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Seiberg–Witten invariants, the topological degree and wall crossing formula

Research Article

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Abstract: Following S. Bauer and M. Furuta we investigate finite dimensional approximations of a monopole map in the case $b_1 = 0$. We define a certain topological degree which is exactly equal to the Seiberg–Witten invariant. Using homotopy invariance of the topological degree a simple proof of the wall crossing formula is derived.

MSC: 57R57, 55M25

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1. Introduction

Seiberg–Witten (SW) invariants were originally introduced to distinguish diffeomorphic but not homeomorphic closed 4-manifolds. Nowadays they have many other applications. For example, C.F. Taubes counted SW invariants for symplectic manifolds and showed that some manifolds do not admit a symplectic structure, cf. [4, Chapter 13]. However, most of the papers dealing with SW theory are very technical and hard to read.

In this paper the author would like to show that SW invariants can be defined in an abstract way. This abstract viewpoint, Section 3, should be more accessible for nonspecialists. The only data which is needed, in a simply connected case, is an S^1-equivariant Fredholm map satisfying certain conditions (A.1)–(A.4). Perturbing this map one can divide by an S^1 action and end up with a section over a non-compact manifold. The topological degree of this section is equal to the SW number if the starting Fredholm map was the monopole map. The monopole map $\mathcal{D}$ is defined in Section 2 (Section 2 also contains some basic definitions to make this article self-contained). Comparing to (A.1)–(A.4) one can see which properties of $\mathcal{D}$ are important.

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Usefulness of the abstract viewpoint is shown in the subsection . There are many proofs of the wall crossing formula, [4, Chapter 9] and references therein. However, the wall crossing formula is also true for any Fredholm map satisfying (A.1)–(A.4). To prove it one needs only the homotopy invariance of the topological degree and does not have to go into details of SW theory or deal with regularity as in other papers. This suggests that some phenomena in gauge theories can be studied in such an abstract way.

The idea that SW invariants depend only on the homotopy class of $\mathcal{D}$ was first noted by S. Bauer and M. Furuta, see [1].

2. Seiberg–Witten equations

2.1. Basic definitions

We recall some basic definitions from differential geometry. For more detailed information we refer to [4] and [5, Chapter 10].

Let G be a compact Lie group and (P,π) a principal G-bundle. To every representation $h\colon G\to \mathrm{Aut}(\mathbb{K}^n)$ we associate a $\mathbb{K}^n$ vector bundle by taking pairs $(p,z)\in P\times\mathbb{K}^n$ and dividing by the relation

$$(p,z)\sim\left(pg^{-1},h(g)z\right). \tag{1}$$

The projection is defined by $\pi_E([p,v])=\pi(p)$. This bundle is denoted by $P\times_{h(G)}\mathbb{K}^n$; if h is the standard representation we omit it in the notation.

Let X be a closed oriented Riemannian 4-manifold. By a $\mathrm{Spin}^c(4)$-structure we mean a principal $\mathrm{Spin}^c(4)$-bundle together with an isomorphism $P\times_{\mathrm{Spin}^c(4)}\mathbb{R}^4\simeq TX$, see [4, Chapter 5.1]. This is a natural definition if we think of a Riemannian structure as an isomorphism $P\times_{\mathrm{O}(4)}\mathbb{R}^4\simeq TX$ and of an orientation as an isomorphism $P\times_{\mathrm{SO}(4)}\mathbb{R}^4\simeq TX$. The group $\mathrm{Spin}^c(4)$ can be identified with

$$\left\{\begin{pmatrix} u_+ & 0\\ 0 & u_-\end{pmatrix} : u_+,u_-\in \mathrm{U}(2),\ \det u_+=\det u_-\right\}\subset \mathrm{U}(4). \tag{2}$$

In addition to the standard $\mathbb{C}^4$ representation we have the $\mathbb{C}^2$ representations which come from a multiplication by u_+,u_- and a U(1) representation from a multiplication by $\det u_+$. We will denote by W,W^+,W^- and L vector bundles associated to the above representations. Vector bundles W^+ and W^- are called the spinor bundles. Notice that the bundle $\mathrm{End}(W)$ is associated to the adjoint representation.

Take a unit vector $v=(t,x,y,z)\in\mathbb{R}^4$. It defines an element in SU(2) by

$$(t,x,y,z)\mapsto Q(v)=\begin{pmatrix} t+ix & -y+iz\\ y+iz & t-ix\end{pmatrix}$$

and an element in $\mathbb{C}^{4\times 4}$ by

$$v\mapsto\Gamma(v)=\begin{pmatrix} 0 & -Q^\dagger\\ Q & 0\end{pmatrix}, \tag{3}$$

where $\dagger$ denotes a hermitian conjugation. It can be easily checked that the map Γ is Spin^c equivariant. Therefore it induces a map from $TX=P\times_{\mathrm{Spin}^c(4)}\mathbb{R}^4$ to $\mathrm{End}(W)=P\times_{\mathrm{Ad}(\mathrm{Spin}^c(4))}\mathbb{C}^{4\times4}$ which is still denoted by Γ. Having this in mind, we can multiply a section of W by a vector field. Directly from the formula (3) we see that this multiplication interchanges W^+ and W^-.

Since the metric gives a canonical isomorphism between TX and T^*X we can as well think of Γ as a map from T^*X. This can be extended to an isomorphism from $\Lambda^*TX\otimes\mathbb{C}$ onto $\mathrm{End}(W)$. The extension is given simply by $\Gamma(\omega_1\wedge\omega_2)=\Gamma(\omega_1)\Gamma(\omega_2)$.

Let us now briefly discuss the notion of a connection on a principal G-bundle P. This would be useful because choosing a connection on P would give us a connection on every associated bundle. Recall that there is an exact sequence of vector bundle homomorphisms

$$0 \to \ker \pi_* \to TP \to \pi^* TX \to 0. \tag{4}$$

Definition 2.1.

A connection on P is a G-equivariant splitting of (4), i.e. a G-equivariant map $A\colon TP \to \ker \pi_*$ such that $A\restriction_{\ker \pi_*} = \mathrm{id}\restriction_{\ker \pi_*}$.

The kernel of A is a subbundle in TP isomorphic to $\pi^* TX$. It is called the horizontal subbundle and denoted by H_A. By definition H_A is such that $H_A \oplus V_A = TP$ where $V_A = \ker \pi_*$. Since $\ker \pi_*$ is canonically isomorphic to $P \times \mathrm{Lie}(G)$, one can think of a connection as a map from TP onto $P \times \mathrm{Lie}(G)$.

Let A and A' be two connections on P. Their difference $a^P = A - A'$ is identically zero on $\ker \pi_*$ and therefore defines an equivariant map from $\pi^* TX$ to $P \times \mathrm{Lie}(G)$ ($\pi^* TX$ is identified with a subbundle of TP by the connection A). By the equivariance, a^P gives a section a of a bundle $P \times_G \mathrm{Lie}(G) \otimes T^* X$ over X, where G acts on $\mathrm{Lie}(G)$ by the adjoint representation. Denote by $C^\infty\big(X, P \times_G \mathrm{Lie}(G) \otimes T^* X\big)$ the space of smooth sections of $P \times_G \mathrm{Lie}(G) \otimes T^* X$.

Fact 2.2.

The space of all connections on P is an affine space modeled on $C^\infty\big(X, P \times_G \mathrm{Lie}(G) \otimes T^* X\big)$.

Elements of $C^\infty\big(X, P \times_G \mathrm{Lie}(G) \otimes T^* X\big)$ are called one forms with values in $\mathrm{Lie}(G)$.

Remark 2.3.

To prove the above fact one has to show that there exists at least one connection. This can be constructed using a partition of unity.

Let E be a bundle associated to P, i.e. $E = P \times_G V$ for some G-representation V, and let s be a section of E. Given a connection A on P we define a connection ∇^A on E as follows. The section s can be viewed as a G-equivariant map from P to V. Denote by $ds\colon TP \to V$ the differential of s. Since H_A is canonically isomorphic to $\pi^* TX$, ds defines a G-equivariant map $(\nabla^A s)^P\colon \pi^* TX \to V$. This, again by the equivariance, gives a section $\nabla^A s$ of $(P \times_G V) \otimes T^* X = E \otimes T^* X$. It can be easily verified that ∇^A, defined in this way, satisfies the standard Leibniz rule

$$\nabla^A(fs) = f \nabla^A s + s \otimes df.$$

A connection ∇^A which is induced by a connection on a principal G-bundle is called a G-connection. Notice that E can be associated to more than one principal bundle. Given a connection on E one can also define a connection on P.

Let us return to the Spin^c structure. In Seiberg–Witten theory we are not interested in all Spin^c connections but only in those being compatible with a given Riemannian structure. This can be expressed as follows. Recall that the group Spin^c is isomorphic to $\mathrm{Spin}(4) \times_{\mathbb{Z}_2} \mathrm{U}(1)$ so $\mathrm{Lie}(\mathrm{Spin}^c(4)) = \mathrm{Lie}(\mathrm{SO}(4)) \oplus \mathrm{Lie}(\mathrm{U}(1))$. With this in mind, a Spin^c connection is uniquely determined by an $\mathrm{SO}(4)$ connection on TX and an $\mathrm{U}(1)$ connection on L. Going the other way, a Spin^c connection induces the $\mathrm{SO}(4)$ and $\mathrm{U}(1)$ connections. We say that a Spin^c connection is compatible with a Riemannian structure if it induces the Levi-Civita connection on TX. The space of all such connections is denoted by $\mathcal{A}$ and is naturally isomorphic to the space of $\mathrm{U}(1)$ connections on L and therefore an affine space modeled on $C^\infty\big(X, P \times_{\mathrm{Spin}^c(4)} \mathrm{Lie}(\mathrm{U}(1)) \otimes T^* X\big) = \Omega^1(X, i\mathbb{R})$. We are ready to define a Dirac operator.

Definition 2.4.

For a given connection $A \in \mathcal{A}(P)$ a Dirac operator $D_A\colon C^\infty(X, W^+) \to C^\infty(X, W^-)$ is a composition of Γ and ∇^A, i.e.,

$$D_A \psi(p) = \sum_{i=1}^{4} \Gamma(e_i) \nabla^A_{e_i} \psi(p),$$

where e_1, e_2, e_3, e_4 is an orthonormal frame in a neighbourhood of p.

$D_A\phi = 0$ is the first of two SW equations. In order to write the second one we have to introduce a quadratic map. Take $\psi = (\psi_1, \psi_2) \in \mathbb{C}^2$ and define $(\psi\psi^*)_0 \in \mathrm{End}(\mathbb{C}^2)$ by a standard formula $(\psi\psi^*)_0(\phi) = \langle\psi, \phi\rangle\psi - |\psi|^2\phi/2$. This is a traceless skew-hermitian endomorphism of $\mathbb{C}^2$. The map that assigns $\psi \in \mathbb{C}^2$ an endomorphism $(\psi\psi^*)_0$ turns out to be Spin^c equivariant and therefore extends to a map from $C^\infty(X, W^+)$ to $\mathrm{End}(W^+) \subset \mathrm{End}(W)$. Denote this map by $\widehat{\sigma}$. Recall that Γ together with the metric gives an isomorphism of $\mathrm{End}(W)$ and $\Lambda^* T^*X \otimes \mathbb{C} = \Omega^*(X, \mathbb{C})$. Thus we have a map $\sigma\colon C^\infty(X, W^+) \to \Omega^*(X, \mathbb{C})$. In fact, the image of σ is equal to $\Omega^{2,+}(X, i\mathbb{R})$ – the space of purely imaginary self-dual two forms.

Seiberg–Witten equations are

$$D_A\psi = 0, \qquad F_+^A = \sigma(\psi),$$

where F_A^+ is the self-dual part of the U(1) connection associated to A. We are interested in finding pairs $(A, \phi) \in \mathcal{A}(P) \times C^\infty(X, W^+)$ that solve these equations.

2.2. Monopole map and gauge transformations

Define a map $\mathcal{D}\colon \mathcal{A}(P) \times C^\infty(X, W^+) \to C^\infty(X, W^-) \oplus \Omega^{2,+}(X, i\mathbb{R})$ by

$$\mathcal{D}(A, \phi) = \big(D_A\phi, F_+^A - \sigma(\psi)\big).$$

Zeros of $\mathcal{D}$ are solutions of the SW equations. This map is equivariant with respect to the action of a gauge transformation, i.e. the group of automorphisms of the principal bundle.

All connections in $\mathcal{A}$ are compatible with a Riemannian structure so we are left with U(1) gauge only. Since U(1) is Abelian, the group of gauge transformations is isomorphic to a group of maps from X to U(1). This group acts on $C^\infty(X, W)$ by a multiplication and on $\mathcal{A}(P)$ by adding $u^{-1}du$ for $u \in \mathcal{G}$. Fix a point $x_0 \in X$. $\mathcal{G}$ decomposes into the product of two smaller subgroups, namely $\mathcal{G}_{x_0} = \{u \in \mathcal{G} : u(x_0) = 1\}$ and $S^1 = \{u \in \mathcal{G} : u = \mathrm{const}\}$.

Fix a connection A_0. Any connection $A \in \mathcal{A}(P)$ can be written in the form $A = A_0 + a$, where $a \in \Omega^1(X, i\mathbb{R})$. Suppose $b_1(X) = 0$ where $b_1(X) = \dim H^1(X, \mathbb{R})$.

Fact 2.5.
In every orbit of the action of $\mathcal{G}_{x_0}$ there is exactly one connection $A = A_0 + a$ such that $d^*a = 0$.

Consider a slight modification of $\mathcal{D}$ (still denoted by $\mathcal{D}$).

$$\mathcal{D}\colon (\Omega^1(X, i\mathbb{R}) \cap \ker d^*) \oplus C^\infty(X, W^+) \to C^\infty(X, W^-) \oplus \Omega^{2,+}(X, i\mathbb{R})$$

is given by the formula

$$\mathcal{D}(a, \phi) = \big(D_{A_0+a}\phi, F_+^{A_0} + d^+a - \sigma(\psi)\big),$$

where d^+ is the projection onto $\Omega^{2,+}(X, i\mathbb{R})$. This is called a monopole map. Notice that we are left only with the action of S^1. $\mathcal{D}$ is a perturbation of a first order elliptic differential operator and therefore extends to a Fredholm map in certain Sobolev completions.

Theorem 2.6 (compactness).
$\mathcal{D}^{-1}(\eta)$ is compact for every $\eta \in \Omega^{2,+}(X, i\mathbb{R})$.

For details see [4, Theorem 7.12]. If $\eta \in \Omega^{2,+}(X, i\mathbb{R})$ is a regular value then $\widehat{\mathcal{M}} = D^{-1}(\eta)$ is a compact finite dimensional manifold. The Seiberg–Witten invariant is an oriented counting of the connected components of $\widehat{\mathcal{M}}$. We will make this more precise in the next section where we will define the Seiberg–Witten number for a larger class of operators to which the monopole map belongs.

3. Abstract viewpoint

3.1. Topological degree and Poincaré duality

Let $s\colon \mathbb{R}^d \to \mathbb{R}^d$ be a proper map. Suppose that 0 is a regular value of s, i.e. $s^{-1}(0)$ consists of a finite number of points $x_1, x_2, \dots, x_l$. We can take pairwise disjoint neighbourhoods $\{U_i\}$ such that $x_i \in U_i$ and $s\restriction_{U_i}$ is a diffeomorphism onto its image for every i.

Choose a form c representing a generator in cohomology with compact support $H^d_c(\mathbb{R}^d, \mathbb{Z})$, such that $\operatorname{supp} c \subset \bigcap_i s(U_i)$, see [2, Chapter 4]. Then

$$\int_{\mathbb{R}^d} s^*(c) = \sum_i \int_{U_i} s^*(c) = \sum_i (-1)^{o(i)} \int_{s(U_i)} c = \sum_i (-1)^{o(i)},$$

where $o(i)$ is equal 1 or -1 depending on whether $s\restriction_{U_i}$ does or does not preserve orientation. Thus one can define the ordinary topological degree as $\langle [\mathbb{R}^d], [s^*(c)]\rangle$.

This has a natural generalization to the vector bundles. Let E be a vector bundle over a manifold Y such that $\operatorname{rank} E = \dim Y = d$ and let $s\colon Y \to E$ be a proper section, i.e. $s^{-1}(0) = s(Y) \cap s_0(Y)$ (where s_0 is a zero section) is compact.

Definition 3.1.

The topological degree $\deg s$ is the number

$$\langle [Y], [s^*(c(E))]\rangle = \int_Y s^*(c(E)),$$

where $c(E)$ denotes the Thom class of E.

Notice that this depends only on a homotopy class of s, i.e. if there exists a homotopy $H\colon Y \times [0,1] \to E$ such that $H^{-1}(0)$ is compact then $\deg H(\cdot\,, 0) = \deg H(\cdot\,, 1)$.

Remark 3.2.

Suppose that Y is compact. Then every section is homotopic to the zero section by linear homotopy. Pullback of $c(E)$ by the zero section is equal to the Euler class. Thus $\deg s$ is equal to the Euler characteristic for every section s.

Suppose we have a splitting $E' = E \oplus F$ for some bundles E, F of ranks $d-2k$ and $2k$ respectively. Let s_E be a proper transversal section of E. By this $M = s_E^{-1}(0)$ is a $2k$ dimensional compact manifold. Let $j\colon M \to Y$ denote an inclusion.

Lemma 3.3.

We have

$$\int_M j^*(e(F)) = \deg s,$$

where $s \in C^\infty(Y, E\oplus F)$, $s = (s_E, s_0)$ and s_0 is the zero section.

Proof. By the Poincaré duality there exists a class $[\eta_M] \in H^{d-2k}_c(Y)$ such that $\int_M j^*(\omega) = \int_Y \omega \wedge \eta_M$ for every $[\omega] \in H^{2k}(Y)$. Set $\omega = e(F)$. Then

$$\int_M j^*(e(F)) = \int_Y e(F) \wedge \eta_M.$$

On the other hand,

$$\deg s = \int_Y s^*(c(E\oplus F)) = \int_Y s_E(c(E)) \wedge s_0(c(F)) = \int_Y s_E(c(E)) \wedge e(F) = \int_Y e(F) \wedge s_E(c(E)).$$

It is sufficient to prove that $\eta_M = s_E(c(E))$. This can be found in [2]. □

Notice that the left hand side of the equality $\int_M j^*(e(F)) = \deg s$ makes sense only when the section s_E is transversal, but $\deg s$ can be defined in general. This is an important observation because it will allow us not to deal with regularity problems in Seiberg–Witten theory.

3.2. Finite dimensional approximations

We can define an analogue to the Seiberg–Witten number for some class of Fredholm operators. Precisely, consider real Hilbert spaces W_1, W_2 and complex Hilbert spaces V_1, V_2. Suppose that $f\colon W_1 \oplus V_1 \to W_2 \oplus V_2$ satisfies the following conditions:

(A.1) $f = L + C$, where L is a linear Fredholm map and C is compact,

(A.2) L and C are S^1 equivariant (here S^1 acts trivially on W_1, W_2 and by complex multiplication on V_1, V_2),

(A.3) $f^{-1}(\eta)$ is compact for any $\eta \in W_2$ (for simplicity we write η for $(\eta, 0) \in W_2 \oplus V_2$),

(A.4) $f\restriction_{W_1} = A$ for some linear $A\colon W_1 \to W_2$ with $\dim\ker A = 0$ and $\dim\operatorname{coker} A > 0$.

If the Fredholm index of f is even we define the SW number to be zero. Suppose $\operatorname{ind} f = 2k+1$. Let $\eta \in W_2$ be a regular value and $\eta \notin \operatorname{Im} A$. Then $\widehat{\mathcal{M}} = f^{-1}(\eta)$ is a compact manifold with free S^1 action. We define

$$\mathrm{SW}(f, \eta) = \int_{\mathcal{M}} \tau^k,$$

where $\mathcal{M} = \widehat{\mathcal{M}}/S^1$ and τ is the Euler class of a circle bundle $\widehat{\mathcal{M}} \to \mathcal{M}$.

It is known that the SW number can be viewed as "counting" solutions with appropriate signs. We will make this remark more precise by defining the SW number as the topological degree of a section of a certain vector bundle. For this purpose we consider finite dimensional approximations of f. Let $W_2^{(n)} \oplus V_2^{(n)}$ be such that

- $\dim\left(W_2^{(n)} \oplus V_2^{(n)}\right) < \infty$ for every n,
- $(\operatorname{Im} L)^\perp \subset W_2^{(1)} \oplus V_2^{(1)}$,
- $W_2^{(n)} \oplus V_2^{(n)} \subset W_2^{(n+1)} \oplus V_2^{(n+1)}$,
- $\overline{\bigcup_{n\geq 1}(W_2^{(n)} \oplus V_2^{(n)})} = W_2 \oplus V_2$.

Let $W_1^{(n)} \oplus V_1^{(n)} = L^{-1}\left(W_2^{(n)} \oplus V_2^{(n)}\right)$. We define $f_n\colon W_1^{(n)} \oplus V_1^{(n)} \to W_2^{(n)} \oplus V_2^{(n)}$ by the formula $f_n = L_n + p_n \circ C$, where

$$L_n = L\restriction_{W_1^{(n)} \oplus V_1^{(n)}}$$

and p_n is the orthogonal projection onto $W_2^{(n)} \oplus V_2^{(n)}$. For a sufficiently large n_0 the map f_{n_0} satisfies conditions (A.1)–(A.4) so the number $\mathrm{SW}(f_{n_0}, \eta)$ is well defined.

Vector spaces $W_1^{n_0}, W_2^{n_0}, V_1^{n_0}, V_2^{n_0}$ are all finite dimensional. Thus without loss of generality one can think of

$$\overline{f} = f_{n_0}\colon \mathbb{R}^n \oplus \mathbb{C}^m \to \mathbb{R}^{n'} \oplus \mathbb{C}^{m'}.$$

The Fredholm index of f is equal to $2k+1$ and so is the index of $\overline{f}$. Thus we have $2k+1 = n + 2m - n' - 2m'$. By (A.4), $n < n'$ so $m > m' + k = m''$. Define $\widehat{f}\colon \mathbb{R}^n \oplus \mathbb{C}^m \to \mathbb{R}^{n'} \oplus \mathbb{C}^{m''}$ by $\widehat{f} = i \circ \overline{f}$, where i is an inclusion from $\mathbb{R}^{n'} \oplus \mathbb{C}^{m'}$ to $\mathbb{R}^{n'} \oplus \mathbb{C}^{m''}$. Clearly, by definition, $\operatorname{ind} \widehat{f} = 1$.

Set $A_{\epsilon,R} = \{z \in \mathbb{C}^m : \epsilon < |z| < R\}$ and $Y'_{\epsilon,R} = B(\mathbb{R}^n, R) \times A_{\epsilon,R}$, where $B(\mathbb{R}^n, R)$ is an open ball of radius R. Suppose that $\eta \in \mathbb{R}^{n'}$ is a regular value and $\eta \notin \operatorname{Im} A$, i.e. $\widehat{\mathcal{M}} = \widehat{f}^{-1}(\eta)$ is a 1-dimensional compact manifold (compactness is due to (A.3)). Since $\widehat{\mathcal{M}} \cap \mathbb{R}^n \times \{0\} = \emptyset$ we can choose ϵ and R so that $\widehat{\mathcal{M}} \subset Y'_{\epsilon,R} = Y'$.

Let E and F be vector bundles defined by the projections $\pi_E\colon \left((Y' \times \mathbb{R}^{n'}) \oplus \mathbb{C}^{m''}\right)/S^1 \to Y'/S^1$ and $\pi_E\colon (Y \times \mathbb{C}^k)/S^1 \to Y/S^1$, respectively. The map $\overline{f}$ defines a section s_E of the bundle E by the formula $s_E([y]) = [y, \overline{f}(y) - \eta]$. In the same way the map $\widehat{f} = i \circ \overline{f}$ defines a section s of $E \oplus F$ which is equal (s_E, s_0). By Lemma 3.3 we have

Theorem 3.4.
In the above situation, $\mathrm{SW}(\widehat{f},\eta)=\deg s$.

It is a general fact [3, Proposition 7.2], that there is a one to one correspondence between homotopy classes of G-equivariant maps from X to Z and homotopy classes of sections of bundle $(X\times Z)/G\to X/G$. Thus by the homotopy invariance of the topological degree we deduce S^1-homotopy invariance of the SW number.

The map $\widehat{f}$ is a composition of $\overline{f}$ with the inclusion map into a bigger space. It is seen on the vector bundles level by adding another vector bundle and defining section to be zero on that bundle. As one might expect this should not carry any additional information about $\overline{f}$ or s_E, respectively. This is exactly the case by the following observation.

Notice that Y has the homotopy type of $\mathbb{C}P^N$. $s^*(c(E\oplus F))$ is an element of $H_c^{2N}(\mathbb{C}P^N,\mathbb{Z})\simeq\mathbb{Z}$. We can decompose the bundle F into k copies of bundles F_1 defined by a projection $\pi_1\colon Y'\times\mathbb{C}\to Y$. We have

$$s^*(c(E\oplus F))=s_E(c(E))\wedge e(F)=s_E(c(E))\wedge e(F_1)\wedge\ldots\wedge e(F_1).$$

On the level of $\mathbb{C}P^N$, the bundle F_1 is the canonical bundle, i.e. $e(F_1)$ is a generator of $H^2(\mathbb{C}P^N,\mathbb{Z})$. Thus in the standard identification of $H^2(\mathbb{C}P^N,\mathbb{Z})$ with $\mathbb{Z}$ elements $s_E(c(E))$ and $s^*(c(E\oplus F))$ define the same number. However, in some cases it seems to be easier to work with $s^*(c(E\oplus F))$ because it naturally occurs as a topological degree.

3.3. The wall crossing formula

The aim of this paragraph is to study η dependence of SW number. Instead of doing it directly, we will use Theorem 3.4 and study $\deg s$ constructed above. Write $\deg(s,\eta)$ for $\deg s$ to emphasise η dependence.

Lemma 3.5.
Let $\xi\colon[0,1]\to\mathbb{R}^{n'}$ *be a path joining* η_0 *with* η_1 *such that* $\xi([0,1])\cap\operatorname{Im}A=\emptyset$. *Then* $\deg(s,\eta_0)=\deg(s,\eta_1)$.

Proof. Let $\widehat{\mathcal{M}}(t)=\widehat{f}^{-1}(\xi(t))$. We can choose such ϵ and R that $\widehat{\mathcal{M}}(t)\subset Y'_{\epsilon,R}=Y'$ for every t. In this case, ξ defines an S^1-equivariant homotopy between $\widehat{f}$ and $\widehat{f}+\eta_0-\eta_1$ and so a proper homotopy of sections on vector bundles level. □

As a corollary we have that $\mathrm{SW}(f,\eta_0)=\mathrm{SW}(f,\eta_1)$ whenever η_0 and η_1 can be joint by a path. This can be always done when $\operatorname{coker}A>1$. Suppose that $\operatorname{coker}A=1$, i.e. $n'=n+1$ (then $m''=m-1$). In this case, there are exactly two connected components of $\mathbb{R}^{n'}-\operatorname{Im}A$.

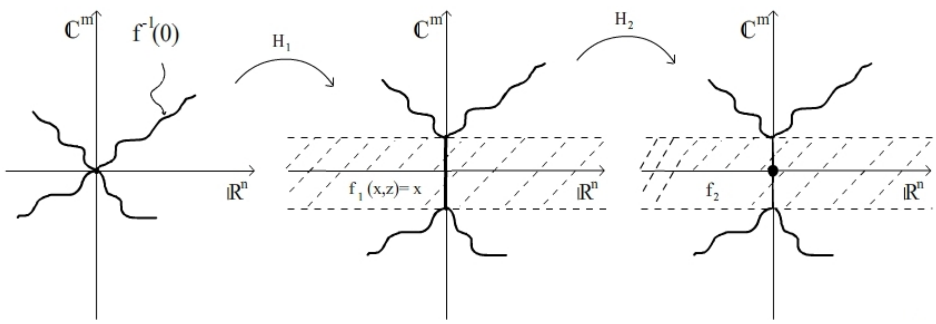
Theorem 3.6 (wall crossing formula).
If η_+ *and* η_- *are in different connected components of* $\mathbb{R}^{n'}-\operatorname{Im}A$, *then* $|\mathrm{SW}(f,\eta_+)-\mathrm{SW}(f,\eta_-)|=1$.

The idea is as follows. Recall that $f(x,0)=Ax$ and so $f^{-1}(0)\cap\mathbb{R}^n=\{0\}=\{x_0\}$. We will find an S^1-homotopic map to f such that x_0 is isolated from other elements of $f^{-1}(0)$. From the construction it would be clear that perturbing this map would produce an S^1 orbit of solutions from x_0 in one case and nothing in the other.

Proof. *Step 1*: Choose such coordinates that A is an inclusion $\mathbb{R}^n\hookrightarrow\mathbb{R}^{n+1}$, i.e. $Ax=x$. Let R be such that $f^{-1}(0)\subset B(\mathbb{R}^n,R)\times B(\mathbb{C}^m,R)=B$ and take any $\epsilon<R$. Define a homotopy $H_1\colon B\times[0,1]\to\mathbb{R}^{n+1}\oplus\mathbb{C}^{m+1}$ by the formula

$$H_1(x,z,t)=\begin{cases}x & \text{for}\quad \|z\|\le t\epsilon,\\ f\left(x,\dfrac{R(\|z\|-t\epsilon)}{\|z\|(R-t\epsilon)}z\right) & \text{for}\quad \|z\|>t\epsilon.\end{cases}\tag{5}$$

Let $f_1=H_1(\cdot,1)$.



Introduce a bump function

$$\theta(r) = \begin{cases} e^{-1/(r(\epsilon-r))} & \text{for} \quad r \in (0,\epsilon), \\ 0 & \text{for} \quad r \notin (0,\epsilon). \end{cases}$$

In order to isolate $x_0 = 0$ from other solutions we define

$$H_2(x,z,t) = f_1(x,z) + t \cdot \theta(\|z\|)\big((0,\dots,0,1) + (z_2,z_3,\dots,z_m)\big) \tag{6}$$

and $f_2 = H_2(\cdot,1)$. Notice that $f_2^{-1}(0) \cap B = \{0\}$.

Step 2: Take such $\eta_\pm = (0,\dots,0,\eta_\pm^{n+1}) \in \mathbb{R}^{n+1}$ that $\eta_- < 0 < \eta_+ < \theta(\epsilon/2)$ and maps $f_2 - \eta_+$ and $f_2 - \eta_-$ are homotopic on $Y'_{\epsilon/2,R}$. Since $f_2^{-1}(\eta_\pm) \cap \mathbb{R}^n = \emptyset$ we can choose such ϵ' that $f_2^{-1}(\eta_\pm) \subset Y'_{\epsilon',R}$.

We want to compute $|\mathrm{SW}(f_2,\eta_+) - \mathrm{SW}(f_2,\eta_-)|$. By Theorem 3.4 it is equal to $|\deg(s_2,\eta_+,Y'_{\epsilon',R}) - \deg(s_2,\eta_-,Y'_{\epsilon',R})|$. Since $\deg(s_2,\eta_\pm,Y'_{\epsilon',R}) = \deg(s_2,\eta_\pm,Y'_{\epsilon',\epsilon/2}) + \deg(s_2,\eta_\pm,Y'_{\epsilon',R})$ and $f_2 - \eta_+$ is S^1 homotopic to $f_2 - \eta_-$, we have

$$\deg(s_2,\eta_+,Y'_{\epsilon',R}) - \deg(s_2,\eta_-,Y'_{\epsilon',R}) = \deg(s_2,\eta_+,Y'_{\epsilon',\epsilon/2}) - \deg(s_2,\eta_-,Y'_{\epsilon',\epsilon/2}).$$

Suppose $f_2(x,z) - \eta = 0$. This is equivalent to

$$\eta_+ = f_2(x,z) = x + \theta(\|z\|)\big((0,\dots,0,1) + (z_2,z_3,\dots,z_m)\big),$$

i.e. $0 = x = z_2 = \ldots = z_m$ and $\theta(\|z\|) = \theta(|z_1|) = \eta_+^{n+1}$. Denote by c the smaller of two solutions of the equation $\theta(x) = \eta_+^{n+1}$. Then

$$f_2^{-1}(\eta_+) \cap Y'_{\epsilon',\epsilon/2} = \{(c \cdot e^{i\phi},0,\dots,0) \in \mathbb{C}^m : \phi \in \mathbb{R}\}.$$

While computing the corresponding degree we have to quotient it by S^1 and thus it gives us exactly one solution.

In a similar way we deduce that $f_2^{-1}(\eta_+) \cap Y'_{\epsilon',\epsilon/2}$ is empty because the equation $\theta(x) = \eta_-^{n+1}$ has no solutions. As a corollary we have

$$\deg(s_2,\eta_+,Y'_{\epsilon',R}) - \deg(s_2,\eta_-,Y'_{\epsilon',R}) = 1.$$

Since η_0,η_1 are in different connected components of $\mathbb{R}^{n+1} - \mathbb{R}^n \times \{0\}$, one of them can be connected by a path with η_+ and the other one with η_-. By Lemma 3.5,

$$\deg(s_2,\eta_0,Y'_{\epsilon',R}) - \deg(s_2,\eta_1,Y'_{\epsilon',R}) = 1. \qquad \square$$

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