Beata PAŁCZYŃSKA, Dariusz ŚWISULSKI
Gdansk University of Technology

doi:

Sparse representation of a non-stationary signal in compressive sensing technique

***Abstract**. The paper presents the application of the compressive sensing technique to reconstruct a non-stationary signal based on compressed samples in the time-frequency domain. A greedy algorithm with different dictionaries to seek sparse atomic decomposition of the signal was applied. The results of the simulation confirm that the use of compressive sensing allows reconstruction of the non-stationary signal from a reduced number of randomly acquired samples, with slight loss of reconstruction quality.*

***Streszczenie.** Przedstawiono zastosowanie techniki oszczędnego próbkowania do rekonstrukcji sygnału niestacjonarnego na podstawie skompresowanych próbek w dziedzinie czas-częstotliwość. Zastosowano nadmiarowy algorytm z różnymi słownikami aby znaleźć rzadką reprezentację sygnału. Wyniki symulacji potwierdzają, że zastosowanie oszczędnego próbkowania pozwala na rekonstrukcję sygnału niestacjonarnego z małej liczby losowo pobranych próbek, z niewielką utratą jakości rekonstrukcji. (**Rzadka reprezentacja sygnału niestacjonarnego w technice oszczędnego próbkowania**).*

Keywords: sparse representation, non-stationary signal, compressive sensing.
Słowa kluczowe: rzadka reprezentacja sygnału, sygnał niestacjonarny, oszczędne próbkowanie.

Introduction

Compressive sensing (CS) is a technique of measuring signals and then reconstructing them with incomplete data (in comparison to classical measurement methods) [1, 2, 3]. It enables sampling below Nyquist frequency, without (or with slight) loss of a reconstructed signal quality. Particularly, CS is used in signal processing to obtain and recover sparse or compressible signals. Sparsity is the inherent property of those signals for which all information, contained in the signal, can be represented only by means of several significant components, compared to the total signal length. There is a base, in which the signal representation has a few components differing from zero. A signal can have sparse representation either in the original domain or in some transform domains. The time–frequency domain provides an ideal base to sparsely represent the non-stationary signals for two main reasons [4, 5]. First, it is extremely difficult to find a sparse representation of a non-stationary signal separately in the time domain as well as in the frequency domain. The second one is related to the fact that recent advances in computational resources enabled fast manipulations of large matrices, which are required for CS of non-stationary signals in the time–frequency domain.

Sparse representation originates from atomic decomposition, which is used for describing functions in mathematics [6]. Atomic decomposition can represent arbitrary signals as a superposition of some optimal elementary waveforms (atoms) that best match the signal major structures, based on a dictionary (a library of atoms). The number of atoms used in the signal representation determines the level of the sparsity.

In this paper, the matching pursuit (MP) algorithm is applied to seek sparse atomic decomposition. Furthermore, two types of dictionaries are used to reconstruct the time-frequency representations of a signal, namely the Gaussian and the chirplet dictionaries. Using only a few samples from the time-frequency domain, the non-stationary signals are recovered in the time-frequency domain based on the idea of the compressive sensing.

The paper is organized as follows. First section reviews the main ideas behind compressive sensing. Next, the approach to obtain compressed samples in the time–frequency domain including compressive sensing of non-stationary signals using time–frequency dictionaries is presented. Then, the exemplary simulation results are shown. Concluding remarks are drawn in the last section.

Review on theory of compressive sensing

The CS signal processing scheme contains both acquisition and reconstruction models (see Fig. 1).

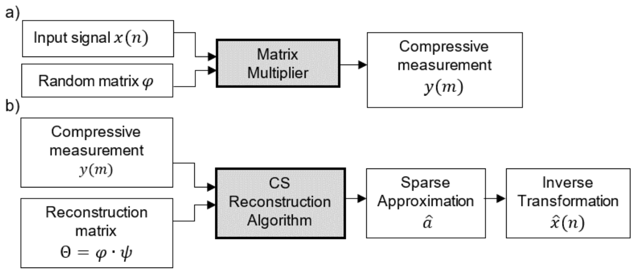

Fig.1. The signal processing scheme based on compressive sensing: acquisition model (a), reconstruction model (b) [7]

In order for CS to be applicable, it is assumed that the input signal $x = [x_1, \cdots, x_N]^T$ of length N can be represented by a linear combination of known basis functions [7]:

$$(1) \qquad x = \sum_{i=1}^{N} a_i \cdot \psi_i = \psi \cdot a$$

where: $a \in R^N$ – transform domain coefficients of x, ψ_i – the column vector in sparse matrix $\psi \in R^{NxN}$.

When the number of non-zero coefficients in x is K that x *is* K - sparse, which means that signal $x \in R^N$ can be transformed in sparse transform matrix ψ to K orthogonal vectors, where $K << N$.

The acquired signal y, containing a set of M random samples ($M << N$) is compressed with following equation [7]:

$$(2) \qquad y = \varphi \cdot x = \varphi \cdot \psi \cdot a = \Theta \cdot a$$

where: $\varphi \in R^{MxN}$ – a measurement matrix, $\Theta \in R^{MxN}$ – a reconstruction (sensing) matrix.

The measurement matrix should be designed as incoherent with the sparsity basis. The generally used measurement matrices in CS can be divided into the random measurement matrices, such as the random Gaussian matrix and the Bernoulli matrix, and the deterministic measurement matrices, such as the Fourier matrix, the Hadamard matrix, and the Toeplitz matrix [8].

Finally, the reconstruction process model is described as follows [7]:

(3) $$\hat{a} = \arg\min \|a\|_1 \quad \textit{subject to } y = \Theta \cdot a$$

where: $\hat{a}$ – the estimate of a, $\|a\|_1$ – denotes the l_1 – norm of a.

The most common used reconstruction algorithms for the above sparse signal recovery are the greedy algorithms, that solve the reconstruction problem by finding the answer iteratively [8]. The widely used algorithm is matching pursuit (MP), especially when the signal is highly sparse, then the MP procedure has a low implementation cost and high speed of recovery [9].

Time–frequency dictionary and MP algorithm

The aim of the reconstruction algorithm is to find the K non-zero coefficients of a which subjects to (2), on condition that a is a K– sparse representation of the signal. An approximation of a compressively sampled signal y is obtained using a linear expansion of atoms $g_{\gamma n}$ selected from a complete and redundant time-frequency dictionary Θ as [11]:

(4) $$y = \sum_{n=1}^{M} a_n g_{\gamma_n}$$

where: a_n – a sparse coefficient of the signal in time frequency domain.

The dictionary Θ is the reconstruction matrix expressed as follows [12]:

(5) $$\Theta = \begin{bmatrix} G_{11} & \cdots & G_{1N} \\ \vdots & \ddots & \vdots \\ G_{M1} & \cdots & G_{MN} \end{bmatrix}$$

and the *i-th* column of the matrix Θ is defined by [12]:

(6) $$g_{\gamma n}(i) = \left[G_{1,i}, \cdots, G_{Mi}\right]^T$$

The Gaussian chirplet atom $g_{\gamma n}$ is a chirp function modulated by a Gaussian envelope, expressing with four-parameter $(t_n, \omega_n, \sigma_n, \beta_n)$, where (t_n, ω_n) denotes the time-frequency center of the chirplet, σ_n is the Gaussian envelope's standard deviation and β_n specifies the chirp rate. When the chirp rate β_n is equal to zero, the elementary function is taking shape of the Gaussian pulse. In practice, the MP, based on the chirplet dictionary, has better resolution than the Gaussian pulse [5]. Furthermore, the convergence rate of the MP algorithm based on the chirplet is faster than one based on the Gaussian dictionary.

Then, rewrite (2) as follows:

(7) $$y = \begin{bmatrix} G_{11} & \cdots & G_{1N} \\ \vdots & \ddots & \vdots \\ G_{M1} & \cdots & G_{MN} \end{bmatrix} \cdot a = \\ = a_1 \cdot g_{\gamma n}(1) + a_2 \cdot g_{\gamma n}(2) + \cdots + a_M \cdot g_{\gamma n}(M)$$

Starting with a null initial model, MP algorithm iteratively builds up an approximation by adjoining at each stage an atom, which best correlates with the current residual signal $R^n y$ (see Tab. 1). The procedure is implemented iteratively until $\|R^M \cdot y\|$ reaches a predefined threshold or $M > K$. Then, the decomposition coefficients can be described by:

(8) $$\hat{a} = \left\langle R^n y, g_{\gamma n} \right\rangle, \quad n = 0, 1, ..., M-1$$

The theorem introduces a necessary condition for correct reconstruction regarding the minimum number of measurements to be acquired. It can be shown that, under the assumption of the restricted isometry property (RIP), the number of random samples M is such that [13]:

(9) $$M \geq C \cdot K \cdot \log\left(\frac{N}{K}\right)$$

where: $C = \frac{1}{2}\log\left(\sqrt{24}+1\right) \approx 0{,}28$.

Table 1. The MP procedure

Initialization	The residual vector is initialized with measurement vector y $i = 0, \quad R^0 y = y$
Atom search	This step finds a column of reconstruction matrix which is maximally correlated with the residual vector $g_{\gamma i} = \arg\max_{g_{\gamma i}} \left\|\left\langle R^i y, g_{\gamma i}\right\rangle\right\|$ $R^{i+1} y = R^i y - \left\langle R^i y, g_{\gamma i}\right\rangle g_{\gamma i}$
Update sparse solution	The signal y is expressed as a sum of atoms that best fit its residues $i = M-1, \quad y = \sum_{i=0}^{M-1} \left\langle R^i y, g_{\gamma i}\right\rangle g_{\gamma i} + R^M y$

Simulation

Simulations were carried out using the program, which was created based on available programs in the LabVIEW environment [14, 15]. It applies a compressive sensing algorithm to recover the signals from a set of single random (scalar) samples, where the signal is K – sparse in a time – frequency domain. Each measurement (sample) represents a random projection of the signal onto a single scalar value. Taking into consideration the basic compressive sensing equation (2), the elements of y are given by:

(10) $$y = \left\langle \varphi, x \right\rangle = \sum_{j=0}^{N-1} \varphi_{i,j} \cdot x_j$$

where: $\varphi_{i,j}$ - the *(i, j)*th entry of the random binary matrix φ, generated by a pseudorandom pattern of ones and zeros that guarantees the Bernoulli distribution.

The following equation defines the probability function of the Bernoulli noise [16]:

(11) $$\Pr[\varphi_{i,j} = z] = p^z \cdot (1-p)^{1-z}, \quad z \in \{0,1\}$$

where: p - the ones probability, which means, e.g. if p is equal to 0,1, each element of Bernoulli noise has a 10% chance of being one and a 90% chance of being zero.

The original signal x consists of two different components: one is a sinusoid of high concentration in the frequency domain and the other is the sum of three damped sinusoids of fine localization in the time. The time-frequency (sparse) domain results for different number of atoms, used in the signal representation are shown in Fig. 2.

The percentage of tested signal's variance (energy) explained by the CS reconstruction defines the accuracy of the reconstruction. To study the effect of noise background, a white Gaussian noise is added at two different signal-to-noise ratio (S/N), 3 dB and 20 dB. Fig. 3 shows results of sparse reconstruction for 70 iterations (measurements) under each noise level. The recovered signal explains about 70% of the signal total energy in the presence of weak noise. In the case of strong noise, the accuracy of CS decomposition is significantly decreasing.

The signal convergence in the sparse domain presents Fig. 4. In the weak noise case, the original signal is well recovered (see Fig. 4a). In the second case, noise contaminates the spectrogram (see Fig. 4b). However, the major time-frequency structures still matches with the true one shown in Fig. 2. A more accurate reconstruction for a noisy signal can be reached by increasing the number of random samples (measurements) in the CS acquisition or the number of atoms used to expand the signal in the MP algorithm.

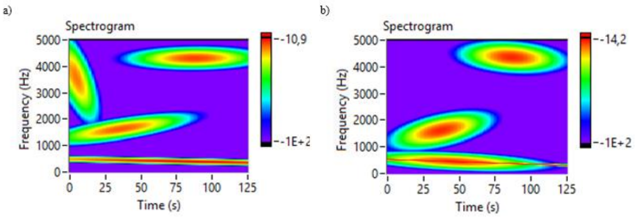

Fig. 2. An adaptive spectrogram of a tested signal for sparsity level equal to 10 (a), 6 (b)

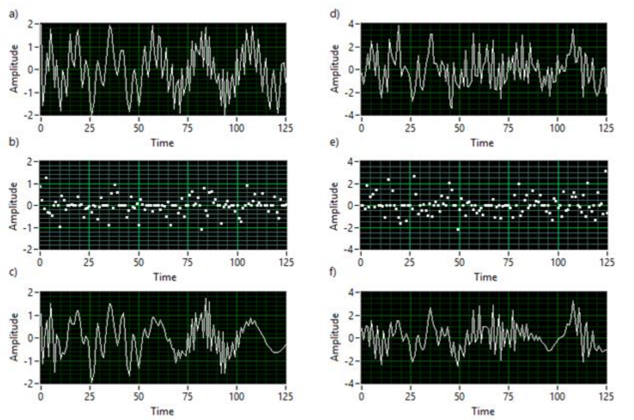

Fig.3. The waveforms of: tested signal (a), noise convoluted sparse signal (b), sparse representation (c) for S/N equals 3 dB. The waveforms of: tested signal (d), noise convoluted sparse signal (e), sparse representation (f) for S/N equals 20 dB

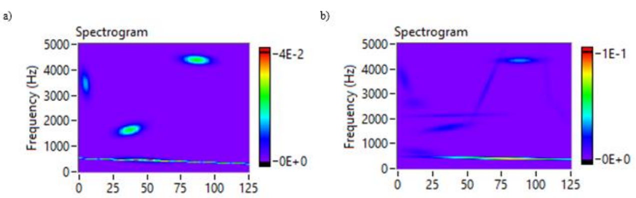

Fig. 4. An adaptive spectrogram of a reconstructed signal for S/N equal to 3 dB (a) and 20 dB (b)

Concluding remarks

The paper presents a short review of sparse representation of non-stationary signal in time-frequency domain. It describes the implementation of CS reconstruction by MP algorithm. Although MP is a heuristic procedure, it affords comparable and more accurate results in recovering the noiseless signal. In the noisy signal reconstruction case, MP processing contains errors that may be unacceptable. The convergence of the MP decomposition is not dependent on the type of atom used. The results demonstrate that the reconstruction of a non-stationary signal can be effectively performed from a small set of random measurements. The dimension of the measurement matrix affects the accuracy of the reconstruction process.

***Authors**: dr inż. Beata Pałczyńska, Politechnika Gdańska, Katedra Metrologii i Systemów Informacyjnych, ul. Narutowicza 11/12, 80-233 Gdańsk, E-mail: beata.palczynska@pg.edu.pl, dr hab. inż. Dariusz Świsulski, Politechnika Gdańska, Katedra Metrologii i Systemów Informacyjnych, ul. Narutowicza 11/12, 80-233 Gdańsk, E-mail: dariusz.swisulski@pg.edu.pl.*

REFERENCES

[1] Candes E., Wakin M., An Introduction to compressive sampling, *IEEE Signal Processing Magazine*, 25 (2008), no 5, 21-30.
[2] Donoho D., Compressed Sensing, *IEEE Transactions on Information Theory*, 52 (2006), 1289-1306
[3] Baraniuk, R. G., Compressive Sensing - Lecture Notes, *IEEE Signal Processing Magazine,* 24 (2007), no 4,118-121
[4] Sejdića, E., Orovićb, I., Stanković, S., Compressive sensing meets time–frequency: An overview of recent advances in time–frequency processing of sparse signals, *Digital Signal Processing,* 77 (2018), 22–35
[5] Ghofrani S., McLernon D.C., Ayatollahi A., Comparing Gaussian and chirplet dictionaries for time-frequency analysis using matching pursuit decomposition, *Proceedings of the 3rd IEEE International Symposium on Signal Processing and Information Technology* (2004), 713-716
[6] Mallat S. G., Zhang Z., Matching pursuit with time frequency dictionaries, *IEEE Trans. Signal Process.,* 41 (1993), no 12, 3397-3415
[7] Rani M., Dhok S. B., Deshmukh R. B., A Systematic Review of Compressive Sensing: Concepts Implementations and Applications, *IEEE Access* (2018)
[8] Candès E., Eldar Y.C., Needell D., Randall P., Compressed sensing with coherent and redundant dictionaries, *Appl. Comput. Harmon. Anal.*, 31 (2011), no 1, 59-73
[9] Candès, E. J., Romberg, J., Tao, T, Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information, *IEEE Transactions on Information Theory,* 52 (2006), 489–509
[10] Sankaranarayanan, A. C., Turaga, P. K., Chellappa, R., Baraniuk, R. G., Compressive acquisition of linear dynamical systems, *SIAM Journal on Imaging Sciences*, 6 (2013), 2109-2133
[11] Pałczyńska B., Identification of the time-vary magnetic field sources based on matching pursuit method, *Energies*, 10 (2017), Issue 5, article no 655
[12] Wang, Z., Lee, I., Sorted random matrix for orthogonal matching pursuit, *International Conference on Digital Image Computing: Techniques and Applications*, (2010), 116–120
[13] Candes E., The restricted isometry property and its implications for compressed sensing, *Science Direct Theory of Signals/ Mathematical Analysis,* 1 (2008), 589-592
[14] Perez L., Compressive Data Acquisition with LabVIEW, *https://forums.ni.com/t5/Example-Programs/Compressive-Data-Acquisition-with-LabVIEW/* (2009) – accessed on 2nd July 2019
[15] Rao T.J.V. S., Raja V.V., Srikanth Ch., Pravallika R., Bharadwaja P.S.K., Compressive data acquisition and reconstruction of multi tone signals in NI LabVIEW, *International Journal of Pure and Applied Mathematics*, 115 (2017), no. 7, 441-446
[16] LabVIEW - Advanced Signal Processing Toolkit - Time Frequency Analysis Tools, *User Manual, National Instruments*, (2005)