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Propagation of acoustic pulses in some fluids with yield stress

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Abstract

This study is devoted to derivation of approximate equations governing acoustic pulse in flows with yield stress, including some time-dependent flows with slow dependence of yield stress and apparent viscosity on time. Modeling of yield stress and apparent viscosity in the vicinity of zero deformation rate allows to consider thixotropic fluid as a Bingham plastic with coefficients dependent on time. The ordering scheme results in equations valid in the leading order with respect to a number of small parameters. The unusual property of domains of a waveform with positive and negative shear stress to propagate with different velocities is discovered. The illustration concerns the bipolar initially acoustic pulse which becomes positive after some time. The proposed theory predicts broadening of a pulse due to the yield stress starting from some time from the beginning of evolution.

Keywords: Non-newtonian liquids, Thixotropic fluids, Acoustics, Bingham plastic

1 Introduction

Many fluids reveal non-newtonian behavior. In contrast with newtonian fluids, the non-newtonian properties are cased by the viscous dissipation of energy due to collisions between large particles, or to the distortion of or the collision between colloidal structures. One of the main types in this group is Bingham plastic. Paints, slurries, pasts and some food products are described by Bingham model, i.e., a yield stress followed by Newtonian behavior with constant viscosity when shear rate differs from zero. Bingham plastics belong to the category of time independent fluids, because their shear rate depends only on a shear stress and is a single valued function of it.

The thixotropic fluids belong to the class of time dependent fluids, i.e. fluids, whose viscosity depends on the duration of flow [1]. As for thixotropic fluid, stress tensor and apparent viscosity depend not only on shear rate at any instant, but on the previous history of a fluid, for example, wether or not it has been stirred recently. That makes studies of the flows of these fluids fairly difficult [1, 2, 3]. The apparent viscosity of thixotropic fluids usually decreases as the shearing continues. The type of negative thixotropic fluids, which apparent viscosity increases with shear rate and duration of shear, are much less common that the thixotropic kind [1]. Particles of the shape of long needles, thin discs or polymeric particles are thixotropic to a greater degree than molecules of uniform shapes such as spheres. Any theory of thixotropy must account for the time dependence. Some thixotropic materials initially behave as Bingham plastics in that they possess a yield value, below which no flow takes place. There are many food products among them: molten chocolate and creams of all sorts, ketchup and other sauces [4, 5]. They are sometimes referred to as thixotropic-plastic fluids, with values of yield stress and apparent viscosity which should be remembered varies depending on shear rate and duration of shear.

The main idea of this study which is devoted to sound propagation in time-dependent flows with threshold stress, is to consider the domain in the vicinity of zero deformation rates because it is only of interest in the weakly nonlinear acoustics. That allows to apply the ordering scheme which starts from establishing of stress tensor as the Taylor series in powers of shear rate with coefficients (namely the yield stress and apparent viscosity) dependent on time. One more simplifying condition will refer to the smallness of variations of these coefficients with time as compared with variations of perturbations in the sound wave. In the considered example of time-dependent foods, that is valid for enough large times. A Bingham plastic may be considered as a simple limit of a thixotropic fluid with constant threshold stress and viscosity. As far as the author knows, acoustics of Bingham plastics is a new domain to study. The ordering procedure allows to establish approximate relations between perturbations in the sound wave and to derive dynamic equations for it. In spite of evident difficulties in analytical description of flow characteristics of time dependent fluids, the simplifying conditions make possible to consider the problem in general and to conclude about distortion of sound waveforms in them.

2 Dynamic equations and stress tensor in a fluid with a yield stress

The continuity, momentum and energy equations describing any viscous fluid flow without external forces read:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla} \cdot (\rho \overrightarrow{v}) = 0 \\
&\frac{\partial \overrightarrow{v}}{\partial t} + (\overrightarrow{v} \cdot \overrightarrow{\nabla})\overrightarrow{v} = \frac{1}{\rho}\left(-\overrightarrow{\nabla} p + Div\,\mathbf{P}\right), \\
&\frac{\partial e}{\partial t} + (\overrightarrow{v} \cdot \overrightarrow{\nabla})e = \frac{1}{\rho}\left(-p(\overrightarrow{\nabla} \cdot \overrightarrow{v}) + \mathbf{P} : Grad\,\overrightarrow{v}\right),
\end{aligned} \tag{1}$$

$\vec{v}$ denotes the displacement (or particle) velocity, ρ, p are density and pressure of a fluid, respectively, e denote its internal energy per unit mass, and x_i, t are the spatial coordinates and time. The operator Div denotes the divergence of a tensor, $Grad$ is a dyad gradient, and $\mathbf{P}$ is the viscous stress tensor.

The thermodynamic function $e(p, \rho)$ complements the system (1). It may be written as series of the excess internal energy $e' = e - e_0$ in powers of excess pressure and density $p' = p - p_0$,

$\rho' = \rho - \rho_0$ (ambient quantities are marked by index 0):

$$e' = \frac{E_1}{\rho_0}p' + \frac{E_2 p_0}{\rho_0^2}\rho' + \frac{E_3}{p_0\rho_0}p'^2 + \frac{E_4 p_0}{\rho_0^3}\rho'^2 + \frac{E_5}{\rho_0^2}\rho' p' + \ldots, \tag{2}$$

where $E_1 = \frac{\rho_0 C_v \kappa}{\beta}$, $E_2 = -\frac{C_p \rho_0}{\beta p_0} + 1$, E_3, E_4, E_5 are dimensionless coefficients, C_v, C_p denote the heat capacities per unit mass under constant volume and pressure, and κ, β mark compressibility and the thermal expansion, respectively:

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T, \beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p. \tag{3}$$

The series (2) allows consideration of a wide variety of fluids in the general form: discrepancies are manifested by the different coefficients for different fluids. Infinitesimal sound velocity in a non-viscous fluid (when $\mathbf{P} = \mathbf{0}$), is

$$c_0 = \sqrt{\frac{(1 - E_2)p_0}{E_1 \rho_0}}. \tag{4}$$

Determination of stress tensor is the main point in studies of any viscous flow. As for studies of weakly nonlinear dynamics of a sound and associated with it phenomena, the ordinary scheme focuses on equations including nonlinear terms of order not higher than second in the acoustic Mach number $M = v_0/c_0$, where v_0 is a typical velocity magnitude in a fluid. We will consider one-dimensional motion along axis OX with one-component shear rate $\dot{\gamma} \equiv \partial v/\partial x$ and shear stress $\mathbf{P}_{x,x} \equiv \sigma$. In this study, the ordering scheme, along with the Mach number, refer to some other small parameters. The first is associated with viscosity η. For sound to be the wave process, it must be weakly attenuated during its period. The second condition requires smallness of the yield stress, σ_0. Hereafter will be shown that the yield stress contributes to the sound speed in a fluid. We will consider the additional term stipulated by the yield stress small compared with the sound speed in a medium without yield stress. To facilitate implementation of the ordering scheme, we will consider the first parameter of order M, and the second one of the same order or smaller,

$$\frac{\eta\Omega}{\rho_0 c_0^2} = O(M), \quad \frac{\sigma_0}{E_1 \rho_0 c_0^2} = O(M^n), \tag{5}$$

where $n \geq 1$, and Ω denotes a characteristic frequency of sound. The first requirement may be easily satisfied by choice of correspondent frequency of sound. It is well-known that sound of higher frequency attenuates stronger. The second demand refers to the properties of the medium itself. Typically, yield stress is not a large quantity in structured liquids, it varies from units till thousands Pa [4]. The denominator in the second condition is expected to be large. For example, $E_1 \rho_0 c_0^2$ in water at normal conditions is about $2 \cdot 10^{10}$ Pa (water is not thixotropic fluid, but in contradistinction to most of them, its thermodynamic properties is well-studied). The typical value of the Mach number varies from 10^{-3} till 10^{-2}. So that, the second parameter from Eq.(5) may be considerably smaller than the first one, but it is of importance since it participates in the linear part of energy equation and hence is responsible for the linear dynamics of sound. Our primary objective is to derive model equations including nonlinear terms at order M^2 outside thermoviscous boundary layers of a viscous fluid. In view of this, the terms including $\dot{\gamma}^0$ and $\dot{\gamma}^1$ only are of importance in the series of stress tensor. In the vicinity of zero shear rate, positive or negative, shear stress may be expanded in the Taylor series and takes the form dependent on the sign of shear rate:

$$\sigma = \begin{cases} \sigma_0 + \eta\dot{\gamma}, & \text{if} \quad \dot{\gamma} > 0, \\ -\sigma_0 + \eta\dot{\gamma}, & \text{if} \quad \dot{\gamma} < 0. \end{cases} \tag{6}$$

If $\dot{\gamma} = 0$, the shear stress may take values from $-\sigma_0$ till σ_0. Eqs (6) describes exactly a Bingham plastic if σ_0 and η are constants [6] (in this case, it is correct not for only small shear rates) and, in general, a thixotropic fluid if σ_0 and η vary with time. That makes possible not to consider dependence of these quantities on shear rate. The figure below illustrates this simple idea.

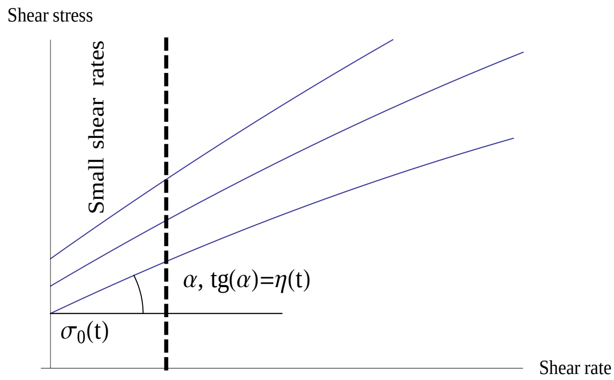

Fig. 1. In the vicinity of zero yield stress, σ_0 and η are functions of time in thixotropic flows, and constants in Bingham plastics.

3 Equations describing dynamics of sound

It is convenient to rearrange formulae in the dimensionless quantities

$$p^* = \frac{p'}{c_0^2 \cdot \rho_0}, \; \rho^* = \frac{\rho'}{\rho_0}, \; v^* = \frac{v}{c_0}, \; x^* = \frac{\Omega x}{c_0}, \; t^* = \Omega t, \; \sigma_0^* = \frac{\sigma_0}{E_1 \rho_0 c_0^2}, \; \eta^* = \frac{\Omega \eta}{\rho_0 c_0^2}. \tag{7}$$

Starting from Eqs (8), the upper indexes (asterisks) denoting the dimensionless quantities will be omitted everywhere in the text. In the dimensionless quantities, Eqs (1) accounting for Eq. (2), read:

$$\begin{aligned} \underbrace{\frac{\partial \rho}{\partial t} + \frac{\partial v}{\partial x}}_{linear} &= \underbrace{-v\frac{\partial \rho}{\partial x} - \rho\frac{\partial v}{\partial x}}_{O(M^2)}, \\ \underbrace{\frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} - \eta\frac{\partial^2 v}{\partial x^2}}_{linear} &= \underbrace{-v\frac{\partial v}{\partial x} + \rho\frac{\partial p}{\partial x}}_{O(M^2)}, \\ \underbrace{\frac{\partial p}{\partial t} + (1+S)\frac{\partial v}{\partial x}}_{linear} &= \underbrace{-v\frac{\partial p}{\partial x} + (D_1 p + D_2 \rho)\frac{\partial v}{\partial x}}_{O(M^2)}, \end{aligned} \tag{8}$$

where

$$S = \begin{cases} -\sigma_0, & \text{if} \quad \left(\frac{\partial v}{\partial x}\right) > 0, \\ \sigma_0, & \text{if} \quad \left(\frac{\partial v}{\partial x}\right) < 0, \end{cases} \tag{9}$$

and

$$D_1 = \frac{1}{E_1}\left(-1 + 2\frac{1-E_2}{E_1}E_3 + E_5\right), \quad D_2 = \frac{1}{1-E_2}\left(1 + E_2 + 2E_4 + \frac{1-E_2}{E_1}E_5\right). \tag{10}$$

3.1 Sound of infinitely small magnitude

The set of equations (8) describes weakly nonlinear dynamics of all possible motions in a fluid, including two acoustic modes, propagating in positive or negative direction of axis OX, and the thermal (non-wave) mode. The linearized version of system (8), which describes motion of infinitely small magnitude, yields in relations connecting dimensionless perturbations of density, pressure and velocity for any mode. It is of importance to derive the dynamic equation governing one from two acoustic branches individually. It would include the first partial derivative with respect to time in contrast with equation describing both these branches, which includes the second partial derivative. If S and η were constants independent on time, relationships for progressive in the positive direction of axis OX sound, would look like

$$p_a = \left(1 + \frac{S}{2}\right) v_a + \frac{\eta}{2}\frac{\partial}{\partial x} v_a, \quad \rho_a = \left(1 - \frac{S}{2}\right) v_a + \frac{\eta}{2}\frac{\partial}{\partial x} v_a. \tag{11}$$

These relations in fact may be obtained by requiring equivalence of all three linearized equations from the set (8) in terms of v_a for the mode propagating in the positive direction of axis OX with the approximately unit velocity. The important and unusual property of sound mode is that its definition depends (by means of S) on the sign of velocity gradient. The linear equation describing the fluid velocity in an acoustic wave propagating in the positive direction of axis OX, with account for (11), takes the form which describes differently cases of positive and negative shear rates:

$$\frac{\partial v_a}{\partial t} + \left(1 - \frac{\sigma_0}{2}\right)\frac{\partial v_a}{\partial x} - \frac{\eta}{2}\frac{\partial^2 v_a}{\partial x^2} = 0, \ \frac{\partial v_a}{\partial x} > 0, \tag{12}$$
$$\frac{\partial v_a}{\partial t} + \left(1 + \frac{\sigma_0}{2}\right)\frac{\partial v_a}{\partial x} - \frac{\eta}{2}\frac{\partial^2 v_a}{\partial x^2} = 0, \ \frac{\partial v_a}{\partial x} < 0.$$

We may readily derive demand on smallness of variations of S and η in time making Eqs (11), (12) valid in the leading order,

$$\frac{\partial S}{\partial t} = O(M^{n+1}), \quad \frac{\partial \eta}{\partial t} = O(M^2). \tag{13}$$

To satisfy above conditions, S and η must vary slower approximately M^{-n} and M^{-1} times than perturbations in the sound wave itself, correspondingly. Eqs (12) in the limit $\eta = 0$ have exact solutions

$$v_a = \varphi\left(x - t + \frac{1}{2}\int_0^t \sigma_0(\tau)d\tau\right), \ \frac{\partial v_a}{\partial x} > 0, \tag{14}$$
$$v_a = \varphi\left(x - t - \frac{1}{2}\int_0^t \sigma_0(\tau)d\tau\right), \ \frac{\partial v_a}{\partial x} < 0,$$

satisfying the initial condition $v_a(x, t = 0) = \varphi(x)$. Velocity of propagation of the waveform domains where shear rates is positive, is smaller comparatively with that of the domains of negative shear rate and does not depend on the absolute value of shear rate. That expands the part of positive particle velocities. Fig.2 illustrates this property. The important peculiarity of the liquid flows with a yield stress which differs them from Newtonian, is that the momentum of one-dimensional pulsed sound is not constant in time. We may rearrange the second equation from Eqs (8) with account for Eqs (11) into the following leading-order equation

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(-p + \eta\frac{\partial v}{\partial x} - \frac{v^2}{2} + \frac{p^2}{2}\right) \approx \frac{\partial}{\partial x}\left(-p + \eta\frac{\partial v}{\partial x}\right). \tag{15}$$

In regard to the bipolar initially impulse sound (the curve marked by $t_1 = 0$ in Fig.2), integrating Eq.(15) over the space coordinate x, and taking into account that for a pulsed sound $\partial v/\partial x(x = \pm\infty) = 0$, yields

$$\frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} v dx\right) = -\left(1 - \frac{\sigma_0}{2}\right) v\,\Big|_{v_{min}}^{v_{max}} + \left(1 + \frac{\sigma_0}{2}\right) v\,\Big|_{v_{min}}^{v_{max}} = \sigma_0(v_{max} - v_{min}), \tag{16}$$

where v_{min} and v_{max} denote minimum and maximum value of the particle velocity in a pulse, correspondingly. Thus the area of the particle velocity profile, which is known to be proportional to the momentum of acoustic pulse [7], is not conserved during its propagation. That concerns both linear and weakly nonlinear regimes of viscous flow. In standard viscous flow, momentum does not vary [7, 8]. At the first stage, when a pulse of zero initial momentum holds bipolar, its width remains constant but its positive parts broadens wile negative become smaller. The amplitude of the negative part reduces (it depends on time), but of the positive part remains constant. A bipolar acoustic pulse of nonzero initial momentum in its propagation, first, transforms into a unipolar pulse of the same duration, and only after that broadening of the pulse starts.

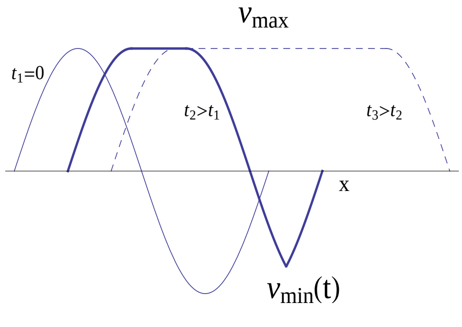

Fig. 2. Schematic dynamics of initially bipolar acoustic pulse in the limit $\eta = 0$.

The momentum of the bipolar pulse slowly increases with time. Starting from time t for which $\int_0^t \sigma_0(\tau)d\tau \approx \pi$, its momentum increases as $M^{n+2}t$, and it begins to broaden. Its energy dissipates due to viscosity which is not considered by this simple example. Account for viscosity would lead to the preferential absorption of high-frequency components in the initial spectrum of a pulse, and as a result, to the expansion of a pulse proportionally to $\sqrt{t}$ and decrease in its energy as $1/\sqrt{t}$ at enough large times [7]. If due to nonlinear distortions the shock wave forms, that leads to the additional absorption of sound energy on its front.

3.2 Weakly nonlinear dynamics of sound

Once linear relationships of perturbation in the sound are established (Eqs(11)), they may be complemented by the second-order nonlinear terms making all three equations from the set (8) equivalent within the accepted accuracy, i.e. including nonlinear terms of order not higher than M^2. They take the form

$$p_a = \left(1 + \frac{S}{2}\right) v_a + \frac{\eta}{2}\frac{\partial}{\partial x}v_a + \frac{1 - D_1 - D_2}{4}v_a^2, \quad \rho_a = \left(1 - \frac{S}{2}\right) v_a + \frac{\eta}{2}\frac{\partial}{\partial x}v_a + \frac{3 + D_1 + D_2}{4}v_a^2, \quad (17)$$

and result in the weakly nonlinear dynamic equations

$$\frac{\partial v_a}{\partial t} + \left(1 - \frac{\sigma_0}{2}\right)\frac{\partial v_a}{\partial x} - \frac{\eta}{2}\frac{\partial^2 v_a}{\partial x^2} + \varepsilon v_a \frac{\partial v_a}{\partial x} = 0, \quad \frac{\partial v_a}{\partial x} > 0, \quad (18)$$

$$\frac{\partial v_a}{\partial t} + \left(1 + \frac{\sigma_0}{2}\right)\frac{\partial v_a}{\partial x} - \frac{\eta}{2}\frac{\partial^2 v_a}{\partial x^2} + \varepsilon v_a \frac{\partial v_a}{\partial x} = 0, \quad \frac{\partial v_a}{\partial x} < 0,$$

with $\varepsilon = \frac{1-D_1-D_2}{2}$ being the parameter of nonlinearity in a fluid. Since the equation of state in a thixotropic fluid, and hence D_1 and D_2, may depend on time, the conditions of validity of Eqs (18) include, along with Eqs (13), requirement of slow variation of D_1 and D_2 in time. In view of they stand by the nonlinear term of order M^2, this demand sounds

$$\frac{\partial D_1}{\partial t} = O(M), \quad \frac{\partial D_2}{\partial t} = O(M). \quad (19)$$

The most simple example, which does not require numerical calculations, considers the nonlinear dynamics of stationary sound without account of viscosity, described by Eqs (18). Fortunately, the family of stationary solutions in the form progressive in the positive direction of axis OX,

$$v_a = -\frac{C_1}{\sqrt{\varepsilon}}\tanh\left(\frac{C_1\sqrt{\varepsilon}}{\eta}(x - t - \frac{1}{2}\int_0^t \sigma_0(\tau)d\tau) + C_2\right) \tag{20}$$

possesses negative $\partial v/\partial x$ for any x. They are exact solutions for the second equation from the set (18) for constant η and approximate ones if η slowly depends on time. C_1 and C_2 are constants originating from double integration. The width of the wave front equals approximately $2\eta/(|C_1|\sqrt{\varepsilon})$. The velocity of this stationary waveform is greater than dimensionless sound velocity 1 in a medium without threshold stress and depends on time. Analytical solution of Eqs (18) when $\eta = 0$, progressive in the positive direction of axis OX, is represented by formulas

$$v_a = \varphi\left(x - t\,(1 + \varepsilon v_a) + \frac{1}{2}\int_0^t \sigma_0(\tau)d\tau\right), \ \frac{\partial v_a}{\partial x} > 0, \tag{21}$$

$$v_a = \varphi\left(x - t\,(1 + \varepsilon v_a) - \frac{1}{2}\int_0^t \sigma_0(\tau)d\tau\right), \ \frac{\partial v_a}{\partial x} < 0.$$

The nonlinear distortions of the sound wave themselves are well-studied [7, 9]. In view of that parameter of nonlinearity ε in fluids is positive, the nonlinearity makes the parts with larger particle velocities to propagate faster. The conditions of the shock wave forming are identical with those in the flows without yield stress, namely the time required for that equals $(\varepsilon M)^{-1}$. Energy of the shock pulse wave decreases in time as $\left(\sqrt{\frac{1}{2} + \varepsilon M t}\right)^{-1}$ due to nonlinear dissipation [7], slower than that of the periodic sound.

4 Yield stress and apparent viscosity in some thixotropic fluids

The idea of considering a weakly nonlinear dynamics of sound, and to use the series of stress tensor $\sigma(t, \dot{\gamma})$ in the vicinity of zero shear rates, $\dot{\gamma} = 0$, may be employed to a wide variety of thixotropic fluids. In order to solve (analytically or numerically) equations of the previous section, one does not require knowledge about complex behavior of a flow at mediate and high shear stresses, only data relating to the low shear stresses, and only two first coefficients in the Taylor series, Eq.(6) which are in fact the threshold stress and apparent viscosity depending exclusively on time. Some analytical formulae based on the kinetic theory are useless, because they do not describe the domain of small shear stresses within satisfactory accuracy. For example, the experimentally measured values of shear stress in stirred yogurt at shear rates less than $100s^{-1}$ would be underestimated by $20 - 50\%$ [10]. The error increased with enlargement of shear rate and was particulary evident at shear rates between 5 and $15s^{-1}$, namely in the domain of interest in weakly nonlinear acoustics. Most experimental investigations of thixotropic fluids consider small shear rates, but do not concern the very area of zero shear rates. We may assume continuity of the dependence of shear stress on shear rate and to evaluate the required quantities σ_0 and η by interpolation of deformation rate towards zero at any instant.

The logarithmic time model proposed by Weltmann and previously used to describe the stress decay behavior of time-dependent foods [5, 11], where in dimensional quantities

$$\sigma = A + B\ln(t/T_m), \tag{22}$$

and T_m is the characteristic parameter to describe structure rebuilding. This model was found to give good correlations when applied to the data. For stirred yogurt, the following power law and logarithmic models were found to best describe the behavior of A and B, respectively, over the full range of shear rates used, i.e., $5-700s^{-1}$ [12]:

$$A = A_1\dot{\gamma}^{A_2}, \quad B = -B_1 \cdot \ln(\dot{\gamma}) - B_2, \tag{23}$$

$A_1 = 37.18$, $A_2 = 0.19$, $B_1 = 2.01$, $B_2 = 0.549$, $T_m = 5s$, where σ is measured in Pa, and shear rate in s^{-1}. Other thixotropic fluid foods (for example, mayonnaise, ketchup, sauces or honey) fit similar equation within enough high accuracy, with obviously somewhat different constants. As for small shear rates, formula (22) are trustable for small, but finite values of shear stress, because it gives infinitely large value of shear stress in zero shear rate in any time. Relatively to yogurt, the lower limit in dimensional quantity is $\dot{\Gamma}_m = 5s^{-1}$.

We propose to extrapolate dimensionless curves by the linear function $\sigma = \sigma_0 + \eta\dot{\gamma}$ in the area $0 \leq \dot{\gamma} \leq \dot{\Gamma}_m/\Omega$. In $\dot{\Gamma}_m$, both experimental and linear functions must have equal value and first partial derivative with respect to shear stress. The Fig.3 below demonstrates the extrapolation of experimental data for stirred yogurt at some values of time.

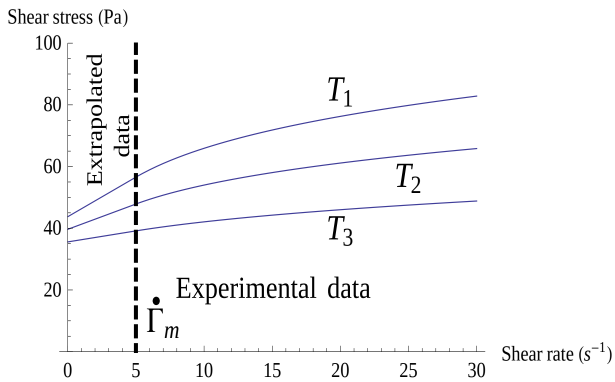

Fig. 3. Extrapolated (for shear rates less or equal than $\dot{\Gamma}_m = 5s^{-1}$) and experimental data for stirred yogurt accordingly to Eqs (22),(23) for $T_1 = 1s$, $T_2 = 10s$ and $T_3 = 100s$.

Above conditions determine non-dimensional η and σ_0:

$$\sigma_0 = C_1 + C_2\ln(t/(\Omega T_m)), \ \eta = C_3 + C_4\ln(t/(\Omega T_m)). \tag{24}$$

Relatively to yogurt, dimensionless coefficients C_1, C_2, C_3, C_4 take the form

$$C_1 = -\frac{A_1(A_2-1)}{E_1\rho_0 c_0^2}(\dot{\Gamma}_m)^{A_2}, \ C_2 = \frac{1}{E_1\rho_0 c_0^2}(B_1 - B_1\ln(\dot{\Gamma}_m) - B_2), \tag{25}$$

$$C_3 = \frac{A_1 A_2 \Omega}{\rho_0 c_0^2}(\dot{\Gamma}_m)^{A_2-1}, \ C_4 = -\frac{B_1\Omega}{\rho_0 c_0^2\dot{\Gamma}_m}.$$

Since σ_0 and η should be small in accordance to Eqs (5), it is sufficient to demand smallness of $\sigma_0(t=0)$ and $\eta(t=0)$,

$$C_1 - \ln(\Omega T_m) = O(M^n), \quad C_3 - \ln(\Omega T_m) = O(M). \tag{26}$$

Both σ_0 and η in the form of (24),(25) are slowly varying functions on time satisfying the requirement (13) if

$$t > \frac{\max(|C_2|, |C_4|)}{\min(M^2, M^{n+1})}. \tag{27}$$

C_2 and C_4 for a common thixotropic fluid are negative. Eqs (24) determine the local sound velocity of domains with positive shear rate as $1-(C_1-C_2+C_2\ln(t/\Omega T_m))/2$, and that of domains with negative shear rate as $1+(C_1-C_2+C_2\ln(t/\Omega T_m))/2$ independently on the absolute value of shear rate, only on its sign. So that, the difference in propagation velocities depends on time, but for enough large times the waveform changes its shape only slowly.

5 Conclusions

Eqs (12), along with (18) and relations of perturbations in the sound wave, Eqs (11) and (17), are the main result of this study. They are valid approximately in Bingham plastics and in some thixotropic fluids. The objective of this study is also to point out unusual properties of sound in these fluids. The equations governing dynamics of sound over them depend on a sign of shear rate. That requires individual evaluation of dynamics of every waveform. The example shows that the yield stress makes parts of different velocity gradients to propagate with different speeds, in general slowly depending on time. That makes a bipolar pulse monopolar with time and leads to additional broadening of it comparatively with that caused by viscosity. The knowledge about variations in the shape of sound wave, and variation of velocity of propagation of stationary waveforms may be useful in evaluations the yield stress itself, where the direct measurements are impossible. Distortion of the waveform differs from that due to nonlinearity in the equations and may be recognized readily. The conclusions, though approximate, follow from analytical formulae. They are valid only for the mobile liquid, at shear stresses small (in sense that are proportional to M) but enough large for experimental data to confirm non-elastic behavior of a medium. The lowest value is usually determined by accuracy of a viscometer, about $10^{-1}s^{-1}$, or even smaller, $10^{-2}s^{-1}$ [4]. If the liquid behavior is confirmed at $\dot{\gamma} \geq 10^{-1}s^{-1}$, that gives for $M = 10^{-3}$ value of $\Omega \geq 10^2 s^{-1}$ which supports enough large gradients of velocity in the sound wave. We should assume sound to be a wave process, weakly attenuating at the wavelength, so that $\frac{\eta(t)\Omega}{\rho_0 c_0^2} << 1$. For preliminary evaluation for yogurt, replacing η with its maximum value, $\eta(0)$, and ρ_0, c_0 by the values in water, one obtains $\Omega << 10^6 s^{-1}$. So, the conclusions are valid for sound with enough high frequency to support shear rates belonging to the area where experiments confirm liquid behavior of a medium, and with enough small frequency for sound not to attenuate strongly over the sound period and to be hence a wave process. At practically zero shear rates, where a medium behaves as solid, the shear stress depends on strain, or, that is very likely in rheological media, both on strain and shear rate. The solution may be thought therefore as consisting of parts with non-zero shear rates stapled continuously with parts of zero shear rates, as it is shown by Fig.2.

The results are based on the idea of treating a thixotropic fluid as a Bingham plastic with parameters depending on time. This way does not distinguish between thinning or thickening flows because takes into account only two first main terms in the Taylor series, Eq.(6). Along with Bingham plastics, the non-Newtonian fluids include shear thinning and shear thickening fluids, whose apparent viscosity decreases or increasing with enlarging shear rate, respectively. The main reason for the sound not to be investigated in time-independent shear thinning or shear thickening fluids, is mathematical difficulty and probably lack of interest to these fluids as the media of sound propagation. As the first approximation, the apparent viscosity may be treated as constant for some narrow domain of shear stresses and hence to the sound frequencies. The experimental data may provide the required parameters describing thixotropic fluid like these for yogurt, which flow characteristics are described by Eqs (24). Eq.(22) which describes time-dependent foods, requires experimental data to establish A, B and T_m. The validity of conclusions is however restricted by important conditions. There are smallness of viscosity for sound to be a wave process, smallness of threshold stress for the waveform speed not to distort considerably from constant, and smallness of temporal variations in structure during the propagation of sound pulse over a medium. That concerns determination of σ_0, η, c_0 and parameter of nonlinearity of sound, ε, in the weakly nonlinear flows. The account for strong dependence of these quantities on time makes the mathematical content of the problem very difficult. The velocity and perturbations of pressure and density in sound are also small. This last condition is usual in linear and weakly nonlinear acoustics. The example at Fig.2 considers only distortions of the wave form originated from the yield stress in view of mathematical difficulties of account for the variable apparent viscosity. Initially bipolar pulse transforms into positive, monopolar one. The similar behavior of acoustic pulses has been observed experimentally and explained theoretically in continuum solid media with hysteresis [14], though the origin of a pulse distortion there is different and associates with hysteretic nonlinearity. The theory developed by Gusev, and that of the present study both predict that a bipolar pulse of nonzero momentum transforms in its propagation into a unipolar pulse. This study concerns homogeneous fluids and flows far from boundaries.

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