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Communication

Propagation in the Open Cylindrical Guide of Arbitrary Cross Section with the Use of Field Matching Method

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***Abstract*—A simple solution to propagation problem in open waveguides and dielectric fibers of arbitrary convex cross section is presented. The idea of the analysis is based on the direct field matching technique involving the usage of the field projection at the boundary on a fixed set of orthogonal basis functions. A complex root tracing algorithm is utilized to find the propagation coefficients of the investigated guides. Different convex shapes of the guides are analyzed and the obtained results are compared with the alternative solutions to verify the validity of the proposed method.**

***Index Terms*—Cylindrical guides, Dielectric fibers, Field matching, Propagation, Root finding.**

I. Introduction

The problems of electromagnetic wave propagation in open waveguides and dielectric fibers are important and complex issues in microwave and optical engineering. Especially the analysis of leaky and complex modes, radiated from the guide, is the most problematic, which arises from the accuracy of modeling of free space. Nowadays, the discrete methods are the most common techniques of the electromagnetic structure analysis, and are willingly implemented in commercial software, due to their flexibility. In these methods, open space modeling is usually realized by the utilization of absorbing boundary conditions (introduction of the perfectly matched layer at the boundary of computational domain [1]). Such an approach in frequency domain leads to the occurrence of a number of spurious modes (artificial solutions), which are difficult to distinguish from the proper ones [2]. In time domain, the analysis requires three-dimensional modeling and a large number of iterations, so it is time and memory consuming [3]. Obviously, for simple structures the analytical expressions for modeling electromagnetic wave propagation can be found. For more complex structures it is more efficient to utilize the integral equation method [4], [5], and techniques based on multipole expansions [6]. However, from the numerical point of view, the use of Green's function can also be complicated, due to the singular points in the computational domain [7].

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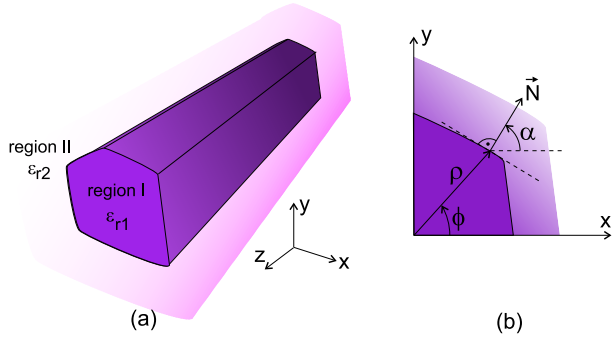

Fig. 1. The geometry of an investigated structure: (a) General view, (b) The definition of angle α.

In this communication, we present a simple and intuitive solution to the propagation problems in open waveguides of arbitrary convex cross section. The method of analysis is adopted from the radiation modeling with finite-difference method [8] and scattering problem with field matching method [9] and is based on the decomposition of the fields in the area of the guide into Fourier-Bessel series with unknown coefficients. The fields are matched at the boundary using their projection on a fixed set of orthogonal basis functions. Such an approach is flexible since it requires only to define the contour representing the waveguide boundary. It does not require the discretization of the domain or the utilization of Green's function (no electric and magnetic currents), therefore its computation and implementation are significantly less complex.

In the proposed approach, the problem boils down to finding the roots, representing propagation coefficients, of simple determinant. For the guided modes in lossless waveguides the problem simplifies to real domain and many standard numerical methods can be applied, such as bisection, secant or Newton's methods. However, for complex or leaky modes and guided modes in lossy medium such methods are ineffective as the solution is a complex number. In the proposed approach to find the guide propagation coefficients we utilize two novel complex root finding algorithms [10], [11]. In the case of roots located at the brunch cut a simple technique involving a pointwise product of all the Riemann sheets can be applied [12].

Several guide geometries and a few guided and leaky modes are considered and the results are discussed. The obtained results are verified by comparison with analytical solution or simulations performed using commercial software employing a finite element-based method [13].

II. Formulation of the problem

The investigated structure is a cylindrical waveguide of arbitrary convex cross section as illustrated in Fig. 1. Here, we consider the problem of electromagnetic wave propagation and the aim of the analysis is to determine the propagation coefficients of guided, leaky and/or complex modes. As the method itself was described in details in [9], here we will only indicate the differences and the required modifications made to adjust this technique for the considered problem.

Two regions of investigation can be distinguished in the structure: region I, located inside the waveguide, and region II, outside. The z components of the electric and magnetic fields in both regions have the following form (suppressing $e^{j\omega t}$ time dependence):

$$F_z^{\mathrm{I}} = \sum_{m=-M}^{M} A_m^F J_m(\kappa_{\mathrm{I}}\rho) e^{jm\phi} e^{-\gamma z} \tag{1}$$

$$F_z^{\mathrm{II}} = \sum_{m=-M}^{M} B_m^F H_m^{(2)}(\kappa_{\mathrm{II}}\rho) e^{jm\phi} e^{-\gamma z} \tag{2}$$

where $F = \{E, H\}$, $\kappa_i^2 = \omega^2\mu_i\varepsilon_i + \gamma^2$ for $i = \{\mathrm{I}, \mathrm{II}\}$, ω is the angular frequency, γ is the mode propagation coefficient, $J_m(\cdot)$ and $H_m^{(2)}(\cdot)$ are Bessel and Hankel functions, respectively, of order m and A_m^F and B_m^F are unknown field coefficients. The utilization of the Hankel function of the second kind satisfies Sommerfeld's radiation condition (representing the outward-traveling wave). Due to the assumed field representation in (1) and (2), only convex shapes of the waveguide can be analyzed. The other components of the electric and magnetic fields (E_ϕ, E_ρ, H_ϕ and H_ρ) can be derived from Maxwell's equations, as in [14].

In order to determine the mode propagation coefficients we need to satisfy the continuity conditions for the tangential field components on the guide surface. Describing the surface of the guide by functions $\rho = \varrho(s)$ and $\phi = \varphi(s)$, where s is the curvilinear coordinate that follows the surface, the continuity conditions for tangential components can be written as follows:

$$F_z^{\mathrm{I}}(\varrho(s), \varphi(s), z) = F_z^{\mathrm{II}}(\varrho(s), \varphi(s), z) \tag{3}$$

$$F_t^{\mathrm{I}}(\varrho(s), \varphi(s), z) = F_t^{\mathrm{II}}(\varrho(s), \varphi(s), z) \tag{4}$$

where $F_t^{(\cdot)}(\cdot) = (\sin\varphi\cos\alpha - \cos\varphi\sin\alpha)F_\rho^{(\cdot)}(\cdot) + (\cos\varphi\cos\alpha + \sin\varphi\sin\alpha)F_\phi^{(\cdot)}(\cdot)$ and $\alpha = \alpha(s)$ is an angle between the x-axis and the normal outgoing vector $\vec{N}$ to the cylinder surface (see Fig. 1).

Similarly, as for scattering problem [9], a projection on the orthogonal set of the functions $w_n(s) = \exp(j2\pi ns/S)/\sqrt{S}$ for $(n = -M \ldots M)$ can be applied in the meaning of the inner product:

$$\langle g|w_n\rangle = \int_0^S g(s) w_n(s)^* \, ds \tag{5}$$

The continuity conditions can then be rewritten in the form of the following matrix equation:

$$\begin{bmatrix} \mathbf{M}^{E,\mathrm{I}} & -\mathbf{M}^{E,\mathrm{II}} \\ \mathbf{M}^{H,\mathrm{I}} & -\mathbf{M}^{H,\mathrm{II}} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} = \mathbf{0} \tag{6}$$

where $\mathbf{A} = [\mathbf{A}^E, \mathbf{A}^H]^T$, with $\mathbf{A}^F = [A_{-M}^F, \ldots, A_M^F]^T$, (vector $\mathbf{B}$ is defined similarly) and matrices $\mathbf{M}^{F,i}$ have the form:

$$\mathbf{M}^{E,i} = \begin{bmatrix} \mathbf{M}_z^{E,i} & \mathbf{0} \\ \mathbf{M}_{t1}^{E,i} & \mathbf{M}_{t2}^{E,i} \end{bmatrix}, \quad \mathbf{M}^{H,i} = \begin{bmatrix} \mathbf{M}_{t1}^{H,i} & \mathbf{M}_{t2}^{H,i} \\ \mathbf{0} & \mathbf{M}_z^{H,i} \end{bmatrix} \tag{7}$$

where the elements of submatrices $\mathbf{M}_{(\cdot)}^{F,i}$ have the following form:

$$\{\mathbf{M}_z^{F,i}\}_{m,n} = \langle \mathcal{Z}_m^i | w_n \rangle \tag{8}$$

$$\{\mathbf{M}_{t1}^{E,i}\}_{m,n} = \left\langle C_\rho \frac{\gamma}{\kappa_i} \mathcal{Z}_m'^i + C_\phi \frac{j\gamma m}{\kappa_i^2 \rho} \mathcal{Z}_m^i \middle| w_n \right\rangle \tag{9}$$

$$\{\mathbf{M}_{t2}^{E,i}\}_{m,n} = \left\langle C_\rho \frac{\omega\mu_i m}{\kappa_i^2 \rho} \mathcal{Z}_m^i + C_\phi \frac{j\omega\mu_i}{\kappa_i} \mathcal{Z}_m'^i \middle| w_n \right\rangle \tag{10}$$

$$\{\mathbf{M}_{t1}^{H,i}\}_{m,n} = \left\langle C_\rho \frac{-\omega\varepsilon_i m}{\kappa_i^2 \rho} \mathcal{Z}_m^i + C_\phi \frac{-j\omega\varepsilon_i}{\kappa_i} \mathcal{Z}_m'^i \middle| w_n \right\rangle \tag{11}$$

$$\{\mathbf{M}_{t2}^{H,i}\}_{m,n} = \left\langle C_\rho \frac{\gamma}{\kappa_i} \mathcal{Z}_m'^i + C_\phi \frac{j\gamma m}{\kappa_i^2 \rho} \mathcal{Z}_m^i \middle| w_n \right\rangle \tag{12}$$

where $\mathcal{Z}_m^i = Z_m(\kappa^i \varrho) e^{jm\varphi}$, with $Z_m = \{J_m, H_m^{(2)}\}$, prime denotes derivative of the function, $C_\rho = \sin\varphi\cos\alpha - \cos\varphi\sin\alpha$ and $C_\phi = \cos\varphi\cos\alpha + \sin\varphi\sin\alpha$.

Nontrivial solutions of the homogenous system (6) exist if its determinant vanishes. The roots of this determinant represent complex propagation coefficients for particular modes. The complex root finding algorithms [10], [11] are utilized to find the solution to the described problem.

It is worth noting that for a fiber of a circular cross section the submatrices (8)-(12) become diagonal and are composed of appropriate Bessel or Hankel functions, or their derivatives with proper coefficients. In this case the procedure boils down to the regular mode matching method and the numerical integration (5) is unnecessary. It is also worth noting that the procedure can be applied for multilayered structures, for which the resultant homogeneous system (described by a matrix equation like the one in (6)) will take more complicated form. However, the efficiency of the proposed procedure decreases in such cases.

III. Results

In order to support the validity of the proposed method several guided and leaky modes in fibers with various cross sections are investigated. For each structure, we start from cylindrical fiber with circular cross section and transform its shape, maintaining constant area of the guide cross section, to obtain different fiber geometries: oval, square and triangle. We consider three different types of waves: transversal magnetic, transversal electric and hybrid modes. In the calculations, we picked the TE_{01}, TM_{01} and HE_{11}, which propagation coefficients for the case of circular fiber are given in Table I.

All the numerical results presented here are obtained with the use of root finding/tracing algorithms [10], [11]. For the guided modes the values of normalized phase coefficients are located in the range determined by the lowest and the highest permittivity of the dielectrics in the structure $\beta_n \in \left[\min_i\{\sqrt{\varepsilon_{r,i}}\}, \max_i\{\sqrt{\varepsilon_{r,i}}\}\right]$. The attenuation coefficient

TABLE I
PROPAGATION COEFFICIENTS OF THE INVESTIGATED WAVES IN CIRCULAR DIELECTRIC FIBER OF RADIUS 0.5 μM WITH $\varepsilon_{r1} = 8.41$ AND $\varepsilon_{r2} = 2.4025$ AT FREQUENCY $f = 10^{14}$ HZ [8].

mode	normalized propagation coefficient	color in figures
TE_{01}	$j1.6255$	red
TM_{01}	$j1.5708$	yellow
HE_{11}	$0.395 + j1.2214$	green(blue)

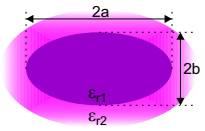

Fig. 2. The geometry of the elliptical fiber cross section.

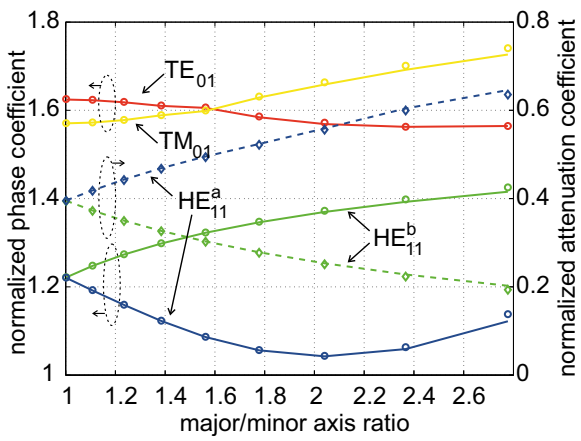

Fig. 3. The propagation coefficients for the elliptical fiber in function of major/minor axis ratio with parameters from Table I. Solid line – normalized phase coefficients; dashed line – normalized attenuation coefficients; circles and diamonds – analytical results.

TABLE II
CONVERGENCE OF THE METHOD FOR THE EXAMPLE FROM FIG. 2 (VALUES OF PROPAGATION COEFFICIENTS OF TM_{01} MODE FOR MAJOR/MINOR AXIS RATIO EQUALS 3). PERCENTAGE ERROR IN BRACKETS WITH RESPECT TO ANALYTICAL SOLUTION.

M	$P = 90$	$P = 180$	$P = 360$	$P = 720$
5	1.7411 (0.97)	1.7417 (0.94)	1.7418 (0.93)	1.7418 (0.93)
6	1.7521 (0.36)	1.7496 (0.50)	1.7491 (0.53)	1.7490 (0.54)
7	1.7522 (0.35)	1.7496 (0.50)	1.7491 (0.53)	1.7490 (0.54)
8	1.7405 (1.02)	1.7518 (0.38)	1.7526 (0.33)	1.7528 (0.32)
9	–	1.7510 (0.42)	1.7527 (0.33)	1.7527 (0.33)
10	–	–	1.7545 (0.22)	1.7562 (0.13)

equals zero for guided modes in lossless structures, however it can be traced in a function of increasing loses. Similarly, the propagation coefficients of leaky/complex modes can be found by tracing them in a function of decreasing frequency (starting from guided modes).

As a first example, an elliptical optical fiber is analyzed. The cross section of the guide is shown in Fig. 2. The analytical solution of such structure is well known and can be found using Mathieu functions for the field expansions. The change of propagation coefficients of investigated modes in function

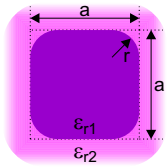

Fig. 4. The geometry of the square fiber cross section with rounded corners. The material parameters as in Table I.

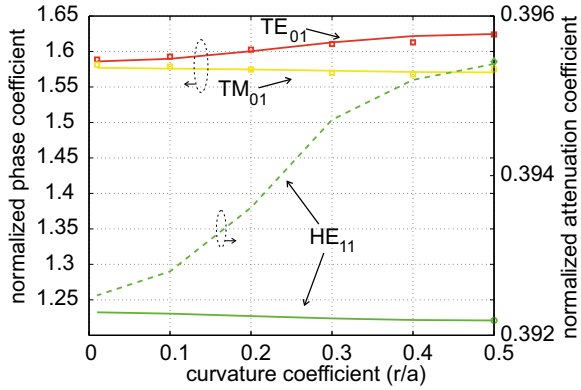

Fig. 5. The propagation coefficients for the square fiber in function of corner curvature. Solid line – normalized phase coefficients; dashed line – normalized attenuation coefficients; squares – HFSS results; circles and diamonds – analytical results.

TABLE III
CONVERGENCE OF THE METHOD FOR THE EXAMPLE FROM FIG. 4 (VALUES OF PROPAGATION COEFFICIENTS). PERCENTAGE ERROR IN BRACKETS WITH RESPECT TO HFSS SOLUTION.

M	$P = 90$	$P = 180$	$P = 360$	$P = 720$
5	1.5763 (0.16)	1.5764 (0.15)	1.5764 (0.15)	1.5765 (0.15)
6	1.5763 (0.16)	1.5765 (0.15)	1.5765 (0.15)	1.5765 (0.15)
7	1.5763 (0.16)	1.5764 (0.15)	1.5764 (0.15)	1.5765(0.15)
8	1.5779 (0.06)	1.5781 (0.04)	1.5781 (0.04)	1.5781 (0.04)
9	1.5779 (0.06)	1.5781 (0.04)	1.5781 (0.04)	1.5781 (0.04)
10	1.5779 (0.06)	1.5781 (0.04)	1.5781 (0.04)	1.5781 (0.04)

of fiber ellipticity (major/minor axis ratios) is calculated and the results are compared with the analytical ones (see Fig. 3). The calculations were performed using $M = 7$ expansion functions, and the integrals in (7) were evaluated from the trapezoidal rule, with $P = 180$ points evenly covering the boundary contour. Such a choice results from convergence analysis, which is presented in Table II. Theoretically, the higher values of M should improve the accuracy of the results as it should better describe the field in the analyzed structure. However, too high values of M results in the increase of calculation time and moreover could lead to the appearance of numerical errors. The value of M is directly connected to number of points P evenly covering the boundary of the structure. The selection of too small number of discretization points P results in ill-conditioning of homogeneous system equation (6) and leads to the increase of the numerical error for higher values of M.

As can be observed, there is a satisfactory agreement between the results obtained from both methods. The analysis

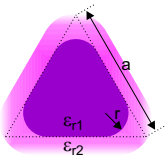

Fig. 6. The geometry of the triangular fiber cross section with rounded corners.

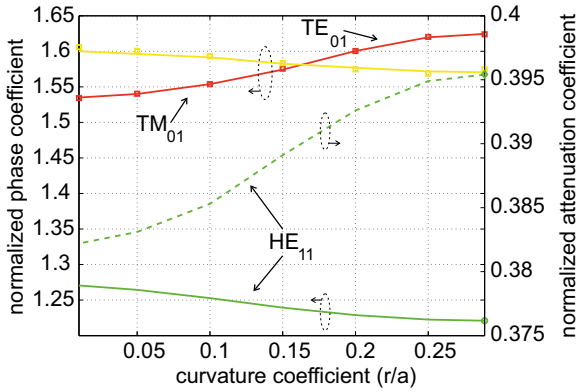

Fig. 7. The propagation coefficients for the triangular fiber in function of corner curvatures. Solid line – normalized phase coefficients; dashed line – normalized attenuation coefficients; squares – HFSS results; circles and diamonds – analytical results.

also shows that the more deformed the structure is (higher major/minor axis ratio) the more expansion functions need to be selected. For the considered case, it was sufficient to use $M = 7$ for ellipse with major/minor axis ratio equals 3. It is also worth noting that the hybrid mode HE_{11} separates into two different waves, due to lack of axial symmetry in elliptical structure.

The second structure is a dielectric fiber with square cross section (rounded corners) with the same material parameters as in the previous example. The analysis is performed in function of corner curvature coefficient (r/a) as illustrated in Fig. 4. The calculated propagation coefficients for different corner curvatures are presented in Fig. 5. The calculations were performed for $M = 10$ and $P = 360$. The convergence analysis is presented in Table III.

Also in this case the obtained results agree with the simulations performed in commercial software HFSS [13]. However, due to the unreliability of modal analysis in discrete methods for leaky modes (modeling of boundary conditions) only the

TABLE IV
CONVERGENCE OF THE METHOD FOR THE EXAMPLE FROM FIG. 6 (VALUES OF PROPAGATION COEFFICIENTS). PERCENTAGE ERROR IN BRACKETS WITH RESPECT TO HFSS SOLUTION.

M	$P = 90$	$P = 180$	$P = 360$	$P = 720$
5	1.5907 (0.61)	1.5910 (0.60)	1.5910 (0.60)	1.5910 (0.60)
6	1.5959 (0.29)	1.5963 (0.26)	1.5963 (0.26)	1.5963 (0.26)
7	1.5959 (0.29)	1.5963 (0.26)	1.5963 (0.26)	1.5963 (0.26)
8	1.5959 (0.29)	1.5963 (0.26)	1.5963 (0.26)	1.5963 (0.26)
9	1.5983 (0.14)	1.5987 (0.12)	1.5987 (0.12)	1.5987 (0.12)
10	1.5983 (0.14)	1.5987 (0.12)	1.5987 (0.12)	1.5987 (0.12)

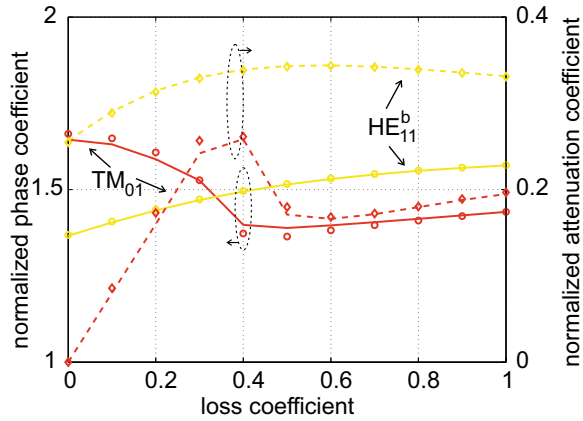

Fig. 8. The propagation coefficients for the elliptical fiber with major/minor axis ratio equals 2 in function of losses l_c. Solid line – normalized phase coefficients; dashed line – normalized attenuation coefficients; circles and diamonds – analytical results.

guided modes are compared.

The third example considers the triangular fiber with rounded corners depicted in Fig. 6. The calculated propagation coefficients for different corner curvatures is presented in Fig. 7. The calculations were performed for $M = 10$ and $P = 360$. The convergence analysis is presented in Table IV.

As in the previous example a good agreement between the calculated results and the commercial software simulations is achieved.

The proposed method can also be utilized to investigate structures with high losses. Therefore, as a last example we consider the elliptical guide with major/minor axis ratio equals 2 and calculate propagation coefficients in function of losses. It was assumed that the inner dielectric is lossy with permittivity $\varepsilon_{r1} = 8.41(1 - jl_c)$ (the l_c is a loss coefficient). The obtained results from the proposed method compared with analytical approach are presented in Fig. 8.

IV. CONCLUSION

The approach presented in this communication utilizing field matching technique is simple and effective. The technique can be successfully used to examine both leaky and guided modes and can be applied for high lossy media. The obtained results are in good agreement with those obtained from analytical approach and a finite element-based method (commercial software). The method does not require the utilization of Green's function, discretization of the computational domain and implementation of absorbing boundary conditions.

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