# Purely Quantum Superadditivity of Classical Capacities of Quantum Multiple Access Channels

Lukasz Czekaj* and Pawel Horodecki†
*Faculty of Applied Physics and Mathematics, Gdansk University of Technology, Gdansk, Poland*
*National Quantum Information Centre of Gdansk, 81-824 Sopot, Poland*
(Received 8 November 2008; published 19 March 2009)

We are studying classical capacities of quantum memoryless multiaccess channels in geometric terms and we are revealing a break of additivity of the Holevo-like capacity. This effect is a purely quantum mechanical one, since, as we point out, the capacity regions of all classical memoryless multiaccess channels are additive. It is the first such effect revealed in the field of classical information transmission via quantum channels.

DOI: 10.1103/PhysRevLett.102.110505 PACS numbers: 03.67.Hk, 89.70.Kn

Quantum channels [1] are among the central concepts of quantum information theory. Many of them allow us to transmit quantum information coherently [2] or transfer classical information, after suitable encoding into quantum states [3]. A multipartite generalization of communication of both types was analyzed through memoryless quantum channels [4,5].

Superadditivity phenomena in classical information transmission with quantum resources were first revealed in a decoding step of the bipartite scenario. Since the well-known observation [6] that entangling (and—as such—nonadditive) receiver measurements may help in an information extraction, these phenomena have been carefully studied [7]. In particular it implies that classical capacity formula under product encoding (in terms of the Holevo function) [3] requires entangling decoding. It was much harder to identify nonadditivity in an encoding step. Since entangling encoding may increase the probability of correct guess of sent signals [8] there is extensive research [9,10] on additivity of the Holevo-function formula mentioned above [3] with respect to encoding. However, additivity of classical capacity $C$ is an even harder problem since channel capacities are asymptotic quantities, i.e., optimized over arbitrary many uses of a channel. On the other hand, nonadditivity in quantum capacities $Q$ has been shown in multiparty scenario [11] following the entanglement activation idea [12]. However, those results concern qubits (not bits) communication and, as such, have no classical analog to be fairly compared with. In addition they have been reached with additional free two-way classical communication support.

Here we construct classes of multiple access (MA) channels exhibiting superadditivity of classical capacity regions which are true asymptotic quantities and involve no additional resource support. This is purely quantum phenomenon since, as we point out, the corresponding regions for classical channels [13] are additive. This is the first effect of this kind in quantum channels theory.

*Capacity region.*—is a set of all rates achievable for channel. For two channels $\Phi_1$, $\Phi_2$ and their capacity regions $\mathcal{C}(\Phi_1)$ and $\mathcal{C}(\Phi_2)$ a geometric (Minkowski) sum $\mathcal{C}(\Phi_1) + \mathcal{C}(\Phi_2) = \{\vec{u}_1 + \vec{u}_2\colon \vec{u}_1 \in \mathcal{C}(\Phi_1), \vec{u}_2 \in \mathcal{C}(\Phi_2)\}$ gives region of achievable rates in case when both channels are used separately, i.e., input states are not correlated across $\Phi_1$, $\Phi_2$ cut. Since the inputs may be correlated: $\mathcal{C}(\Phi_1) + \mathcal{C}(\Phi_2) \subseteq \mathcal{C}(\Phi_1 \otimes \Phi_2)$. The converse inclusion defines additivity.

*Classical memoryless multiple access channel.*—has many senders and one receiver. It is given by $p(y|x_1, \ldots, x_n)$. Senders do not cooperate and transmit their messages independently [probability of sending message $(x_1, \ldots, x_n)$ has form $p_1(x_1)\ldots p_n(x_n)$]. The capacity region is given by the set of inequalities [13]:

$$R(S) \leq I(X(S)\colon Y|X(S^C)), \tag{1}$$

where $S$ represents all possible subsets of senders $S \subseteq \{X_1, \ldots, X_n\}$. $S^C$ stands for complement of $S$ and $R(S)$ is sum of transmission rates $R(S) = \sum_{X_i \in S} R(X_i)$ from senders $X_i \in S$ to the single receiver $Y$. Product channel of two MA channels is given by $p(y|x_1, \ldots, x_n) = p(y_1|x_{1,1}, \ldots, x_{1,n})p(y_2|x_{2,1}, \ldots, x_{2,n})$. Capacity regions of the channels treated separately are determined by $R(S_1) \leq I(X(S_1)\colon Y_1|X(S_1^C))$, $R(S_2) \leq I(X(S_2)\colon Y_1|X(S_2^C))$. The following inequality $I(X(S)\colon Y|X(S^C)) \leq I(X(S_1)\colon Y_1|X(S_1^C)) + I(X(S_2)\colon Y_2|X(S_2^C))$ can be proven leading to the geometric additivity of the capacity regions. Here $S = S_1 \cup S_2$ with $S_1$, $S_2$ being the subsets of the first (second) group of senders. The Fig. 1 illustrates additivity of the regions for two exemplary classical channels.

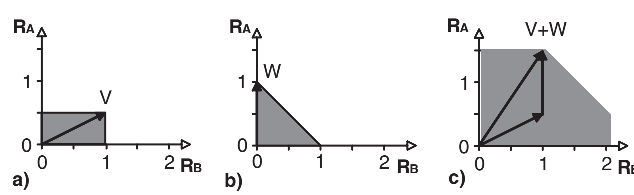


FIG. 1. (a) Two senders pass their messages to the same receiver through independent binary symmetric channels with $H(p) = 0.5$ and $H(p) = 0$. (b) XOR gate. (c) Minkowski sum of the two previous regions illustrating the additivity rule.

 

*Capacity region $\mathcal{C}$ of quantum memoryless multiple access channel.*—In quantum case of two senders and one receiver one defines the classical-quantum channel (cqc) state $\rho = \sum_{ij} p_i q_j e_i \otimes e_j \otimes \Phi(\varrho_i \otimes \varrho_j)$, where $e_i = |e_i\rangle\langle e_i|$ is a projector onto element of the standard basis of classical part belongs to first (second) sender say Alice (Bob) while $\{p_i, \varrho_i\}$, $\{q_j, \varrho_j\}$ represent the ensembles of states send through the channel toward the receiver Charlie. Receiver is allowed to perform POVM measure to recover classical information encoded in quantum states. The capacity region $\mathcal{C}$ for given cqc state is described by the same formulas like in classical case 1 where $I(AB{:}\,C)$, $I(A{:}\,C|B)$, $I(B{:}\,C|A)$ are (conditional) mutual information $I(AB{:}\,C) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$ $[I(A{:}\,C|B) = \sum_j p_j I(A{:}\,C|B = j)]$ of shared cqc state $\rho$ [4,14].

*Noisy toy model channel.*—Consider the case of two senders Alice and Bob and the channel $\Phi^p$ (depicted schematically on Fig. 2) that allows Alice to send a four-level quantum system while Bob is supposed to send only one qubit system.

The capacity of the channel $\Phi^{p=1}$, called here the toy model channel, can be easily found as follows. Let us put partial trace instead of depolarization (they both lead to the same capacity regions). Now if Alice sends fixed state, say $|0\rangle$ then Bob message is not affected which gives rise to the following rate vector $(R_A, R_B) = (0, 1)$. On the other hand if Bob sends fixed pure state, say $|0\rangle$, then Alice may not affect it (sending $|0\rangle$) or may alter with Pauli matrix $\sigma_x$ (sending $|1\rangle$). This corresponds to the rate vector $(R_A, R_B) = (1, 0)$. Clearly sum of the rates cannot exceed one (since Charlie gets only one qubit). Time sharing eventually gives the triangle capacity region $\mathcal{C}(\Phi^{p=1}) = \{(R_A, R_B){:}\ R_A + R_B \le 1\}$.

We also introduce a trivial identity channel $\Psi^{\mathrm{id}}$ that transmits ideally single qubits form Alice and Bob, respectively, (Alice part is intriduced for better geometrical visualization only). Obviously it has the square capacity region $\mathcal{C}(\Psi^{\mathrm{id}}) = \{(R_A, R_B){:}\ R_A \le 1, R_B \le 1\}$.

Now we shall find the capacity region $\mathcal{C}(\Phi^{p=1} \otimes \Psi^{\mathrm{id}})$. Bob may send fixed maximally entangled state, say $|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Then Alice may alter it with her four states $|0\rangle, \ldots, |3\rangle$ and send four independent messages to Charlie. She may also send one additional bit by ideally transmitting part of $\Psi^{\mathrm{id}}$. This gives totally 3 bits of Alice rate ie. $(R_A, R_B) = (3, 0)$. Again, since Charlie gets three qubits, by Holevo bound, sum of the Alice and Bob rates can not exceed 3 bits. By similar argumentation maximal information that Bob can send amounts to 2 bits. Hence $\mathcal{C}(\Phi^{p=1} \otimes \Psi^{\mathrm{id}})$ is described by

$$R_B \le 2, \qquad R_A + R_B \le 3, \tag{2}$$

which is clearly grater than the geometric sum $\mathcal{C}(\Phi^{p=1}) + \mathcal{C}(\Psi^{\mathrm{id}})$ as illustrated in Fig. 3. This example which explores the kind of remote dense coding on Alice part shows how easily that nonadditivity of capacity regions of different channels may naturally occur in MA channel.

For all $\Phi^p$ with $0 < p < 1$ we have $R_A < 2$; hence, one can observe that nonadditivity of the capacity region takes place also for $p < 1$.

Note that in the case $\Phi^{p=1}$ we have single letter formulas for all the three capacities, i.e., entangled signals send across inputs of the same channels will not help. We have $\mathcal{C}^{(n)}(\Phi) \equiv \frac{1}{n}\mathcal{C}(\Phi^{\otimes n}) = \mathcal{C}(\Phi)$. As we shall see subsequently this is not always true.

*The presence of nontrivial noise: When single letter formula does not work.*—Consider a noisy version of $\Phi^p$, i.e., with $p \in (0, 1)$. We will show that then one has $\mathcal{C}^{(1)}(\Phi^p) \subsetneq \mathcal{C}^{(2)}(\,{}^{p})$. For simplicity we focus here on the Alice transmission rate.

Following (1) the bound on the Alice transmission rate in single use of the channel may be expressed by

$$\begin{aligned} R_A &\le \max \chi(\{p_i, \Phi^p(u_i \otimes v)\} \\ &= \max\Big[ S\Big(\Phi^p\Big(\sum_i p_i u_i \otimes v\Big)\Big) - \sum_i p_i S(\Phi^p(u_i \otimes v))\Big] \end{aligned} \tag{3}$$

since the Holevo function can be always saturated on pure state ensembles. The maximum is taken over all Alice ensembles $\{p_i, u_i\}$ and all Bob pure states $\{v\}$. We shall

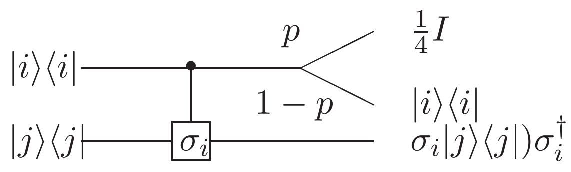


FIG. 2. Circuit model of channel $\Phi^p$ with depolarizing noise. The controlled Pauli matrices $\sigma_i \in \{I, \sigma_x, \sigma_y, \sigma_z\}$ are involved.

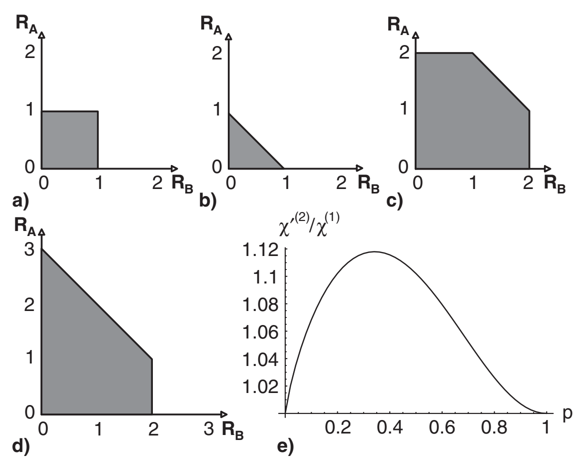


FIG. 3. Illustration of (a) toy model channel capacity region; (b) identity channel capacity region; (c) Minkowski sum of the regions (a) and (b); (d) capacity region of product of two channels which is greater than sum (c); (e) quotient of Alice's Holevo-like capacity for entangled and product coding for the noisy version of the toy model channel (see the main text).

prove that $R_A$ is bounded by $\chi^{(1)} = H((2-p)/8, (2-p)/8, (2-p)/8, (2-p)/8, p/8, p/8, p/8, p/8) - H(1-(3p/4), p/4, p/4, p/4)$. This can be seen from two facts: (i) the total entropy, i.e., the first term in the above bound is maximized by the Alice maximally mixed state (ii) all the terms in the second part $S(\Phi^p(u_i \otimes v))$ have the same minimum for $|u_i\rangle = |i\rangle$ (standard orthonormal basis). Hence maximally mixed ensemble of Alice orthogonal states $\{|i\rangle\}$ at the same time maximizes the first and minimizes the second (averaged) term in (3) reaching the expected bound.

Fact (i) can be proved by the observation that total entropy can be seen as a concave function of average state $\rho$ and so it is enough to prove that the it has a critical point, i.e., its derivative along any (traceless) direction $\Delta$ vanishes at $\rho = I/4$. One can use formula [15]:

$$\frac{\partial S(\varrho + \alpha\delta)}{\partial \alpha}\bigg|_{\alpha=0} = -\mathrm{Tr}[\delta \log_2 \varrho], \tag{4}$$

where $\varrho = \Phi^{(p)}(\frac{1}{4}\mathrm{I} \otimes v)$ and $\delta = \Phi^{(p)}(\Delta \otimes v)$ to prove that derivations vanishes via sequence of not difficult, though tedious calculations which are presented elsewhere [16].

Fact (ii) is a consequence of theorem [17] saying that minimal output entropy of the tensor product of depolarizing channel with identity channel is saturated by product pure states and the observation that in our case a product channel (composed of depolarizing channel and identity) follows an entangling unitary operation.

Consider now the case when we have two uses of the channel $\Phi^p \otimes \Phi^p$ and Bob sends just maximally entangled state $|\psi_+\rangle$ while Alice sends products of two maximally mixed ensembles like the one used before. The achieved Alice rate $\chi'^{(2)}$ (we use primed Holevo function to stress that the used ensemble may not be optimal) can be easily computed as a Holevo function of the ensemble of 16 states $\Phi(e_i \otimes e_j \otimes \psi_+)$ with equal probabilities and it amounts to $\chi'^{(2)} = -(\frac{3}{8}(2-p)p\log_2\frac{1}{64}(2-p)p + \frac{1}{8}(4-6p+3p^2)\times \log_2\frac{1}{64}(4-6p+3p^2)) - H(1-(3p/4), p/4, p/4, p/4)$ where the first term contributes to superadditivity with respect to communication form Alice to Charlie.

On Fig. 3(e) the quotient $\chi'^{(2)}/\chi^{(1)}$ is depicted showing that the maximal possible rate of sending information by Alice implies nonadditivity $\mathcal{C}^{(1)}(\Phi^{(p)}) \subsetneq \mathcal{C}^{(2)}(\ ^{(p)})$.

*Three senders channel with classical-like type of noise.*—Here we shall consider another type of MA channel with three senders $A_1$, $A_2$, $B$ and one receiver $C$. $A_1$ and $A_2$ send qubits while $B$ sends four-level system. The channel is depicted on Fig. 4.

We shall consider a configuration similar to one presented for case $\Phi^{p=1}$. We introduce trivial identity channel $\Psi_{\mathrm{id}}$ that transmits ideal single qubits from $A_1$ and $A_2$ to receiver. For the case (i) when $A_1$ and $A_2$ sends single selected product state we immediately get: $R_B^{(1)} \leq 1$. If one allows (ii) entangled coding across many uses of $\Gamma$,

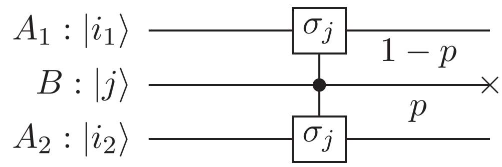


FIG. 4. Circuit model of channel $\Gamma^p$. Up unitarity (between $A_1$ and $B$) occur with probability $1-p$ and down with $p$. The cross sign stands for partial trace.

the (regularized) rate $R_B$ will be bounded by 1.81. In the third scenario (iii) senders $A_1$, $A_2$ send Bell state ($|\Psi_+\rangle$) with their first (second) qubits transferred down the channel $\Gamma$ ($\Psi^{\mathrm{id}}$) and then transmission rate $R_B^{\mathrm{ent}}$ becomes $2 > 1.81$ leading to expected nonadditivity. (i) Follows from Holevo bound for lines $A_1$ and $A_2$ and fact that the additional information which unitary operation (up or down) was performed can only increase the capacity of $\Gamma$; (iii) we get immediately by superdense coding. Note that thanks to numerical analysis of (i) and (iii) for the channel $\Gamma^{p=0.5}$, we found even if $B$ achieves maximal transmission rate, both $A_1$ and $A_2$ can still get some nonzero rates.

Here we present only the main parts underlying estimation of the regularized rate $R_B^{(n)}$ (ii). Technical details were presented elsewhere [16]. Receiver output is invariant under the von Neumann measurement in standard basis on system $B$ that proceeds the action of $\Gamma^p$; therefore, message $B$ may always be chosen to be in the standard basis and we may simulate channel $\Gamma^p$ as a classical channel $\Lambda^p$: $B \mapsto B_1 \otimes B_2$ followed by unitarity $U = U_1 \otimes U_2$, where $U_i$ works on subsystem $A_iB_i$ ($i$ specifies sender $A_1$ or $A_2$). After unitarity we trace out subsystem $B_1B_2$. Channel $\Lambda^p$ maps index $i \in B$ to $(0, i)$ or $(i, 0)$ with the probability $p$ (respectively, $1-p$). Suppose now we have $n$ copies of $\Gamma^p$ at our disposal. Sender $A_i$ is allowed to prepare any state $|\Psi_{A_i}\rangle \in A_i^{\otimes n}$ on his subsystem. At the same time sender $B$ sends random vector variable $(b^1, \ldots, b^n)$ which is mapped to $(b_1^1, \ldots, b_1^n)(b_2^1, \ldots, b_2^n)$. All possible codings determined by choices of $|\Psi_{A_i}\rangle$ and further (random) unitary action are of the form

$$B_i^n \ni (b_i^1, \ldots, b_i^n) \mapsto \rho_i^{(b_i^1, \ldots, b_i^n)} \in A_i^{\otimes n} \qquad i = 1, 2. \tag{5}$$

Hence the receiver gets eventually the product state $\rho_1 \otimes \rho_2$. His task is to perform collective POVM on the letter to get maximum information about $B^n$. The result of POVM is recorded in $\hat{B}^n$. Using techniques similar to those from classical source coding with side information [13] and exploiting separable correlations among $B^n$, $B_i^n$, $(A_i)^{\otimes n}$ one gets

$$R_B^{(n)} \leq \max_{p(B^n)} \frac{1}{n}(H(B^n) - H(B^n|B_2^n) + n) \tag{6}$$

$$\leq \max_{p(B)} I(B: B_2) + 1. \tag{7}$$

Without loss of generality we assume that $p \leq 0.5$ [16]. Therefore, thanks to numerical calculation, we get $R_B^{(n)} < 1.81$ for all $p \leq 0.5$. The result is independent on $n$, hence we get finally that regularized $R_B < 1.81$

*Conclusions.*—We provided constructions of MA channels that exhibit superadditivities of classical capacity regions. First they are nonadditive in the sense that $\mathcal{C}^{(1)} \subsetneq \mathcal{C}^{(2)}$, i.e., entanglement across two inputs of the same channel helps. Even more striking, unlike bipartite channel capacities [18], the presented capacity regions break additivity rule if supplied with identity channel. We showed that both types of nonadditivity have no classical analog. The results are first examples of nonadditivity of classical capacities where (i) no additional resources are involved, (ii) classical analogs are additive. They also provide first quantum improvement of bits transmission rate that does not require quantum memory, since entanglement particles are sent "in parallel." The cumulative bipartitelike rate ($R_A + R_B$) is still additive here; however, the proof of the results shows that multiparty scenarios may be arena of efficient applications of tools known from bipartite case. The simplicity of our initial (toy model) channel breaking additivity with identity channel resembles classical butterfly-effect with two senders and two receivers [19]. Note that one can ask the same question for other multiuser scenarios. In particular, general questions about additivity of entanglement assisted classical capacities is open [22] (extension of our approach to that case does not work). Analysis of the presented effects in continuous variable domains is an interesting issue which will be considered elsewhere.

This work was prepared under the EU IP SCALA. We thank L. Pankowski for drawing our attention to classical butterfly effect. For numerical calculations GSL v1.9 and CGAL v3.3.1 were used.

*Note added.*— Some time after posting the present result [16] concerning classical capacities $C$, we learned of an analogous result for quantum capacity $Q$ in a fundamental one-sender–one-receiver scenario [20]. The recent surprising result [21] breaks additivity of bipartite Holevo capacity in the single-use regime. Here we deal with channel capacities which by definition are asymptotic quantities.

*lczekaj@mif.pg.gda.pl
†pawel@mif.pg.gda.pl

[1] C. H. Bennett and P. W. Shor, Science **303**, 1784 (2004).
[2] C. H. Bennett *et al.*, Phys. Rev. A **54**, 3824 (1996); H. Barnum *et al.*, Phys. Rev. A **57**, 4153 (1998).
[3] A. S. Holevo, IEEE Trans. Inf. Theory **44**, 269 (1998); B. Schumacher and M. D. Westmoreland, Phys. Rev. A **56**, 131 (1997).
[4] A. Winter, IEEE Trans. Inf. Theory **47**, 3059 (2001).
[5] J. Yard *et al.*, arXiv:cs/0508031; M. Horodecki *et al.*, Nature (London) **436**, 673 (2005); J. Yard *et al.*, arXiv: quant-ph/0603098; M. Demianowicz and P. Horodecki, Phys. Rev. A **74**, 042336 (2006).
[6] A. Peres and W. K. Wootters, Phys. Rev. Lett. **66**, 1119 (1991).
[7] M. Sasaki *et al.*, Phys. Lett. A **236**, 1 (1997); A. S. Holevo, Russ. Math. Surv. **53**, 1295 (1998); M. Sasaki *et al.*, Phys. Rev. A **58**, 146 (1998); M. Sasaki*et al.*, *ibid.* **58**, 159 (1998); J. R. Buck *et al.*, Phys. Rev. A **61**, 032309 (2000); S. Usami *et al.*, in *Quantum Communication, Computing and Measurement*, edited by P. Tombesi and O. Hirota (Kluwer Academic/Plenum Publishers, Dordrecht, 2001), Vol. 3, p. 35.
[8] C. H. Bennett *et al.*, in *Quantum Communication, Computing and Measurement, Proc. QCM96*, edited by O. Hirota, A. S. Holevo, and C. M. Caves (Plenum, New York, 1997), p. 79.
[9] P. W. Shor, Commun. Math. Phys. **246**, 453 (2004).
[10] C. King and M. B. Ruskai, IEEE Trans. Inf. Theory **47**, 192 (2001); arXiv:quant-ph/9911079; G. G. Amosov *et al.*, Probl. Inf. Transm. **36**, 25 (2000); C. King and M. B. Ruskai, J. Math. Phys. (N.Y.) **42**, 87 (2001); C. King, J. Math. Phys. (N.Y.) **43**, 1247 (2002); C. King, arXiv:quant-ph/0103156; K. M. R. Audenaert and S. L. Braunstein, Commun. Math. Phys. **246**, 443 (2004); C. King *et al.*, Linear Algebra Appl. **404**, 367 (2005); R. Alicki and M. Fannes, Open Syst. Inf. Dyn. **11**, 339 (2004); A. S. Holevo and M. E. Shirokov, Theory Probab. Appl. **50**, 86 (2006); V. Giovannetti and S. Lloyd, Phys. Rev. A **69**, 062307 (2004); V. Giovannetti *et al.*, J. Math. Phys. (N.Y.) **46**, 042105 (2005); M. Hayashi *et al.*, Quantum Inf. Comput. **5**, 13 (2005); N. Datta and M. B. Ruskai, J. Phys. A: Math. Gen **38**, 9785 (2005); C. King *et al.*, Linear Algebra Appl. **404**, 367 (2005); I. Devetak *et al.*, Commun. Math. Phys. **266**, 3763 (2006); P. Hayden and A. Winter, Commun. Math. Phys. **284**, 263 (2008); C. King *et al.*, Markov Processes Relat. Fields **13**, 391 (2007); T. Cubitt *et al.*, Commun. Math. Phys. **284**, 281 (2008).
[11] W. Dur *et al.* Phys. Rev. Lett. **93**, 020503 (2004).
[12] P. Horodecki *et al.*, Phys. Rev. Lett. **82**, 1056 (1999).
[13] T. M. Cover and J. A. Thomas, *Elements of Information Theory* (Wiley and Sons, New York, 1991).
[14] A. E. Allahverdyan and D. B. Saakian, in: *Quantum Computing and Quantum Communications* (Palm Springs, CA, 1998), p. 276; *LNCS* (Springer, Berlin, 1999) p. 1509.
[15] P. W. Shor, Math. Program. **97**, 311 (2003).
[16] E-print of current document with detailed calculations, arXiv:0807.3977.
[17] C. King, arXiv:quant-ph/0204172v2.
[18] B. Schumacher and M. D. Westmoreland, arXiv:quant-ph/0004045v1.
[19] R. Ahlswede *et al.*, IEEE Trans. Inf. Theory **46**, 1204 (2000).
[20] G. Smith and J. Yard, Science **321**, 1812 (2008)
[21] M. B. Hastings, arXiv:0809.3972.
[22] C. H. Bennett *et al.*, Phys. Rev. Lett. **83**, 3081 (1999).