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Response to David Steigmann's discussion of our paper

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We thank Professor David Steigmann for his discussion, in particular his proof on the general validity of his internal power expression (see equation (8) below) and his expression for the variation of the geodesic curvature. His proof allows us to clarify two misleading statements in Duong et al. [1] and confirm that its formulation is fully consistent with the formulation of Steigmann [2]. However, some of our original statements made in Remark 4.9 are not wrong: the third term in equation (8) can lead to spurious constitutive models for in-plane bending, which was our original concern.

The discussion below follows the notation in Duong et al. [1]. This translates to the notation in Steigmann's discussion as follows: $c_\gamma \equiv p_\gamma$, $c^0_\gamma \equiv M_\gamma$, $\ell^\alpha \equiv l^\alpha$, $\boldsymbol{c}_0 \equiv \mathbf{M}$, $\boldsymbol{c} \equiv \mathbf{p}$, $\boldsymbol{\ell} \equiv \mathbf{l}$, $\kappa_{\mathrm{g}} \equiv \eta_l$, and $\kappa^0_{\mathrm{g}} \equiv \eta_L$.

1. Concerning Steigmann's discussion of Remark 4.8

We agree with Steigmann that our statement in Remark 4.8 is incorrect. This error stems from our incomplete proof provided in Appendix 5 of Duong et al. [1], which is based on the incorrect assumption that the variation of the fiber director, δc_α, is not solely expressible through the variation of the metric, $\delta a_{\alpha\beta}$. Instead,

$$\delta c_\gamma = \frac{1}{2} c_\gamma \, c^{\alpha\beta} \, \delta a_{\alpha\beta} \,, \tag{1}$$

i.e., δc_α is solely expressible through $\delta a_{\alpha\beta}$. Equation (1) follows from the variations $\delta J = J\, a^{\alpha\beta}\, \delta a_{\alpha\beta}/2$, $\delta\lambda = L^{\alpha\beta}\delta a_{\alpha\beta}/(2\lambda)$, the identity $a^{\alpha\beta} = \ell^{\alpha\beta} + c^{\alpha\beta}$ that follows from equation (8) in Duong et al. [1], and the relation

$$c_\gamma = J\, \lambda^{-1}\, c^0_\gamma \,, \tag{2}$$

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given in Steigmann's discussion. Here, $c_\alpha := \boldsymbol{c} \cdot \boldsymbol{a}_\alpha$, $J = \det_{\rm s} \boldsymbol{F}$, $\lambda = \|\boldsymbol{F}\boldsymbol{L}\|$, $c^0_\alpha := \boldsymbol{c}_0 \cdot \boldsymbol{A}_\alpha$, $c^{\alpha\beta} := c^\alpha c^\beta$, $c^\alpha := \boldsymbol{c} \cdot \boldsymbol{a}^\alpha$, $\ell^{\alpha\beta} := \ell^\alpha \ell^\beta$, $\ell^\alpha := \boldsymbol{\ell} \cdot \boldsymbol{a}^\alpha$, $L^{\alpha\beta} := L^\alpha L^\beta$, and $L^\alpha := \boldsymbol{L} \cdot \boldsymbol{A}^\alpha$. We thank Steigmann for providing relation (2). We were not aware of this relation in our derivation of Appendix 5 [1], since we had used

$$\delta c_\alpha = -\ell^\beta_\alpha\, \boldsymbol{c} \cdot \delta \boldsymbol{a}_\beta + \boldsymbol{c} \cdot \delta \boldsymbol{a}_\alpha \tag{3}$$

(see equation (213) of [1]) instead of equation (1). We note here that equation (3) is not wrong. Instead, it is equivalent to equation (1). This can be shown by applying the identities $\boldsymbol{c} \cdot \delta \boldsymbol{a}_\alpha = \delta^\beta_\alpha\, \boldsymbol{c} \cdot \delta \boldsymbol{a}_\beta$ and $\delta^\beta_\alpha = \ell^\beta_\alpha + c^\beta_\alpha$ (following from equation (8) in [1]) to equation (3) and using the symmetry of $c^{\alpha\beta}$.

Relation (2) can also be used to show that the geodesic curvature given in equation (6) of Steigmann's discussion,

$$\kappa_{\rm g} = J\, \lambda^{-3}\, L^{\alpha\beta}\, c^0_\gamma\, S^\gamma_{\alpha\beta} - J\, \lambda^{-3}\, L^{\alpha\beta}\, c^0_{\alpha;\beta}\,, \tag{4}$$

is identical to equation (51) in Duong et al. [1], i.e.,

$$\kappa_{\rm g} = \ell^{\alpha\beta}\, c_\gamma\, S^\gamma_{\alpha\beta} + \lambda^{-1}\, c_\alpha\, \ell^\beta\, L^\alpha_{;\beta}\,. \tag{5}$$

This follows from equation (2), $L^\alpha = \lambda\, \ell^\alpha$ and $L^\alpha\, c^0_{\alpha;\beta} = -c^0_\alpha\, L^\alpha_{;\beta}$ (cf. equation (38) in [3]).

With variation (1), we can now continue the derivation in Appendix 5 of Duong et al. [1]: starting from equation (5) and using equation (1) together with equations (208) and (210) in [1], we arrive at the expression

$$\dot{\kappa}_{\rm g} = \ell^{\alpha\beta}\, c_\gamma\, \dot{S}^\gamma_{\alpha\beta} + \frac{1}{2}\, \kappa_{\rm g}\, (a^{\alpha\beta} - 3\, \ell^{\alpha\beta})\, \dot{a}_{\alpha\beta}\,, \tag{6}$$

which is also found in Steigmann's discussion. That is, the rate of geodesic curvature, $\dot{\kappa}_{\rm g}$, is fully expressible in terms of $\dot{a}_{\alpha\beta}$ and $\dot{S}^\gamma_{\alpha\beta}$. Since these are symmetric w.r.t. α and β, the stress symmetrization employed in the power balance of Steigmann [2] is indeed general, contrary to what was written in Remark 4.8. We thank Professor Steigmann for pointing out this error.

Equation (6) allows us to confirm that our internal power expression given in equation (107) of Duong et al. [1],

$$P_{\rm int} = \frac{1}{2} \int_{\mathcal{R}_0} \tau^{\alpha\beta}\, \dot{a}_{\alpha\beta}\, {\rm d}A + \int_{\mathcal{R}_0} M_0^{\alpha\beta}\, \dot{b}_{\alpha\beta}\, {\rm d}A + \int_{\mathcal{R}_0} \bar{\mu}_0\, \dot{\kappa}_{\rm g}\, {\rm d}A\,, \tag{7}$$

is equivalent to the internal power expression given in Steigmann [2] (cf. equation (63) there), which reads

$$P_{\rm int} = \frac{1}{2} \int_{\mathcal{R}_0} \tau^{\alpha\beta}\, \dot{a}_{\alpha\beta}\, {\rm d}A + \int_{\mathcal{R}_0} M_0^{\alpha\beta}\, \dot{b}_{\alpha\beta}\, {\rm d}A + \int_{\mathcal{R}_0} \bar{M}^{\alpha\beta}_{0\gamma}\, \dot{S}^\gamma_{\alpha\beta}\, {\rm d}A\,, \tag{8}$$

in our notation. This follows from inserting equation (6) into equation (7) and redefining the stress as

$$\tau^{\alpha\beta} \leftarrow \tau^{\alpha\beta} + \bar{\mu}_0\, \kappa_{\rm g}\, (a^{\alpha\beta} - 3\, \ell^{\alpha\beta}). \tag{9}$$

Equation (7) is also equivalent to

$$P_{\rm int} = \frac{1}{2} \int_{\mathcal{R}_0} \tau^{\alpha\beta}\, \dot{a}_{\alpha\beta}\, {\rm d}A + \int_{\mathcal{R}_0} M_0^{\alpha\beta}\, \dot{b}_{\alpha\beta}\, {\rm d}A + \int_{\mathcal{R}_0} \bar{M}_0^{\alpha\beta}\, \dot{\bar{b}}_{\alpha\beta}\, {\rm d}A\,, \tag{10}$$

the internal power expression proposed in Duong et al. [1] (see equation (113) there). Therefore, the internal power expressions given by Steigmann [2] (i.e., equation (8)) and our expression in equation (10) are equivalent. However, the last term in equation (8) can lead to spurious constitutive models for in-plane bending, as is shown below.

2. Concerning Steigmann's discussion of Remark 4.9

Steigmann's discussion of Remark 4.9 concerns two points: the well-definedness of the in-plane bending power term and its parametrization-dependence.

We agree that the internal power of Steigmann (see equation (8)) is not wrong and does not miss any contributions in the present context. This was shown above. Also it does not depend on the surface parameterization (i.e., the choice of curvilinear coordinate system), as Steigmann rightly points out.

However, there is still an issue with the last term in equation (8). It can lead to constitutive models for in-plane bending that are not well-defined for some deformations.

Steigmann's expression (8) leads to the constitutive equations for the membrane and bending stresses

$$\tau^{\alpha\beta} = 2\frac{\partial W}{\partial a_{\alpha\beta}}\,, \quad M_0^{\alpha\beta} = \frac{\partial W}{\partial b_{\alpha\beta}}\,, \quad \bar{M}_{0\gamma}^{\alpha\beta} = \frac{\partial W}{\partial S_{\alpha\beta}^{\gamma}}\,, \tag{11}$$

based on the stored energy function $W = W\big(a_{\alpha\beta}, b_{\alpha\beta}, S^{\gamma}_{\alpha\beta}\big)$. Our expression (10), on the other hand, leads to the constitutive equations,

$$\tau^{\alpha\beta} = 2\frac{\partial W}{\partial a_{\alpha\beta}}\,, \quad M_0^{\alpha\beta} = \frac{\partial W}{\partial b_{\alpha\beta}}\,, \quad \bar{M}_0^{\alpha\beta} = \frac{\partial W}{\partial \bar{b}_{\alpha\beta}}\,, \tag{12}$$

based on the stored energy function $W = \hat{W}\big(a_{\alpha\beta}, b_{\alpha\beta}, \bar{b}_{\alpha\beta}\big)$.

In general, the in-plane bending behavior of fibers should be related to a change of their curvature κ_{g}. Thus, the in-plane bending energy should be a function of κ_{g}. This means that in general the in-plane bending energy cannot be a function of $S^{\gamma}_{\alpha\beta}$ alone, as it does not fully describe κ_{g} according to equation (4). The problem with the third terms in equations (8) and (11) is that they suggest precisely that. Indeed, such a function has been proposed in Steigmann and Dell'Isola [3], cf. equation (60) and used in the computational model of Schulte et al. [4], cf. equation (42). It can lead to spurious bending moments as is shown in the example below. In our formulation, this problem does not appear, since κ_{g} is fully described by $\bar{b}_{\alpha\beta}$, i.e., $\kappa_{\mathrm{g}} = \bar{b}_{\alpha\beta}\,\ell^{\alpha\beta}$, cf. equation (50) of Duong et al. [1].

The issue can be illustrated by the following example (see Figure 1). We consider a ring-shaped domain described by the parameterization

$$\boldsymbol{X}(r,\phi) = r\cos\phi\,\boldsymbol{e}_1 + r\sin\phi\,\boldsymbol{e}_2\,, \quad 0 \le \phi < 2\pi\,, \quad r_1 \le r \le r_2\,, \tag{13}$$

with a circular fiber located at $r = r_0$. The domain is deformed by applying the displacement field

$$\boldsymbol{u}(r,\phi) = \frac{a_1}{2}\left(r^2 - r_0^2\right)\boldsymbol{e}_1 + \frac{a_2}{2}\left(r^2 - r_0^2\right)\boldsymbol{e}_2\,, \tag{14}$$

where a_1 and a_2 are the parameters of $\boldsymbol{u}$ that have sufficiently small magnitude to avoid singularities in the deformation $\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u}$. For any admissible a_i, displacement $\boldsymbol{u}$ is zero along the fiber, and hence does not change its curvature, which remains equal to $1/r_0$. (The fiber is only stretched in lateral direction.) Without curvature change, no fiber bending moment should appear. However, according to the model of Steigmann and Dell'Isola [3], the third term in equation (11) generates the fiber bending moment (see Appendix 1)

$$\bar{m}_0 \sim a_1\cos\phi + a_2\sin\phi\,, \tag{15}$$

which increases with a_i, see Figure 1(c). On the contrary, for the constitutive model proposed in Duong et al. [1], no fiber bending moment appears. The reason for the spurious moment in equation (15) lies in the constitutive model of Steigmann and Dell'Isola [3] that is based on $S^{\gamma}_{\alpha\beta}$ alone. As seen in equation (4)—and illustrated by the example in section 7.1 of Duong et al. [1]—$S^{\gamma}_{\alpha\beta}$ yields an incomplete curvature description. The constitutive model of Steigmann and Dell'Isola [3] is a natural choice following from equation (8). Therefore, equation (8), which itself is correct, can suggest spurious bending models.

As the example shows, the last term in equation (11) can generate a fiber moment that depends inconsistently on the deformation, seen here through the surface deformation parameters a_i. However, the third term in equation (8) does not depend on the surface parametrization, as was written imprecisely in Remark 4.9.

3. Concerning the remaining points raised by Steigmann

Steigmann claims that our "force and torque balance laws are postulated on the basis of free-body diagrams without reference to the Principle of Virtual Power." This is not true. Our formulation constructs the force

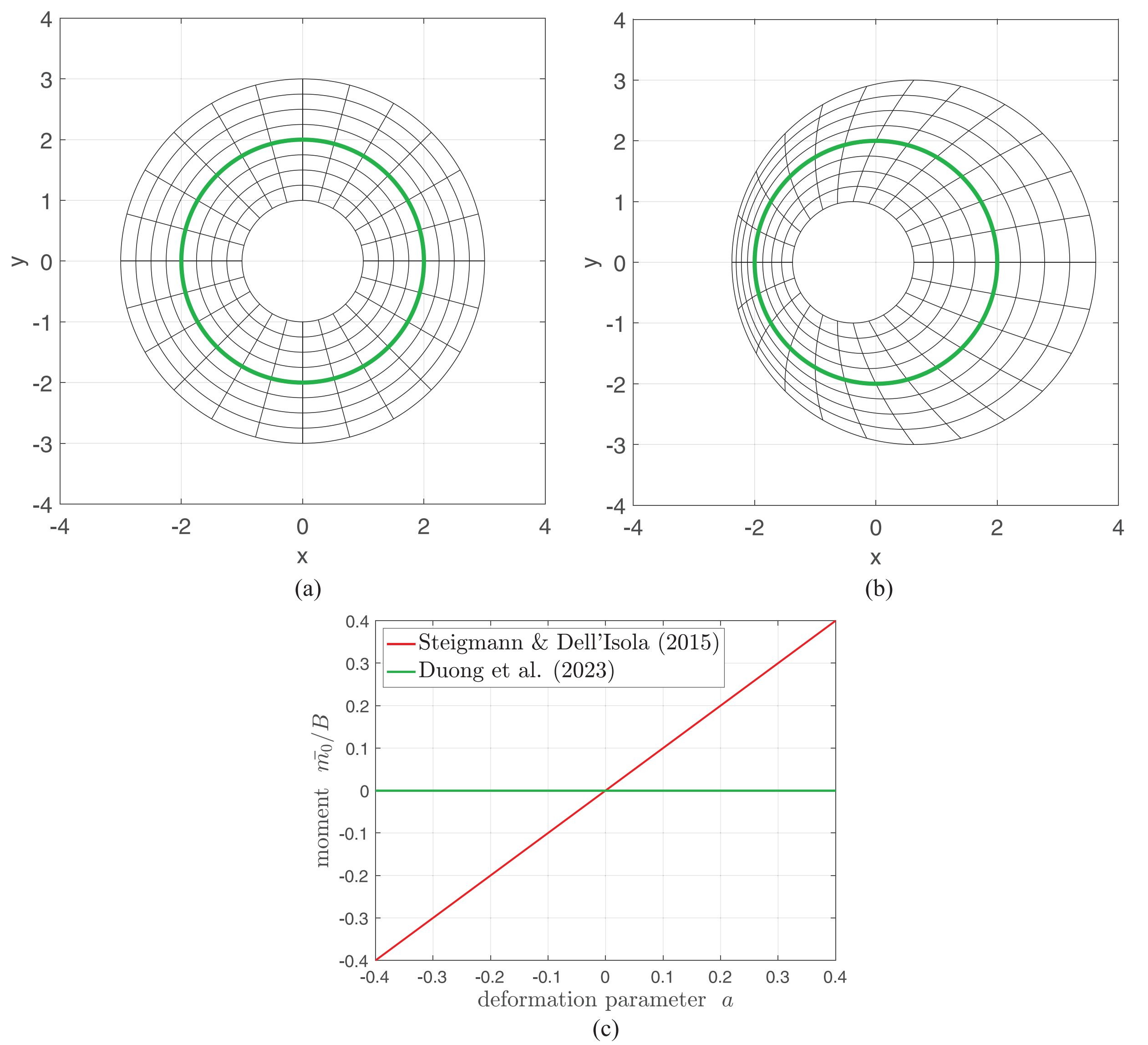

Figure 1. Deformation example of equations (13) and (14). (a) Initial configuration for $r_1 = 1$, $r_0 = 2$, and $r_2 = 3$. (b) Deformed configuration for $a_1 = 0.25$ and $a_2 = 0$. (c) Corresponding fiber bending moment $\bar{m}_0$ according to the constitutive models of Steigmann and Dell'Isola [3] and Duong et al. [1]. The former predicts an increase of $\bar{m}_0$ with the deformation, even though the fiber curvature does not change. Here, $a := a_1 \cos\phi + a_2 \sin\phi$.

and torque balance laws and the constitutive equations systematically from linear momentum balance, angular momentum balance, and the mechanical power balance in sections 3.3, 3.4, and 3.5 of Duong et al. [1]. The latter balance is mathematically equivalent to the principle of virtual power.

Steigmann further states that our formulation does "not make explicit the famous Kirchhoff boundary conditions and corner forces established by Kirchhoff." Such corner forces are fully contained in our formulation. They appear when the boundary traction $\boldsymbol{T}$ is replaced by its effective counterpart and the domain boundary of the shell surface is not smooth, which is shown in Appendix 2. The corner forces are part of the external virtual work, and so the internal virtual work expression presented in Duong et al. [1] is unaffected.

Singular boundary fibers are indeed not studied in our work. But they do not affect the examples presented in Duong et al. [1, 5].

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Appendix 1

Derivation of the bending moments in the example of Figure 1

Introducing $c := \cos\phi$ and $s := \sin\phi$, the deformation defined by equations (13) and (14) has the tangent vectors along parameters r and ϕ,

$$\boldsymbol{a}_1 = \boldsymbol{x}_{,r} = (c + a_1\, r)\,\boldsymbol{e}_1 + (s + a_2\, r)\,\boldsymbol{e}_2\,, \quad \boldsymbol{a}_2 = \boldsymbol{x}_{,\phi} = (-s\,\boldsymbol{e}_1 + c\,\boldsymbol{e}_2)\,r\,, \tag{16}$$

and their derivatives

$$\boldsymbol{a}_{1,1} = a_1\,\boldsymbol{e}_1 + a_2\,\boldsymbol{e}_2\,, \quad \boldsymbol{a}_{1,2} = -s\,\boldsymbol{e}_1 + c\,\boldsymbol{e}_2\,, \quad \boldsymbol{a}_{2,2} = -(c\,\boldsymbol{e}_1 + s\,\boldsymbol{e}_2)\,r\,. \tag{17}$$

From this follows

$$[a_{\alpha\beta}] = \begin{bmatrix} (c + a_1\, r)^2 + (s + a_2\, r)^2 & (a_2\, c - a_1\, s)\, r^2 \\ (a_2\, c - a_1\, s)\, r^2 & r^2 \end{bmatrix}, \tag{18}$$

$$\det[a_{\alpha\beta}] = g^2\, r^2\,, \quad g := 1 + (a_1\, c + a_2\, s)\, r, \tag{19}$$

and

$$[a^{\alpha\beta}] = \frac{1}{g^2\, r^2} \begin{bmatrix} r^2 & (a_1\, s - a_2\, c)\, r^2 \\ (a_1\, s - a_2\, c)\, r^2 & (c + a_1\, r)^2 + (s + a_2\, r)^2 \end{bmatrix}, \tag{20}$$

which leads to the dual tangent vectors

$$\boldsymbol{a}^1 = \frac{c\,\boldsymbol{e}_1 + s\,\boldsymbol{e}_2}{g}\,, \quad \boldsymbol{a}^2 = \frac{-(s + a_2\, r)\,\boldsymbol{e}_1 + (c + a_1\, r)\,\boldsymbol{e}_2}{g\, r}\,. \tag{21}$$

The Christoffel symbols thus become

$$[\Gamma^1_{\alpha\beta}] = \frac{1}{g} \begin{bmatrix} a_1\, c + a_2\, s & 0 \\ 0 & -r \end{bmatrix}, \quad [\Gamma^2_{\alpha\beta}] = \frac{1}{g\, r} \begin{bmatrix} a_2\, c - a_1\, s & g \\ g & (a_2\, c - a_1\, s)\, r^2 \end{bmatrix}. \tag{22}$$

The corresponding quantities for the reference configuration ($\boldsymbol{A}_\alpha$, $\boldsymbol{A}_{\alpha,\beta}$, $A_{\alpha\beta}$, $A^{\alpha\beta}$, $\boldsymbol{A}^\alpha$, and $\bar{\Gamma}^\gamma_{\alpha\beta}$) follow from this with $a_1 = a_2 = 0$ and $g = 1$. Thus,

$$\begin{aligned} [S^1_{\alpha\beta}] &:= [\Gamma^1_{\alpha\beta} - \bar{\Gamma}^1_{\alpha\beta}] = \frac{a_1\, c + a_2\, s}{g} \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}, \\ [S^2_{\alpha\beta}] &:= [\Gamma^2_{\alpha\beta} - \bar{\Gamma}^2_{\alpha\beta}] = \frac{a_2\, c - a_1\, s}{g\, r} \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}, \end{aligned} \tag{23}$$

which is only zero for vanishing a_i. Further $J = \sqrt{\det[a_{\alpha\beta}]/\det[A_{\alpha\beta}]} = g$.

For the fiber at $r = r_0$, the fiber direction (before and after deformation) is

$$\boldsymbol{L} = \boldsymbol{\ell} = \frac{\boldsymbol{a}_2}{\|\boldsymbol{a}_2\|} = -s\,\boldsymbol{e}_1 + c\,\boldsymbol{e}_2\,, \tag{24}$$

such that

$$L^1 = \ell^1 := \boldsymbol{\ell}\cdot\boldsymbol{a}^1 = 0\,, \quad L^2 = \ell^2 := \boldsymbol{\ell}\cdot\boldsymbol{a}^2 = \frac{1}{r_0}\,. \tag{25}$$

With this, the curvature invariant of Steigmann and Dell'Isola [3] follows as

$$\boldsymbol{g}_\mathrm{L} := L^\alpha\,L^\beta\,S^\gamma_{\alpha\beta}\,\boldsymbol{a}_\gamma = a_1\,\boldsymbol{e}_1 + a_2\,\boldsymbol{e}_2\,, \tag{26}$$

such that fiber bending energy of Steigmann and Dell'Isola [3],

$$W_\mathrm{SD} := \frac{B}{2}\,\boldsymbol{g}_\mathrm{L}\cdot\boldsymbol{g}_\mathrm{L} = \frac{B}{2}\big(a_1^2 + a_2^2\big)\,, \tag{27}$$

increases with a_1 and a_2. Here, B is the material constant for in-plane bending.

The fiber director follows from $\boldsymbol{\ell}$ by a counterclockwise rotation of 90°, i.e.,

$$\boldsymbol{c} = -c\,\boldsymbol{e}_1 - s\,\boldsymbol{e}_2\,. \tag{28}$$

Thus,

$$c_1 := \boldsymbol{c}\cdot\boldsymbol{a}_1 = g_0 := g|_{r=r_0}\,, \quad c_2 := \boldsymbol{c}\cdot\boldsymbol{a}_2 = 0\,. \tag{29}$$

With this and $\boldsymbol{L}_{,2} = -c\,\boldsymbol{e}_1 - s\,\boldsymbol{e}_2$, the two contributions of the geodesic curvature in equation (4) become

$$\kappa_\mathrm{g}^\Gamma := \ell^\alpha\,\ell^\beta\,S^\gamma_{\alpha\beta}\,c_\gamma = -a_1\,c - a_2\,s\,, \tag{30}$$

and

$$\kappa_\mathrm{g}^\mathrm{L} := \lambda^{-1}\,c_\alpha\,\ell^\beta\,\boldsymbol{A}^\alpha\cdot\boldsymbol{L}_{,\beta} = c_1\,\ell^2\,\boldsymbol{A}^1\cdot\boldsymbol{L}_{,2} = \frac{1}{r_0} + a_1\,c + a_2\,s\,, \tag{31}$$

for the present case, where there is no fiber stretch ($\lambda = 1$). The full geodesic curvature thus is

$$\kappa_\mathrm{g} = \kappa_\mathrm{g}^\Gamma + \kappa_\mathrm{g}^\mathrm{L} = \frac{1}{r_0}\,, \tag{32}$$

which is the expected value for the example. This is equal to the geodesic curvature in the reference configuration (i.e., $\kappa_\mathrm{g}^0 = 1/r_0$). All the relative curvature measures of Duong et al. [1] are thus zero, and the in-plane bending energy of Duong et al. [1],

$$W = \frac{B}{2}\big(\kappa_\mathrm{g} - \kappa_\mathrm{g}^0\big)^2\,, \tag{33}$$

vanishes for all a_i. In our formulation, the in-plane bending moment in the fiber is

$$\bar{m}_0 = \frac{\partial W}{\partial \kappa_\mathrm{g}}\,, \tag{34}$$

which gives $\bar{m}_0 = B\big(\kappa_\mathrm{g} - \kappa_\mathrm{g}^0\big) = 0$, here. Equation (34) follows from equations (82b), (108c), and (110c) of Duong et al. [1] with $\boldsymbol{\nu} = \boldsymbol{\ell}$ and $\bar{m}_0 = J\,\bar{m}$.

The corresponding (equivalent) bending moment according to formulation (8) is

$$\bar{m}_0 = \bar{M}^{\alpha\beta}_{0\,\gamma}\,\ell_\alpha\,\ell_\beta\,c^\gamma\,. \tag{35}$$

This follows from inserting equation (6) and $\bar{\mu}_0 = \bar{m}_0$ into equation (7) and comparing corresponding terms with equation (8). Here, $\ell^\alpha\ell_\alpha = 1$ and $c^\alpha c_\alpha = 1$ have been used. For the constitutive model of Steigmann and Dell'Isola [3] follows

$$\bar{M}^{\alpha\beta}_{0\,\gamma} = B\,L^{\alpha\beta}\,L^{\mu\eta}\,S^\delta_{\mu\eta}\,a_{\gamma\delta} \tag{36}$$

from equations (11c) and (27). Equation (35) then yields

$$\bar{m}_0^{\mathrm{SD}} = B\,\lambda^2\, S^{\gamma}_{\alpha\beta}\, L^{\alpha}\, L^{\beta}\, c_{\gamma}, \tag{37}$$

for general fibers with $L^{\alpha} = \lambda\,\ell^{\alpha}$. With equations (23), (25), and (29) and with $\lambda = 1$ of the present example, this becomes $\bar{m}_0^{\mathrm{SD}} = B\,(a_1\,c + a_2\,s)$, leading directly to equation (15).

Appendix 2

Corner forces in our formulation

Corner forces appear when the external virtual work expression (here for a single fiber family)

$$G_{\mathrm{ext}} = \int_{\mathcal{S}} \delta\boldsymbol{x}\cdot\boldsymbol{f}\,\mathrm{d}a + \int_{\partial\mathcal{S}} \delta\boldsymbol{x}\cdot\boldsymbol{T}\,\mathrm{d}s + \int_{\partial\mathcal{S}} \delta\boldsymbol{n}\cdot\boldsymbol{M}\,\mathrm{d}s + \int_{\partial\mathcal{S}} \delta\boldsymbol{c}\cdot\bar{\boldsymbol{M}}\,\mathrm{d}s, \tag{38}$$

(cf. equation (135.3) in [1]) is rewritten following the procedure used in our previous work ([6], cf. section 6.3). This starts with the balance of angular momentum

$$\frac{D}{Dt}\int_{\mathcal{R}} \rho\,\boldsymbol{x}\times\boldsymbol{v}\,\mathrm{d}a = \int_{\mathcal{R}} \boldsymbol{x}\times\boldsymbol{f}\,\mathrm{d}a + \int_{\partial\mathcal{R}} \boldsymbol{x}\times\boldsymbol{T}\,\mathrm{d}s + \int_{\partial\mathcal{R}} \hat{\boldsymbol{m}}\,\mathrm{d}s\,, \tag{39}$$

(cf. equation (87) in [1]), where $\hat{\boldsymbol{m}} := m_{\tau}\,\boldsymbol{\tau} + m_{\nu}\,\boldsymbol{\nu} + \bar{m}\,\boldsymbol{n}$ is the complete boundary bending moment defined in equation (66) in Duong et al. [1]. Using $\boldsymbol{\nu} = \boldsymbol{\tau}\times\boldsymbol{n}$, $\boldsymbol{n} = -\boldsymbol{\tau}\times\boldsymbol{\nu}$, and $\boldsymbol{\tau} := \partial\boldsymbol{x}/\partial s = \boldsymbol{x}'$, the last two terms in equation (39) can be rewritten as

$$\begin{aligned}\boldsymbol{x}\times\boldsymbol{T} + \hat{\boldsymbol{m}} &= \boldsymbol{x}\times\boldsymbol{T} + m_{\nu}\boldsymbol{x}'\times\boldsymbol{n} - \bar{m}\boldsymbol{x}'\times\boldsymbol{\nu} + m_{\tau}\,\boldsymbol{\tau}\\ &= \boldsymbol{x}\times\boldsymbol{T} + (m_{\nu}\,\boldsymbol{x}\times\boldsymbol{n})' - \boldsymbol{x}\times(m_{\nu}\,\boldsymbol{n})' - (\bar{m}\boldsymbol{x}\times\boldsymbol{\nu})' + \boldsymbol{x}\times(\bar{m}\,\boldsymbol{\nu})' + m_{\tau}\,\boldsymbol{\tau}\\ &= \boldsymbol{x}\times\;\boldsymbol{T} - (m_{\nu}\,\boldsymbol{n})' + (\bar{m}\,\boldsymbol{\nu})'\Big] + m_{\tau}\,\boldsymbol{\tau} + (m_{\nu}\,\boldsymbol{x}\times\boldsymbol{n})' - (\bar{m}\,\boldsymbol{x}\times\boldsymbol{\nu})'\,.\end{aligned} \tag{40}$$

The last expression shows that the moment components m_{ν} and $\bar{m}$ contribute to the effective boundary traction

$$\hat{\boldsymbol{t}} := \boldsymbol{T} - (m_{\nu}\,\boldsymbol{n})' + (\bar{m}\,\boldsymbol{\nu})'\,. \tag{41}$$

With this, $\delta\boldsymbol{x}'\cdot\boldsymbol{n} = -\boldsymbol{\tau}\cdot\delta\boldsymbol{n}$ and $\delta\boldsymbol{x}' = \tau^{\alpha}\,\delta\boldsymbol{a}_{\alpha}$ (following from $\boldsymbol{\tau}\cdot\boldsymbol{n} = 0$ and $\boldsymbol{x}' = \tau^{\alpha}\boldsymbol{a}_{\alpha}$ with $\tau^{\alpha} := \partial\xi^{\alpha}/\partial s$ being fixed), we can write

$$\delta\boldsymbol{x}\cdot\boldsymbol{T} = \delta\boldsymbol{x}\cdot\hat{\boldsymbol{t}} + \delta\boldsymbol{n}\cdot m_{\nu}\,\boldsymbol{\tau} + \delta\boldsymbol{a}_{\alpha}\cdot\bar{m}\,\tau^{\alpha}\,\boldsymbol{\nu} + (m_{\nu}\,\delta\boldsymbol{x}\cdot\boldsymbol{n})' - (\bar{m}\,\delta\boldsymbol{x}\cdot\boldsymbol{\nu})'. \tag{42}$$

Inserting equation (42) into equation (38) and using $\boldsymbol{M} = m_{\tau}\,\boldsymbol{\nu} - m_{\nu}\,\boldsymbol{\tau}$ (cf. equation (58) from [6]), $\bar{\boldsymbol{M}} = -\bar{m}\,\boldsymbol{\ell}$ and $\delta\boldsymbol{c}\cdot\boldsymbol{\ell} = -\ell^{\alpha}\boldsymbol{c}\cdot\delta\boldsymbol{a}_{\alpha}$ (cf. equations (76c) and (212) from [1]) gives

$$\begin{aligned}G_{\mathrm{ext}} = &\int_{\mathcal{S}} \delta\boldsymbol{x}\cdot\boldsymbol{f}\,\mathrm{d}a + \int_{\partial\mathcal{S}} \delta\boldsymbol{x}\cdot\hat{\boldsymbol{t}}\,\mathrm{d}s + \int_{\partial\mathcal{S}} \delta\boldsymbol{n}\cdot m_{\tau}\,\boldsymbol{\nu}\,\mathrm{d}s + \int_{\partial\mathcal{S}} \delta\boldsymbol{a}_{\alpha}\cdot\bar{m}\left(\tau^{\alpha}\boldsymbol{\nu} + \ell^{\alpha}\,\boldsymbol{c}\right)\mathrm{d}s\\ &+ \;\delta\boldsymbol{x}\cdot m_{\nu}\,\boldsymbol{n}\Big] - \;\delta\boldsymbol{x}\cdot\bar{m}\,\boldsymbol{\nu}\Big],\end{aligned} \tag{43}$$

where the last two terms are jump terms that appear at corners on $\partial\mathcal{S}$ due to the out-of-plane boundary moment m_{ν} and the in-plane boundary moment $\bar{m}$. Effectively, these moments act as point forces at those corners. Contrary to m_{ν}, $\bar{m}$ also contributes to another term in G_{ext}—the fourth term in equation (43). This is also seen in the formulation of Steigmann [2], cf. equation (87).