of optimum control command often exceeds the possibilities of classical piezoelectric actuators. On the other hand, it is very difficult to apply electrodynamic actuators when the rotor turns. To avoid these problems, a magnetic bearing has been applied to the optimum control of the rotor vibrations.

The rotor with a magnetic bearing is a structurally unstable system. Thus the controller should not only to control the rotor vibrations, but also to stabilize them.

Therefore, the problem of optimization leading to Riccati equation for the unstable system has been solved.

In the paper, one presents an efficient way of finding the weight coefficients of the integral performance index for unstable, multi-degrees-of-freedom system. In this way, the chosen modal forms would have the previously assumed dynamic properties and the performance index would take the minimum value. The presented method gives the possibility to control the selected vibration forms of the system.

2. Mathematical description of the dynamics of mechanical systems

The simplified model of the structure, shown in Fig. 2 and Fig. 5, is obtained by the rigid finite element method [1], [5], [6], [10]. It depends on dividing the structure into non-flexible mass elements called rigid finite elements (RFE) which are connected by massless spring-damping elements (SDE).

The equations of motion are derived in two stages [1], [5], [6], [10]. In the first stage, the differential equation of the system free of constraints is determined, while in the second stage two subsystems with known and unknown motion are separated. The system free of constraints is the one whose constraints have been replaced by their reactions. Thus, the number of freedom degrees of the system free of constraints equals

$$\tilde{n} = n + r, \tag{1}$$

where: n – number of degrees of freedom of the system with constraints, r – number of constrains (i.e. fixed degrees of freedom).

The equation of motion of the system free of constraints (with $\tilde{n}$ degrees of freedom) can be expressed in the form

$$\tilde{\boldsymbol{M}} \cdot \ddot{\tilde{\boldsymbol{q}}} + \tilde{\boldsymbol{L}} \cdot \dot{\tilde{\boldsymbol{q}}} + \tilde{\boldsymbol{K}} \cdot \tilde{\boldsymbol{q}} = \tilde{\boldsymbol{f}}, \tag{2}$$

where: $\tilde{\boldsymbol{M}}$, $\tilde{\boldsymbol{L}}$, $\tilde{\boldsymbol{K}}$ – matrices of inertia, damping and stiffness of the system free of constraints, $\tilde{\boldsymbol{q}}$, $\tilde{\boldsymbol{f}}$ – vectors of generalised displacements and forces of the system free of constraints.

Some components of the vector $\tilde{\boldsymbol{q}}$ are given in time domain (i.e. kinematic excitations), but the other ones – the unknown time functions, refer to displacements of the system. These components can be arranged in such a way that the ones of the vector of unknown displacements $\boldsymbol{q}$, will be first, and the components of the vector of kinematic excitations $\boldsymbol{z}$ will follow them. Thus, we can write

$$\tilde{\boldsymbol{q}} = col(\boldsymbol{q}, \boldsymbol{z}), \tag{3}$$

where: $\boldsymbol{q}$ – vector of unknown generalised displacements $(n \times 1)$, $\boldsymbol{z}$ – vector of given kinematic excitations $(r \times 1)$.

By analogy, the vector of generalised forces is

$$\tilde{\boldsymbol{f}} = col(\boldsymbol{p}, \boldsymbol{r}), \tag{4}$$

where: $\boldsymbol{p}$ – vector of given generalised forces (force excitations) $(n \times 1)$, $\boldsymbol{r}$ – vector of unknown generalised forces (constraint reactions) $(r \times 1)$.

It is possible to divide the equation of motion (2) into matrix blocks as a consequence of some arrangement of components of the generalised displacements vector (3) and the generalised forces vector (4), which leads to [1], [5], [6]

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{M}'' \\ \boldsymbol{M}''^{T} & \boldsymbol{M}' \end{bmatrix} \cdot \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \ddot{\boldsymbol{z}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{L} & \boldsymbol{L}'' \\ \boldsymbol{L}''^{T} & \boldsymbol{L}' \end{bmatrix} \cdot \begin{bmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{z}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{K} & \boldsymbol{K}'' \\ \boldsymbol{K}''^{T} & \boldsymbol{K}' \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{z} \end{bmatrix} = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{r} \end{bmatrix} \tag{5}$$

where: $\boldsymbol{M}$, $\boldsymbol{L}$, $\boldsymbol{K}$ – matrices of inertia, damping and stiffness of subsystem with unknown motion $(n \times n)$, $\boldsymbol{M}'$, $\boldsymbol{L}'$, $\boldsymbol{K}'$ – matrices of inertia, damping and stiffness of subsystem with given motion $(r \times r)$, $\boldsymbol{M}''$, $\boldsymbol{L}''$, $\boldsymbol{K}''$ – matrices of inertia, damping and stiffness of the interaction subsystem $(n \times r)$.

The equation (5) can be written as a system of two equations:

$$\boldsymbol{M} \cdot \ddot{\boldsymbol{q}} + \boldsymbol{L} \cdot \dot{\boldsymbol{q}} + \boldsymbol{K} \cdot \boldsymbol{q} = \boldsymbol{h}(t), \tag{6}$$

$$\boldsymbol{r} = \boldsymbol{M}''^{T} \cdot \ddot{\boldsymbol{q}} + \boldsymbol{M}' \cdot \boldsymbol{z}'' + \boldsymbol{L}''^{T} \cdot \dot{\boldsymbol{q}} + \boldsymbol{L}' \cdot \dot{\boldsymbol{z}} + \boldsymbol{K}''^{T} \cdot \boldsymbol{q} + \boldsymbol{K}' \cdot \boldsymbol{z}\ , \tag{7}$$

where

$$\boldsymbol{h}(t) = \boldsymbol{p} - \boldsymbol{M}'' \cdot \ddot{\boldsymbol{z}} - \boldsymbol{L}'' \cdot \dot{\boldsymbol{z}} - \boldsymbol{K}'' \cdot \boldsymbol{z} \tag{8}$$

is the vector of reduced force-kinematic excitations.

The solution to equation (6) yields the vector of generalised displacements $\boldsymbol{q}$. Subsequently, the vector of constraint reactions $\boldsymbol{r}$ is determined on the basis of (7).

3. State-space representation of the mathematical model

The equation of motion (6)÷(8) can be written in a state space form as:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A} \cdot \boldsymbol{x} + \boldsymbol{B} \cdot \boldsymbol{u}, \\ \boldsymbol{y} &= \boldsymbol{C} \cdot \boldsymbol{x}, \end{aligned} \tag{9}$$

where: $\boldsymbol{x} = col(\boldsymbol{x}_1, \boldsymbol{x}_2) = col(\boldsymbol{q}, \dot{\boldsymbol{q}})$ – state vector $(2n \times 1)$,

$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1} \cdot \boldsymbol{K} & -\boldsymbol{M}^{-1} \cdot \boldsymbol{L} \end{bmatrix}$ – state matrix $(2n \times 2n)$,

$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{M}^{-1} \end{bmatrix}$ – input matrix $(2n \times k)$, $\boldsymbol{u} = \boldsymbol{h}(t)$ – input vector $(k \times 1)$.

The system described by equation (9) can be transformed into the modal state-space representation, which is characterized by the block-diagonal submatrices [2], [3], [9]

$$\boldsymbol{A}_{mi} = \begin{bmatrix} -\zeta_i \omega_{ni} & \omega_{di} \\ -\omega_{di} & -\zeta_i \omega_{ni} \end{bmatrix}, \quad i = 1, 2, \ldots, n, \tag{10}$$

where: n – number of degrees of freedom, ζ_i – damping coefficient, ω_{ni} – natural frequency of non-damped vibrations, ω_{di} – frequency of damped vibrations.

The modal state-space representation of the block-diagonal form submatrices (10) can be obtained by applying the linear transformation of the state variables

$$\boldsymbol{x} = \boldsymbol{T}_d \cdot \boldsymbol{T}_b \cdot \boldsymbol{x}_m = \boldsymbol{T}_m \cdot \boldsymbol{x}_m, \tag{11}$$

where: $\boldsymbol{T}_d$ – eigenvectors of matrix $\boldsymbol{A}$, which transform matrix $\boldsymbol{A}$ into diagonal form, $\boldsymbol{T}_b$ – matrix that transforms complex diagonal form of $\boldsymbol{A}$ into real block-diagonal form. It takes the form [8]

$$T_b = \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} & 0 & \dots & 0 \\ \frac{1}{2} & \frac{j}{2} & 0 & \dots & \dots \\ 0 & 0 & 1 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{bmatrix} \tag{12}$$

and is created in such a way that number 1 is inserted instead of the real eigenvalues of $\boldsymbol{A}$, while in the other ones, according to complex conjugate eigenvalues of $\boldsymbol{A}$, 2×2 block-matrix, as in (12).

Putting equation (11) into (9) yields a modal state-space representation, i.e.:

$$\begin{aligned} \dot{\boldsymbol{x}}_m &= \boldsymbol{A}_m \cdot \boldsymbol{x}_m + \boldsymbol{B}_m \cdot \boldsymbol{u}, \\ \boldsymbol{y} &= \boldsymbol{C}_m \cdot \boldsymbol{x}_m, \end{aligned} \tag{13}$$

where: $\boldsymbol{A}_m = \boldsymbol{T}_m^{-1} \cdot \boldsymbol{A} \cdot \boldsymbol{T}_m$, $\boldsymbol{B}_m = \boldsymbol{T}_m^{-1} \cdot \boldsymbol{B}$, $\boldsymbol{C}_m = \boldsymbol{C} \cdot \boldsymbol{T}_m$.

4. Optimum control of selected modal forms with assumed previously dynamic properties

For the system described by the equation (9) the integral performance index can be defined as [4]

$$I = \frac{1}{2} \int_0^{t_k} (\boldsymbol{x}^T \cdot \boldsymbol{Q} \cdot \boldsymbol{x} + r^2 u) dt, \tag{14}$$

where: $\boldsymbol{Q}$ – weighing matrix, r – positively defined input weighing coefficient.

The control input command $u(t)$ that minimizes the integral performance index (14), for a general mechanical system shown in Fig. 1, is expressed by the following relationship [3], [4], [8]

$$u(t) = -\boldsymbol{R} \cdot \boldsymbol{x}, \tag{15}$$

where $\boldsymbol{R}$ is the feedback matrix gain given by

$$\boldsymbol{R} = \frac{1}{r} \boldsymbol{B}^T \cdot \boldsymbol{S}, \tag{16}$$

whereas $\boldsymbol{S}$ is a solution of the Riccati equation [3], [4], [7]

$$\boldsymbol{S}\cdot\boldsymbol{A}-\frac{1}{r}\boldsymbol{S}\cdot\boldsymbol{B}\cdot\boldsymbol{B}^T\cdot\boldsymbol{S}+\boldsymbol{Q}+\boldsymbol{A}^T\cdot\boldsymbol{S}=0. \tag{17}$$

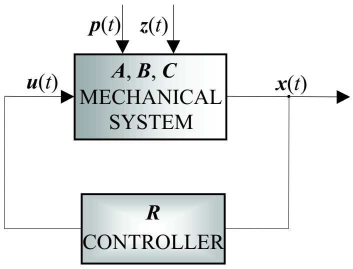

Fig. 1. General scheme of vibration control

It can be proved that the control input (15) is optimal, i.e. it minimizes the index (14), when the characteristic equation of the closed-loop transfer function

$$D(s)=\det(s\boldsymbol{I}-(\boldsymbol{A}-\boldsymbol{B}\cdot\boldsymbol{R})), \tag{18}$$

and the characteristic equation of the open-loop transfer function

$$D_0(s)=\det(s\boldsymbol{I}-\boldsymbol{A}) \tag{19}$$

are related by the equation [4]

$$D(-s)D(s)=D_0(-s)D_0(s)+\frac{1}{r}\boldsymbol{L}_0^T(-s)\cdot\boldsymbol{Q}\cdot\boldsymbol{L}_0(s), \tag{20}$$

where

$$\boldsymbol{L}_0(s)=\boldsymbol{C}\cdot(s\boldsymbol{I}-\boldsymbol{A})_{adj}\cdot\boldsymbol{B} \tag{21}$$

is the numerator of the open-loop transfer function of (9).

5. General procedure for computation of weight coefficients

One can choose the weight coefficients of the integral performance index (14) such that it takes the minimum value, and the system exhibits the dynamic properties prescribed earlier.

For this purpose, for the multi degree-of-freedom system [3] one should:

1. Transform the equation (9) into the block-diagonal form (13).
2. Reduce the system by removing from $\boldsymbol{A}_m$, $\boldsymbol{B}_m$ rows and columns that are insignificant for the system dynamics, and from $\boldsymbol{T}_m^{-1}$ only the insignificant rows according to $\boldsymbol{A}_m$.

3. Calculate the weight coefficients of $\boldsymbol{Q}_{mr}$, from the equation (20) for the reduced system ($\boldsymbol{A}_{mr}$, $\boldsymbol{B}_{mr}$) substituting $\boldsymbol{A}$ and $\boldsymbol{B}$ in (18) and (19) by $\boldsymbol{A}_{mr}$ and $\boldsymbol{B}_{mr}$ and $\boldsymbol{C}$ in (21) by $\boldsymbol{I}_{mr}$.
4. Calculate $\boldsymbol{Q}$ for the full order system from the equation $\boldsymbol{Q} = \boldsymbol{T}_{mr}^{-T} \cdot \boldsymbol{Q}_{mr} \cdot \boldsymbol{T}_{mr}$. The latter is obtained by putting (11) into (14).

6. Illustrative examples

6.1. Example 1

The simplified two-degrees-of-freedom system consists of two rigid masses connected by spring and damping elements (Fig. 2). General co-ordinates of the system are displacements of RFE-s: q_1 and q_2. The system is excited by force $p(t) = \delta(t)$, where $\delta(t)$ is the Dirac delta function (Fig. 2). The control signal $u(t)$ has also a force character.

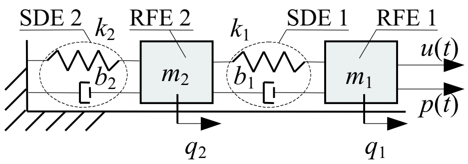

Fig. 2. Model of the structure under consideration

Table 1.

The parameter values of the model

i	m_i [kg]	b_i [Ns/m]	k_i [N/m]
1	2	3	15000
2	1	3	10000

The parameters values of the model are defined in Table 1.
The mathematical model of such system can be written in the form of (9), where:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -7500 & 7500 & -2.5 & 2.5 \\ 15000 & -25000 & 5 & -8 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{22}$$

and its eigenvalues are put in Table 2.

Table 2.

Eigenvalues of the system shown in Fig. 2

Modal form	Eigenvalues	Dynamic properties
1	−4.863636 + 173.1367i	$\varsigma = 0.02808$
	−4.863636 − 173.1367i	$\omega_n = 173.205$
2	−0.386363 + 49.9985i	$\varsigma = 0.0077272$
	−0.386363 − 49.9985i	$\omega_n = 50$

The problem consists in computation of the matrix weight coefficient $\boldsymbol{Q}$ of the integral performance index (14) to shift the modal form 2 (Tab. 2) to the place which is characterized by the dynamics properties: $\varsigma = 0.25$, $\omega_n = 100$ and to attain the minimum value of performance index (14). Further, one proceeds in accordance with the description given at the fifth point. Transformation to the state-space modal representation (13) yields:

$$
\begin{gathered}
\boldsymbol{A}_m = \begin{bmatrix} -0.386 & -49.998 & 0 & 0 \\ 49.998 & -0.386 & 0 & 0 \\ 0 & 0 & -4.863 & -173.136 \\ 0 & 0 & 173.136 & -4.863 \end{bmatrix}, \quad \boldsymbol{B}_m = \begin{bmatrix} -0.49176 \\ -0.00369 \\ -0.28748 \\ -0.00846 \end{bmatrix}, \\
\boldsymbol{C}_m = \begin{bmatrix} 1.285e-4 & -1.663e-2 & 4.743e-5 & -1.825e-3 \\ 7.898e-5 & -1.109e-2 & -1.537e-4 & 5.474e-3 \\ -0.83188 & 0 & -0.316239 & 663.634e-6 \\ -0.55459 & 336.1e-6 & 0.9486617 & 0 \end{bmatrix}, \\
\boldsymbol{T}_m = \begin{bmatrix} 128.563e-6 & -166.371e-4 & 474.391e-7 & -182.519e-5 \\ 789.876e-7 & -110.915e-4 & -153.798e-6 & 547.494e-5 \\ -831.882e-3 & 0 & -316.239e-3 & 663.634e-6 \\ -554.591e-3 & 336.1003e-6 & 948.661e-3 & 0 \end{bmatrix}, \\
\boldsymbol{T}_m^{-1} = \begin{bmatrix} -704.452e-4 & 957.323e-4 & -983.53e-3 & -327.8438e-3 \\ -49.1773 & -16.39349 & -738.346e-5 & -265.985e-5 \\ -237.595e-4 & 617.735e-4 & -574.973e-3 & 862.458e-3 \\ -99.62683 & 149.439508 & -169.202e-4 & 235.689e-4 \end{bmatrix}.
\end{gathered} \tag{23}
$$

Reducing the system to the form that should be shifted leads to the following matrices:

$$\boldsymbol{A}_{mr} = \begin{bmatrix} -0.386 & -49.998 \\ 49.998 & -0.386 \end{bmatrix}, \quad \boldsymbol{B}_{mr} = \begin{bmatrix} -0.49176 \\ -0.00369 \end{bmatrix},$$

$$\boldsymbol{C}_{mr} = \begin{bmatrix} 128.563e-6 & -166.371e-4 \\ 789.876e-7 & -110.915e-4 \end{bmatrix}, \tag{24}$$

$$\boldsymbol{T}_{mr}^{-1} = \begin{bmatrix} -704.452e-4 & 957.323e-4 & -983.5308e-3 & -327.8438e-3 \\ -49.1774 & -16.393498 & -738.346e-5 & -265.985e-5 \end{bmatrix}.$$

The open-loop characteristic equation (19) for the reduced system takes the form

$$D_0(s) = \det(s\boldsymbol{I}_{mr} - \boldsymbol{A}_{mr}) = s^2 + 0.77272724789s + 2500, \tag{25}$$

whereas for the closed loop it becomes

$$D(s) = \det(s\boldsymbol{I}_{mr} - (\boldsymbol{A}_{mr} - \boldsymbol{B}_{mr} \cdot \boldsymbol{R}_{mr})) = s^2 + 2\varsigma\omega_n s + \omega_n^2 = s^2 + 50s + 10000. \tag{26}$$

From equation (21) one can obtain

$$\boldsymbol{L}_0 = \boldsymbol{I}_{mr} \cdot (s\boldsymbol{I}_{mr} - \boldsymbol{A}_{mr})_{adj} \cdot \boldsymbol{B}_{mr} = \begin{bmatrix} -0.4917654s - 542e-5 \\ -24.58897 - 369.173e-5s \end{bmatrix}. \tag{27}$$

Considering that

$$\boldsymbol{Q}_{mr} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \begin{bmatrix} q_1 & q_2 \end{bmatrix}, \tag{28}$$

and substituting (25)÷(27) into (20), and then comparing the coefficient according to the same powers of s in (20) one obtains:

$$q_1 = 2.9561 + 227.354j,$$
$$q_2 = 393.771 - 0.050106j,$$

or

$$\boldsymbol{Q}_{mr} = \begin{bmatrix} -51681.177 + 1344.155i & -1152.629 - 89525.735i \\ sym & 155055.981 + 39.4605i \end{bmatrix}. \tag{29}$$

From the equation $\boldsymbol{Q} = \boldsymbol{T}_{mr}^{-T} \cdot \boldsymbol{Q}_{mr} \cdot \boldsymbol{T}_{mr}$, it is possible to obtain the weighing matrix $\boldsymbol{Q}$ for the full order system. The solution of the Riccati equation (17) for the full order system gives

$$\boldsymbol{S} = \begin{bmatrix} 1995910.512 & 661743.999 & 30011.174 & 10011.786 \\ & 219397.443 & 9981.032 & 3329.683 \\ & & 196.918 & 65.76 \\ sym & & & 21.9606 \end{bmatrix} \tag{30}$$

and from (16)

$$\boldsymbol{R} = \begin{bmatrix} 15005.588 & 4990.516 & 98.459 & 32.88 \end{bmatrix}. \tag{31}$$

The eigenvalues of the system with the optimum feedback matrix gain $\boldsymbol{R}$ (31) are put into Table 3.

Table 3.

Eigenvalues of the system with the optimum controller

Modal form	Eigenvalues	Dynamic properties
1	−4.863 + 173.136i	ς = 0.02808
	−4.863 − 173.136i	ω_n = 173.205
2	−25.001 + 96.824i	ς = 0.25001
	−25.001 − 96.824i	ω_n = 100

Computer simulation have been performed using the *Matlab* package and the results are shown in Fig. 3 and Fig. 4.

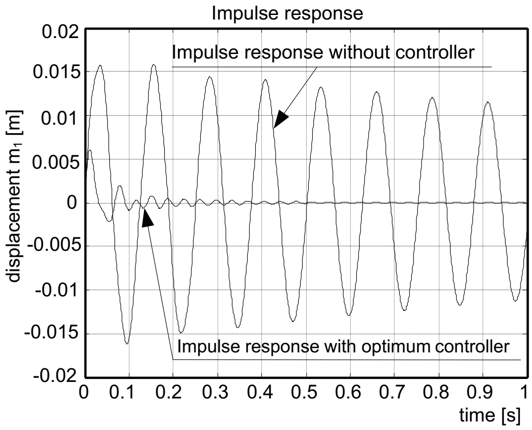

Fig. 3. Displacement of m_1 with and without optimum controller

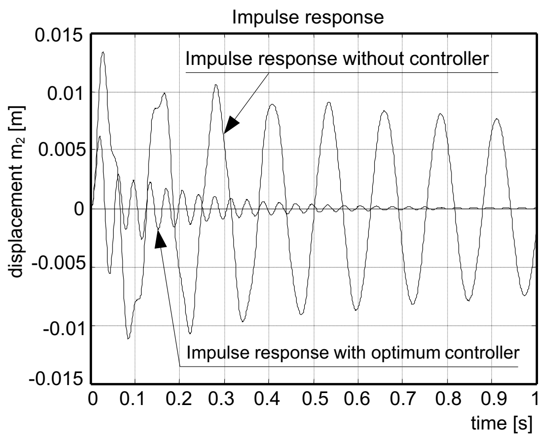

Fig. 4. Displacement of m_2 with and without optimum controller

6.2. Example 2

The model of the laboratory rotor with a magnetic bearing consists of 5 RFEs and 4 SDEs (Fig. 5) [3]. SDEs no. 5 and 6 approach the oscillatory bearing, while SDEs no. 7 and 8 – the magnetic bearing. Each RFE has two degrees of freedom, in other words, displacement in the transverse direction y and angular displacement around axis x.

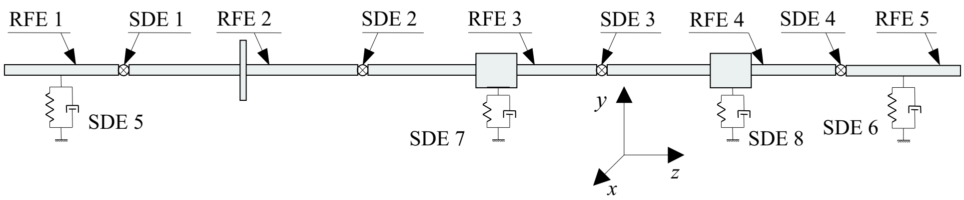

Fig. 5. The discrete model of the laboratory rotor

The task depends in this case on minimizing the vibrations of RFE 3 by the acting force, being generated with magnetic bearing at point RFE 3 or RFE 4.

The integral performance index (14) has been defined for the discrete model with the magnetic bearing.

The structure with a magnetic bearing is unstable (Fig. 7). Thus, the modal form (13) of it has been reduced to the unstable mode. The reduced matrix equation (13) for that model is described by the triple $\boldsymbol{A}_{mr}$, $\boldsymbol{B}_{mr}$, $\boldsymbol{C}_{mr}$. Appropriate matrices are as follows:

$$\boldsymbol{A}_{mr} = \begin{bmatrix} 427.72 & 0 \\ 0 & -432.5 \end{bmatrix}, \quad \boldsymbol{B}_{mr} = \begin{bmatrix} 60.075 \\ 61.693 \end{bmatrix},$$
$$\boldsymbol{C}_{mr} = \begin{bmatrix} 197.0590 & 5.6049 \\ -204.9874 & -5.5223 \end{bmatrix}. \tag{32}$$

Because of the large size of the matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ for that system, only its reduced form has been presented in the paper. The full order mathematical model can be found in [3].
The denominator of the open-loop transfer function (19) for the reduced system equals

$$D_0(s) = \det(s\boldsymbol{I}_{mr} - \boldsymbol{A}_{mr}) = s^2 + 4.78s - 184988.75. \tag{33}$$

Assuming that the dynamics properties of the closed-loop reduced system are equal to: $\varsigma = 0.25$, $\omega_n = 263$, the characteristic equation takes the form

$$D(s) = \det(s\boldsymbol{I}_{mr} - (\boldsymbol{A}_{mr} - \boldsymbol{B}_{mr} \cdot \boldsymbol{R}_{mr})) = s^2 + 2\varsigma\omega_n s + \omega_n^2 = s^2 + 131.5s + 69169. \tag{34}$$

Then one can compute $\boldsymbol{L}_0$ from (21), i.e.

$$\boldsymbol{L}_0(s) = \boldsymbol{I}_{mr} \cdot (s\boldsymbol{I}_{mr} - \boldsymbol{A}_{mr})_{adj} \cdot \boldsymbol{B}_{mr} = \begin{bmatrix} 60.075s + 25982.483 \\ 61.693s - 26387.17 \end{bmatrix}. \tag{35}$$

After taking for granted that

$$\boldsymbol{Q}_{mr} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \begin{bmatrix} q_1 & q_2 \end{bmatrix}, \tag{36}$$

and substituting (33)÷(36) into (20), and then comparing the coefficients according to the same powers of s one can obtain the non-linear equations from which the matrix $\boldsymbol{Q}_{mr}$ is

$$\boldsymbol{Q}_{mr} = \boldsymbol{q} \cdot \boldsymbol{q}^T = \begin{bmatrix} -9.12i \\ -2.478 \end{bmatrix} \cdot \begin{bmatrix} -9.12i & -2.478i \end{bmatrix} = \begin{bmatrix} -83.172 & -22.598 \\ -22.598 & -6.14 \end{bmatrix}. \tag{37}$$

In this case, the system is unstable, so the matrices $\boldsymbol{Q}_{mr}$ and $\boldsymbol{Q}$ should be non-positively defined in order to solve the Riccati equation.

Putting $\boldsymbol{A}_{mr}$, $\boldsymbol{B}_{mr}$, $\boldsymbol{Q}_{mr}$ into the equation (17) for reduced system gives the solution

$$\boldsymbol{S}_{mr} = \begin{bmatrix} 0.13884802810511 & -0.03848709780093 \\ -0.03848709780093 & -0.02341158781770 \end{bmatrix}, \tag{38}$$

and the from (16)

$$\boldsymbol{R}_{mr} = \begin{bmatrix} 5.96694014639143 & -3.75644017996119 \end{bmatrix}. \tag{39}$$

Now one can calculate the weight matrix for the full order system from equation $\boldsymbol{Q} = \boldsymbol{T}_{mr}^{-T} \cdot \boldsymbol{Q}_{mr} \cdot \boldsymbol{T}_{mr}$, and the feedback gain matrix $\boldsymbol{R}$. For this purpose, one can apply the calculation procedure analogous to that of the first example.

Because of a large size of the matrices $\boldsymbol{S}$ and $\boldsymbol{R}$ for the full order system, they are not presented in the paper, however they can be found in [3].

It can be proved that the eigenvalues with the feedback matrix gain $\boldsymbol{R}$ are unchanged except one unstable mode, which has been shifted to the assumed previously dynamic properties: $\varsigma = 0.25$, $\omega_n = 263$.

The results of computer simulation by using *Matlab* package are shown in Figs. 6÷9.

The response of the rotor without the controller (Fig. 6) illustrates the displacements of RFEs nos. 2, 3, 4. Their phases are consistent, while the responses of RFEs no. 2 and 4 pertaining to the optimum controller (Fig. 9) are in anti-phase. This means that the optimum controller fulfils its task, because owing to its action, the displacement of the RFE no. 3, located between RFE no. 2 and RFE no. 4, is suppressed. Moreover, the displacement level of RFE no. 3 is significantly reduced (Fig. 8).

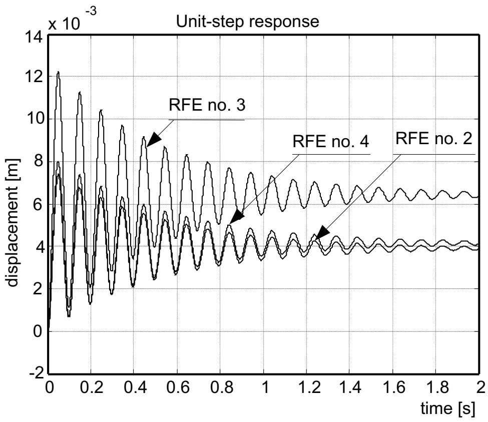

Fig. 6. Unit-step response of the rotor without the magnetic bearing

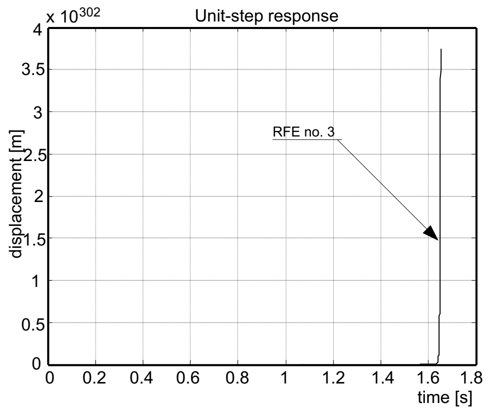

Fig. 7. Unit-step response of the rotor with the magnetic bearing and without controller

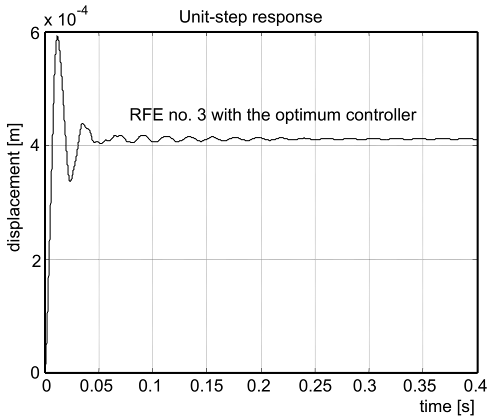

Fig. 8. Unit-step response of the rotor with the magnetic bearing and with the optimum controller

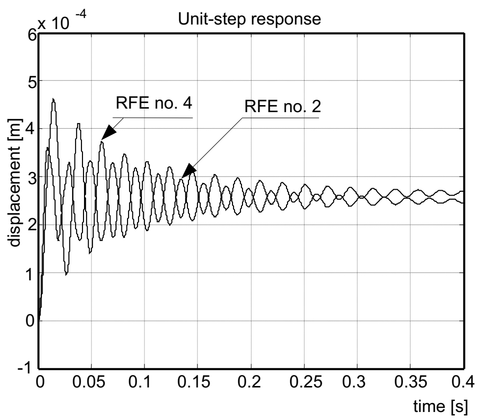

Fig. 9. Unit-step response of the rotor with the magnetic bearing and with the optimum controller

7. Conclusion

The results of the computer simulation and the numerical calculations have proved that the proposed method of choosing the weight coefficients of an integral performance index for the stable and unstable system with multi-degrees-of-freedom is effective. It can be applied to determine the optimal controller coefficients dependent on state space variables, to control the chosen forms of the structure.

Time plots of vibration presented in the above examples showed that the level of vibrations was significantly reduced due to the control performance.

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REFERENCES

[1] Gawroński W., Kruszewski J., Ostachowicz W., Tarnowski J., Wittbrodt E.: The finite element method in the dynamics of structures [in Polish]. Arkady, Warsaw 1984.
[2] Gawronski W.: Dynamics and Control of Structures. Modal approach, Springer, New York 1999.
[3] Hein R.: Optimum control of transverse rotor vibrations [in Polish], PhD Thesis, Gdansk University of Technology, Gdańsk 2003.
[4] Kaczorek T.: Synthesis of linear stationary systems by the state-space method [in Polish], PWN, Warsaw 1975.
[5] Kruszewski J., Gawroński W., Wittbrodt E., Najbar F., Grabowski S.: The rigid finite element method [in Polish]. Arkady, Warsaw 1975.
[6] Kruszewski J., Sawiak S., Wittbrodt E.: The rigid finite element method in dynamics of structures. WNT, Warsaw 1999 [in Polish].
[7] Sawicki J. T., Wittbrodt E.: Optimization of actively controlled mechanical system with harmonic disturbances [in Polish]. Scientific Journal, IMP PAN, Gdansk 1990.
[8] Solheim O. A.: Design of optimal control systems with prescribed eigenvalues. International Journal Control, 1972, Vol. 15, No. 1, pp. 143÷160.
[9] Takahashi Y., Rabins M. J., Auslander D. M.: Control and Dynamic Systems. Addison-Wesley Publishing Company, 1970.
[10] Wittbrodt E., Adamiec-Wójcik F., Wojciech S.: Dynamics of Flexible Multibody Systems. Rigid Finite Element Method. Foundation of Engineering Mechanics. Springer-Verlag Berlin Heidelberg 2006.

Optymalne sterowanie wybranymi formami drgań układów mechanicznych

S t r e s z c z e n i e

W artykule przedstawiono zagadnienie optymalnego sterowania wybranymi formami drgań układów mechanicznych. Dla zaprezentowania sposobu i procedury wyznaczania optymalnych nastaw regulatora i współczynników wagowych wskaźnika jakości posłużono się dwoma przykładami liczbowymi. W pierwszym z nich rozważano uproszczony układ mechaniczny składający się

z dwóch sztywnych elementów skończonych połączonych za pomocą elementów sprężysto-tłumiących, zaś w drugim wirnik laboratoryjny, sterowany łożyskiem magnetycznym. W obydwu przypadkach zdefiniowano całkowy wskaźnik jakości, aby minimalizować poziom drgań w wybranych miejscach układów.

Układ z łożyskiem magnetycznym jest strukturalnie niestabilny. Z tego względu przedstawiono sposób doboru współczynników wagowych całkowego wskaźnika jakości dla układu niestabilnego taki, aby wybrane formy modalne miały założone wcześniej własności dynamiczne, a wskaźnik przyjmował wartość minimalną. Rezultaty symulacji komputerowych wykazały, że proponowany sposób jest skuteczny i umożliwia sterowanie wybranymi formami drgań układów mechanicznych.