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Some Progress on Total Bondage in Graphs

Nader Jafari Rad · Joanna Raczek

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Abstract The total bondage number $b_t(G)$ of a graph G with no isolated vertex is the cardinality of a smallest set of edges $E' \subseteq E(G)$ for which (1) $G - E'$ has no isolated vertex, and (2) $\gamma_t(G - E') > \gamma_t(G)$. We improve some results on the total bondage number of a graph and give a constructive characterization of a certain class of trees achieving the upper bound on the total bondage number.

Keywords Domination number · Total domination · Total bondage · Tree

Mathematics Subject Classification (2000) 05C69 · 05C05

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n. We denote the *open neighborhood* of a vertex v of G by $N_G(v)$ or just $N(v)$, and its *closed neighborhood* by $N_G[v] = N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \bigcup_{v\in S} N(v)$ and $N[S] = \bigcup_{v\in S} N[v]$. The *degree* $\deg_G(x)$ (or just $\deg(x)$) of a vertex x denotes the number of neighbors of x in G. The maximum and minimum degree of a vertex of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A set of vertices S in G is a *dominating set*

N. J. Rad
Department of Mathematics, Shahrood University of Technology, Shahrood, Iran
e-mail: n.jafarirad@shahroodut.ac.ir

J. Raczek (✉)
Department of Applied Physics and Mathematics, Gdansk University of Technology, Narutowicza 11/12, Gdańsk 80-233, Poland
e-mail: Joanna.Raczek@pg.gda.pl

if $N[S] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A set of vertices S in a graph G without isolated vertex is a *total dominating set*, or just TDS, if $N(S) = V(G)$. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS of G. We refer to a $\gamma_t(G)$-set in a graph G as a minimum cardinality TDS of G. For references on domination in graphs see for example [6,7].

With K_n we denote the *complete graph* on n vertices, with $P_n = (v_1, v_2, \ldots, v_n)$ the *path* on n vertices, with C_n the *cycle* of length n, and with $K_{p,q}$ the *complete bipartite graph*, which one partite set is of cardinality p and another partite set is of cardinality q. We also let $g(G)$ be the *girth* of G, that is the length of a shortest cycle in G. Denote by $d_G(u, v)$ the *distance* between u and v in G. The *diameter* of G is defined as $diam(G) = \max\{d_G(u, v) : u, v \in V(G)\}$.

If S is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of G induced by S. We recall that a leaf in a graph is a vertex of degree one and a support vertex is one that is adjacent to a leaf. Let $S(G)$ be the set of all support vertices in a graph G. A *pendant edge* is an edge incident with a leaf.

The *bondage number* $b(G)$ of a nonempty graph G is the minimum cardinality among all sets of edges $E' \subseteq E(G)$ for which $\gamma(G - E') > \gamma(G)$. This concept was introduced by Bauer, Harary, Nieminen and Suffel in [1], and has been further studied for example in [4,5,13]. For more information on this topic we refer the reader to the survey article by Dunbar, Haynes, Teschner and Volkmann [3].

Kulli et al. in [9] introduced the concept of *total bondage* in graphs. The *total bondage number* $b_t(G)$ of a graph G with no isolated vertex is the cardinality of a smallest set of edges $E' \subseteq E(G)$ for which (1) $G - E'$ has no isolated vertex, and (2) $\gamma_t(G - E') > \gamma_t(G)$. In the case that there is no subset of edges E' such that (1) and (2) both hold, we define $b_t(G) = \infty$. The total bondage number was further studied in [12]. Other types of bondage in trees give raise to many interesting problems. This was studied for example in [10] and [11].

In this paper we continue the study of total bondage in graphs. Since determining the exact number of the total bondage number for general graphs is a hard problem [8], in this paper we establish bounds on this number in some classes of graphs. In Sect. 2 we state some known and preliminary results. In Sect. 3 we first obtain an improved upper bound for the total bondage number of a tree and then we give an interesting constructive characterization of a certain class of trees achieving equality for the upper bound. In Sect. 4 we obtain some upper bounds on $b_t(G)$ in terms of maximum and minimum degrees, which in particular improve similar previous bounds.

2 Known and Preliminary Results

In this section we give some known and preliminary results which we use in the next sections. We begin with the following exact values of total bondage number of paths and cycles.

Proposition 1 *(Kulli et al.* [9]*) For $n \geq 2$,*

$$b_t(P_n) = \begin{cases} \infty & \text{if } n \leq 3 \\ 1 & \text{if } n \geq 4,\ n \not\equiv 2 \ (mod\ 4) \\ 2 & \text{if } n \geq 4,\ n \equiv 2 \ (mod\ 4) \end{cases}.$$

Proposition 2 *(Kulli et al.* [9]*) For $n \geq 3$,*

$$b_t(C_n) = \begin{cases} \infty & \text{if } \ n = 3 \\ 2 & \text{if } \ n \geq 4,\ n \not\equiv 2 \ (mod\ 4) \\ 3 & \text{if } \ n \geq 4,\ n \equiv 2 \ (mod\ 4) \end{cases}.$$

Sridharan et al. in [12] obtained the following upper bounds for the total bondage number of graphs.

Theorem 1 *(Sridharan et al.* [12]*) Let G be a connected graph of order $n \geq 4$. Then,*

1. $b_t(G) \leq n - 1$ *if* $g(G) \geq 5$,
2. $b_t(G) \leq n - 2$ *if* $g(G) = 4$,
3. $b_t(G) \leq n - 2$ *if there is a triangle which at least one of its vertices is a support vertex in G,*
4. $b_t(G) \leq n - 1$ *if there is a triangle which at least one of its vertices is of degree two in G.*

Theorem 2 *(Sridharan et al.* [12]*) If T is a tree on n vertices and $T \neq K_{1,n-1}$, then*

$$b_t(T) \leq \min\left\{\Delta(T), \frac{n-1}{3}\right\}.$$

We shall improve Theorems 1 and 2. The following observations are easily verified.

Remark 1 If p non-pendant edges can be removed from a graph G to obtain a graph H without an isolate vertex and with $b_t(H) = t$, then $b_t(G) \leq p + t$.

Remark 2 The total bondage number of a graph G does not change if we add a leaf to a support vertex of G.

Remark 3 Let x be a support vertex of a graph G adjacent to at least two leaves. Then the total bondage number of G does not change if we remove from G a leaf adjacent to x.

The following is a direct consequence of Remark 2.2 of [12].

Proposition 3 *For a graph G, $b_t(G) = \infty$ if and only if each connected component of G is either C_3 or contains only pendant edges.*

Let $\mathcal{B}$ be the class of all graphs G with no isolated vertex such that $b_t(G) \neq \infty$. From now on all graphs G considered in the rest of the paper belong to $\mathcal{B}$.

3 Trees

In this section we study total bondage number of trees $T \in \mathcal{B}$. We improve a previous upper bound for the total bondage number of a tree. Then we present a constructive characterization for a certain class of trees achieving equality for the upper bound. Some of the results presented in this section reference to lemmas presented in Sect. 4. First we have the following upper bound for caterpillars. We recall that a *caterpillar* is a tree with the property that the removal of its leaves results in a path.

Proposition 4 *For a caterpillar* T, $b_t(T) \leq 2$.

Proof Let T be a caterpillar. Since $T \in \mathcal{B}$, $diam(T) \geq 3$. If $diam(T) \in \{3, 4\}$, then obviously $b_t(T) = 1$. So assume that $diam(T) \geq 5$. Let $(x, x_1, x_2, x_3, \ldots, x_{diam(T)})$ be a diametrical path, where x is a leaf. Surely $\gamma_t(T - \{x_1x_2, x_3x_4\}) > \gamma_t(T)$ and the result follows. □

If T is a tree with maximum degree two, then by Proposition 1, $b_t(T) \leq 2$. In the following we give a sharp upper bound for the total bondage number of trees with maximum degree at least three.

Theorem 3 *For any tree* T *with maximum degree at least three,*

$$b_t(T) \leq \Delta(T) - 1.$$

Proof Let T be a tree on n vertices, and let $\Delta(T) \geq 3$. By Remarks 2 and 3, in what follows, we simply consider only trees in which every support vertex is adjacent to exactly one leaf. Since $T \in \mathcal{B}$, $diam(T) \geq 3$. If $diam(T) = 3$, then obviously $b_t(T) = 1$. So assume that $diam(T) \geq 4$. Let $P = (x, x_1, x_2, x_3, \ldots, x_{diam(T)})$ be a diametrical path. Note that x is a leaf. Suppose to the contrary that $b_t(T) \geq \Delta(T)$. Since $b_t(T) \geq 3$, $\gamma_t(T - x_1x_2) = \gamma_t(T)$. This implies that for any mimimum TDS S of $T - x_1x_2$, $x_2 \notin S$. As a consequent $\deg(x_2) = 2$, since P is the longest path of T. By Lemma 7, $\deg(x_3) \geq 3$. If $diam(T) = 4$, then $\gamma_t(T) = 3$, and $\gamma_t(T - x_1x_2) = 4$, which contradicts $b_t(T) \geq \Delta(T)$. So we suppose that $diam(T) \geq 5$. We consider the following cases.

1. x_3 is a support vertex. Let A be the set of all pendant edges incident with x_3. Let T_1 be a graph obtained from $T - x_1x_2$ by removing each edge incident with x_3 with exception of x_2x_3 and the edges in A. Let H_1 be the component of T_1 containing x_3. It is obvious that H_1 is a star. Let S be a $\gamma_t(T_1)$-set, and let $S_1 = S \cap V(H_1)$. It follows that $(S - (S_1 \cup \{x\})) \cup \{x_2, x_3\}$ is a TDS for T of cardinality less than $\gamma_t(T)$, a contradiction.
2. x_3 is not a support vertex. First suppose that there is a support vertex $y_1 \neq x_4$ where y_1 is adjacent to x_3. Let y_2 be the leaf adjacent to y_1. If y_1 is adjacent to a support vertex y_3 and y_4 is a leaf adjacent to y_3, then the path $P' = (y_4, y_3, y_1, x_3, x_4, ..., x_{diam(T)})$ is a diametrical path. Then similar to what we observed for $\deg(x_2)$, we obtain $\deg(y_1) = 2$, a contradiction. So any vertex of $N(y_1) - \{x_3\}$ is a leaf. Let S_2 be a $\gamma_t(T - \{x_3y_1, x_1x_2\})$-set. Since x_2 is a leaf in

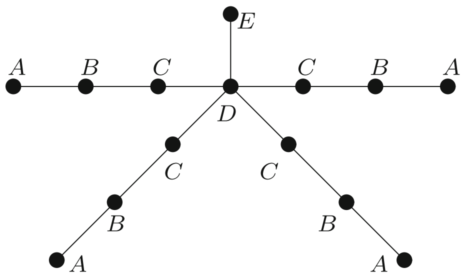

Fig. 1 Tree R_5

$T - \{x_3y_1, x_1x_2\}$, we have $x_3 \in S_2$. However, $\gamma_t(T - \{x_3y_1, x_1x_2\}) = \gamma_t(T)$. Let T_2 be the component of $T - \{x_3y_1, x_1x_2\}$ that contains y_1. Let $v \neq y_1$ be a vertex such that $v \in V(T_2) \cap S_2$. Then $(S_2 - \{x, v\}) \cup \{x_2\}$ is a TDS for T of cardinality less than $\gamma_t(T)$. This contradiction implies that no vertex in $N(x_3) - \{x_4\}$ is a support vertex. Let T_3 be the component of $T - x_3x_4$ containing x_3, and let T_4 be obtained from T_3 by removing all leaves of T_3. So any leaf of T_4 is at distance two from x_3. Furthermore, with a similar discussion as in the proof for $\deg(x_2) = 2$, we observe that T_4 is a tree obtained from $K_{1,\deg(x_3)-1}$ by subdividing any edge. It is obvious that $\gamma_t(T - \{x_1x_2, x_3x_4\}) = \gamma_t(T)$. Let S_3 be a $\gamma_t(T - \{x_1x_2, x_3x_4\})$-set containing as small number of leaves as possible. Then $x_3 \in S_3$, since x_2 is a leaf in $T - \{x_1x_2, x_3x_4\}$. Further, $V(T_4) \subseteq S_3$. Now $(S_3 - \{x, x_3\}) \cup \{x_2\}$ is a TDS for T of cardinality less than $\gamma_t(T)$, a contradiction. □

We note that Theorem 3 improves Theorem 2 in the case when $\Delta(T) < \frac{n(T)+2}{3}$.

3.1 Characterization of Extremal Trees

In this subsection we obtain a constructive characterization for a certain class of trees achieving equality for the upper bound of Theorem 3. We will characterize all trees T_k, where each edge is incident with a support vertex, with $k = \Delta(T) \geq 4$ and having $b_t(T) = \Delta - 1$. By Remarks 2 and 3, in what follows, we simply consider only trees for which every support vertex is adjacent to exactly one leaf. Denote by $\mathcal{T}_k$ the set of all trees T_k in which every support vertex is adjacent to exactly one leaf, each edge is incident with a support vertex and having $b_t(T) = k - 1$, where $k = \Delta(T) \geq 4$.

In [12] a tree H_k with $\Delta(H_k) = k + 1$ and $b_t(H_k) = k$ was introduced as follows. Let x be the central vertex of $K_{1,k-1}$ for some $k \geq 4$ and let H_k be a tree obtained from $K_{1,k-1}$ by subdividing each edge twice and adding a new vertex y and joining y to x. (Note that y is a leaf in H_k and x is a support vertex adjacent to y). In this paper we label H_k with vertex labels $\{A, B, C, D, E\}$ to get a labeled tree R_k as follows. Let $l(y) = E$, let each leaf except of y has label A, let each support vertex of degree 2 has label B, let the vertex of degree $k - 1$ has label D and let every other vertex has label C. It is straightforward to see that $\gamma_t(R_k) = 2k - 1$, $k = \Delta(R_k)$ and $b_t(R_k) = k - 1 = \Delta(R_k) - 1$. Moreover each edge of R_k is incident with a support vertex. See R_5 in Fig. 1.

We describe a procedure to build a family $\mathcal{R}_k$ of trees T_k with vertex labels belonging to the set $\{A, B, C, D, E\}$ and with $k = \Delta(T_k) \geq 4$ as follows. Let $T_k \in \mathcal{R}_k$ and let $x \in V(T_k)$. We call x an *active vertex* if $l(x) = D$ and there is a path P_3 attached to x. Let $\mathcal{R}_k$ be such that:

1. Contains R_k for $k \geq 4$, described as above;
2. Is closed under the following three operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$:
   - **Operation** $\mathcal{O}_1$. Assume T_k^1 and T_k^2 are two trees belonging to $\mathcal{R}_k$. Let x^1 and x^2 be active vertices belonging to T_k^1 and T_k^2, respectively and let P_3^1 and P_3^2 be the paths on three vertices attached to x^1 and x^2, respectively. Then remove P_3^1 and P_3^2 and add the edge x^1x^2 to obtain a new tree with maximum degree k.
   - **Operation** $\mathcal{O}_2$. Assume T_k^1 and T_k^2 are two trees belonging to $\mathcal{R}_k$ and let x_1^1, x_1^2 be vertices with label C belonging to T_k^1, T_k^2, respectively, such that $d_{T_k^1}(x_1^1) + d_{T_k^2}(x_1^2) - 1 \leq k$ and $d_{T_k^1}(x_2^1) + d_{T_k^2}(x_2^2) - 2 \leq k$, where $l(x_2^1) = l(x_2^2) = B$ and x_2^1 is adjacent to x_1^1 and x_2^2 is adjacent to x_1^2. Denote by x_3^1 and x_3^2 the leaves adjacent to x_2^1 and x_2^2, respectively. Identify vertices x_j^1 and x_j^2 into one vertex x_j for each $j \in \{1, 2, 3\}$ to obtain a new tree with maximum degree k. Let $l(x_1) = C, l(x_2) = B$ and $l(x_3) = A$.
   - **Operation** $\mathcal{O}_3$. Assume T_k^1 and T_k^2 are two trees belonging to $\mathcal{R}_k$ and let x_1^1, x_1^2 be vertices with label B belonging to T_k^1, T_k^2, respectively, such that $d_{T_k^1}(x_1^1) + d_{T_k^2}(x_1^2) - 1 \leq k$. Denote by x_2^1 and x_2^2 the leaves adjacent to x_1^1 and x_1^2, respectively. Identify vertices x_j^1 and x_j^2 into one vertex x_j for each $j \in \{1, 2\}$ to obtain a new tree with maximum degree k. Let $l(x_1) = B, l(x_2) = A$.

We first prove that $b_t(T_k) = \Delta(T_k) - 1$ for each tree T_k belonging to $\mathcal{R}_k$. To this aim we first make some observations, which follow immediately from the way in which each tree in the family $\mathcal{R}_k$ is constructed.

Remark 4 If $T_k \in \mathcal{R}_k$ and $v \in V(T_k)$, then

1. $l(v) \in \{A, E\}$ if and only if v is a leaf;
2. $l(v) \in \{B, D\}$ if and only if v is a support vertex;
3. If $l(v) = B$, then exactly one neighbour of v has label A and every other neighbour of v has label C;
4. If $l(v) = C$, then each neighbour of v is a support vertex. Moreover, exactly one neighbour of v has label B and every other neighbour of v has label D;
5. If $l(v) = D$, then exactly one neighbour of v has label E and every other neighbour of v has label C or D. Moreover, v has k neighbours altogether;
6. Each edge of T_k is incident with a support vertex, e.g. a vertex with label B or D;
7. If $l(v) \in \{B, D\}$, then v belongs to every minimum TDS of T_k;
8. If $l(v) \in \{A, C\}$, then v belongs to some minimum TDS of T_k and each neighbour of v belongs to every minimum TDS of T_k;
9. If $l(v) = E$, then v belongs to no minimum TDS of T_k;
10. $\gamma_t(T_k) = 2|\{u \in V(T_k) : l(u) = B\}| + |\{u \in V(T_k) : l(u) = D\}|$.

Lemma 1 *For $k \geq 4$, if a tree T_k belongs to the family $\mathcal{R}_k$, then T_k without labels on vertices belongs to $\mathcal{T}_k$.*

Proof Let T_k be a tree belonging to the family $\mathcal{R}_k$. Clearly $\Delta(T_k) = k$ and each edge of T_k is incident with a support vertex, so it suffices to justify that $b_t(T_k) = k - 1$. Suppose $b_t(T_k) < k - 1$. Let $F \subseteq E(T_k)$ be such that $\gamma_t(T_k) < \gamma_t(T_k - F)$ and $|F| = b_t(T_k) \leq k - 2$. By Remark 4, F contains only edges vw of three types:

- $l(v) = l(w) = D$;
- $l(v) = B$ and $l(w) = C$;
- $l(v) = C$ and $l(w) = D$.

For any $vw \in F$ since $|F| = b_t(T_k) \leq k - 2$, we conclude that $\gamma_t(T_k - (F - \{vw\})) = \gamma_t(T_k)$ and $\gamma_t(T_k - F) > \gamma_t(T_k - (F - \{vw\}))$. We consider three cases for labels of v and w.

Case 1. Assume first that $l(v) = l(w) = D$. Remark 4 implies that both v and w have in $T_k - (F - \{vw\})$ at least two neighbours with labels in $\{C, D\}$ such that if a vertex has label C, then is adjacent to a support vertex with label B and if a vertex has label D, then it is adjacent to a vertex with label C or D, which is not a leaf. Thus, $\gamma_t(T_k - F) = \gamma_t(T_k - (F - \{vw\}))$, which is impossible.

Case 2. Assume now that $l(v) = B$ and $l(w) = C$. Since $|F| = b_t(T_k) \leq k-2$, we conclude that $\gamma_t(T_k - (F - \{vw\})) = \gamma_t(T_k)$ and $\gamma_t(T_k - F) > \gamma_t(T_k - (F - \{vw\}))$. Since vw is a non–pendant edge, Remark 4 implies that w is adjacent in $T_k - (F - \{vw\})$ to at least one vertex with label D, say y, such that y has at least one neighbour in $T_k - (F - \{vw\})$, except of w, with label in $\{C, D\}$ such that if it has label C, then it is adjacent to a support vertex with label B and if a vertex has label D, then it is adjacent to a vertex with label C or D, which is not a leaf. Thus, $\gamma_t(T_k - F) = \gamma_t(T_k - (F - \{vw\}))$, which is impossible.

Case 3. Lastly, assume that $l(v) = C$ and $l(w) = D$. Since $|F| = b_t(T_k) \leq k - 2$, we conclude that $\gamma_t(T_k - (F - \{vw\})) = \gamma_t(T_k)$ and $\gamma_t(T_k - F) > \gamma_t(T_k - (F - \{vw\}))$. Since vw is a non–pendant edge, Remark 4 implies that v is adjacent in $T_k - (F - \{vw\})$ to at least one vertex with label in $\{B, D\}$, say $y \neq w$, such that y has at least one neighbour in $T_k - (F - \{vw\})$. Moreover, w has at least one neighbour, different from v, with label in $\{C, D\}$ such that if a vertex has label C, then it is adjacent to a support vertex with label B and if a vertex has label D, then is adjacent to a vertex with label C or D, which is not a leaf. Thus, $\gamma_t(T_k - F) = \gamma_t(T_k - (F - \{vw\}))$, which is impossible.

We conclude that $b_t(T_k) \geq k - 1$ and Theorem 3 implies the desired result. □

Lemma 2 *If every support vertex of a tree T is adjacent to exactly one leaf, $\Delta(T) \geq 4$ and $b_t(T) = \Delta(T) - 1$, then*

(a) $\deg_T(x) = \deg_T(y) = \Delta(T) - 1$ *for each adjacent support vertices x, y;*
(b) $d_T(x, y) \geq 3$ *for each pair of support vertices x, y of degree 2.*

Proof (a) It is an immediate consequence of Lemma 4.

(b) Let x, y be two support vertices of degree 2 and denote by x', y' the two leaves adjacent to x and y, respectively. If x and y are adjacent, then T is a path on 4 vertices, which is a contradiction. Thus suppose $d(x, y) = 2$ and let z be the vertex adjacent to both x and y. Since T is not a path, $\deg_T(z) \geq 3$. Let S be a smallest TDS of $T - xz - yz$. Then x, x', y, y' belong to S, but on the other hand, $(S - \{x', y'\}) \cup \{z\}$ is a smaller TDS of T. □

Lemma 3 *If a tree T_k belongs to the family $\mathcal{T}_k$ for some $k \geq 4$, then it is possible to label vertices of T_k with labels $\{A, B, C, D, E\}$ in such a way T_k belongs to the family $\mathcal{R}_k$.*

Proof Let T_k be a tree belonging to the family $\mathcal{T}_k$. Then, by the definition of $\mathcal{T}_k$, every support vertex of T_k is adjacent to exactly one leaf, each edge of T_k is incident with a support vertex and $b_t(T_k) = \Delta(T_k) - 1 = k - 1$, where $k = \Delta(T_k) \geq 4$. Since $b_t(T_k) = \Delta(T_k) - 1 \geq 3$, clearly $diam(T_k) \geq 4$. Let $P = (x, x_1, x_2, x_3, \ldots, x_{diam(T_k)})$ be a diametrical path. Since $\gamma_t(T_k - x_1x_2) = \gamma_t(T_k)$, we conclude that for any minimum TDS S of $T_k - x_1x_2$, $x_2 \notin S$. Since P is a longest path of T_k it follows that $\deg(x_2) = 2$. By Lemma 7, $\deg(x_3) \geq 3$. If $diam(T_k) = 4$, then $\gamma_t(T_k) = 3$ and $\gamma_t(T_k - x_1x_2) = 4$, which contradicts $b_t(T_k) = \Delta(T_k) - 1$. So we assume that $diam(T_k) \geq 5$. Since each edge of T_k is incident with a support vertex, we conclude that x_3 is a support vertex. Then by Lemma 5,

$$\begin{aligned}\Delta(T_k) - 1 = k - 1 = b_t(G) &\leq \deg(x_1) + \deg(x_2) \\ &\quad + \deg(x_3) - 5 = 2 + 2 + \deg(x_3) - 5.\end{aligned}$$

Hence $\deg(x_3) = \Delta(T_k)$. Moreover, if there is a leaf at distance 3 from x_3, say (z, z_1, z_2, x_3) is the path from the leaf $z \neq x$ to x_3, where $z_2 \neq x_2$, then similar to what we observed for x_1 and x_2, we obtain $\deg_{T_k}(z_1) = \deg_{T_k}(z_2) = 2$.

Suppose x_3 is adjacent to a support vertex, say y', and let y'' be the leaf adjacent to y'. Then $\deg_{T_k}(y') = 2$. Let S be a minimum TDS of $T_k - x_3y'$ containing as small number of leaves as possible. Then $S - \{y''\}$ is a TDS of T_k, implying $b_t(T_k) = 1$, which is impossible. For the same reasons, $diam(T_k) \geq 6$. Hence, the component of $T_k - x_3x_4$ containing x_3 is the tree R_{k-1} and if $diam(T_k) = 6$, then $T_k = R_k$ and thus $T_k \in \mathcal{R}_k$. Therefore in what follows we assume $diam(T_k) \geq 7$. We follow by induction on the number $s(T_k)$ of vertices of degree $\Delta(T_k)$ in T_k. If $s(T_k) = 1$, then obviously $T_k = R_k$. Thus assume that for every tree T'_k with $s(T'_k) < s(T_k)$, if $T'_k \in \mathcal{T}_k$, then it is possible to label the vertices of T'_k in such a way $T'_k \in \mathcal{R}_k$.

1. If x_4 is a support vertex, then by our assumptions, x_4 is adjacent to exactly one leaf and by Lemma 4 we obtain that x_4 is of degree k. Denote by T^{x_3} and T^{x_4} the two components of $T_k - x_3x_4$ containing x_3 and x_4, respectively. Attach P_3 to x_3 to obtain a tree $T_k^{x_3}$ and attach $P_3 : q_1 - q_2 - q_3$ to x_4 to obtain a tree $T_k^{x_4}$. Clearly $T_k^{x_3} = R_k$ and $\Delta(T_k^{x_4}) = k$. Moreover, $b_t(T_k^{x_3}) = k - 1$. Suppose $b_t(T_k^{x_4}) < k - 1$. In this situation let $F \subseteq E(T_k^{x_4})$ be such that $\gamma_t(T_k^{x_4}) < \gamma_t(T_k^{x_4} - F)$ and $|F| = b_t(T_k^{x_4}) \leq k - 2$.
   If $x_4q_3 \in F$, then for $F' = (F - \{x_4q_3\}) \cup \{x_3x_4\}$ we would obtain $\gamma_t(T_k) < \gamma_t(T_k - F')$ contradicting that $b_t(T_k) = k - 1$. Thus $x_4q_3 \notin F$.
   If $q_2q_3 \in F$ and $x_4q_3 \notin F$, then since x_4 is a neighbour of at least two leaves in $T_k^{x_4} - F$, Remark 3 implies that $b_t(T_k^{x_4} - F) = b_t((T_k^{x_4} - F) - q_3)$. Hence for $F'' = (F - \{q_2q_3\}) \cup \{x_4q_3\}$ we would have $\gamma_t(T_k^{x_4}) < \gamma_t(T_k^{x_4} - F'')$ and again we would obtain a contradiction.
   If $q_2q_3 \notin F$ and $x_4q_3 \notin F$, then clearly $\gamma_t(T_k) < \gamma_t(T_k - F)$, which is impossible. Therefore, $b_t(T_k^{x_4}) = k - 1$ and by induction hypothesis, $T_k^{x_4}$ is in $\mathcal{R}_k$. Since x_4 is

an active vertex in $T_k^{x_4}$ and x_3 is an active vertex in $T_k^{x_3}$, we conclude that T_k may be obtained from $T_k^{x_3}$ and $T_k^{x_4}$ by Operation $\mathcal{O}_1$.

2. If x_4 is not a support vertex, then each neighbour of x_4 is a support vertex since each edge is incident with a support vertex. Thus, x_5 is a support vertex and let x_5' be the leaf adjacent to x_5.

   (a) Assume additionally x_4 is a neighbour of a support vertex of degree 2, say y, and let y' be the leaf adjacent to y. Let $\deg_{T_k}(x_4) \geq 4$ and denote by z a neighbour of x_4 different from x_3, x_5, y. Since x_4z is incident with a support vertex, we obtain that z is a support vertex. Denote by z' the leaf adjacent to z. If $\deg_{T_k}(z) = 2$, then by Lemma 2, $b_t(T_k) < 3$. Hence x_4 is a neighbour of exactly one support vertex of degree 2.
   Thus assume $\deg_{T_k}(z) \geq 3$. Then by Lemma 2, each neighbour of z is not a support vertex and z belongs to a longest path of T_k. Moreover, denote by S a minimum total dominating set of $T - (A \cup \{x_4y\})$, where A is a set of all non-pendant edges incident with z except for zx_4. Without loss of generality we may assume that z, x_4, y, y' belong to S. Further, $S - \{y'\}$ is a smaller TDS of T_k, so $b_t(T_k) \leq |A|+1$. Since $|A| = \deg_{T_k}(z)-2$ and $b_t(T_k) = \Delta(T_k)-1$, we conclude that $\deg_{T_k}(z) = \Delta(T_k)$. By similar reasoning, $\deg_{T_k}(x_5) = \Delta(T_k)$. Denote by T^z and $T_k^{x_4}$ the two components of $T_k - zx_4$ containing z and x_4, respectively. Attach P_3 to z to obtain a tree T_k^z. Clearly $T_k^z = R_k$ and $\Delta(T_k^{x_4}) = k$. Moreover, $b_t(T_k^z) = k-1$.
   Suppose $b_t(T_k^{x_4}) < k-1$. In this situation let $F \subseteq E(T_k^{x_4})$ be such that $\gamma_t(T_k^{x_4}) < \gamma_t(T_k^{x_4} - F)$ and $|F| = b_t(T_k^{x_4}) \leq k-2$. Then $\gamma_t(T_k - (F \cup \{x_4z\})) > \gamma_t(T_k - F)$, so $|F| = k-2$. Let S be a minimum TDS of $T_k - F$. Since x_4 is not isolated in $T_k^{x_4} - F$, x_4 has at least two neighbours in $T_k - F$. Further, each such neighbour is a support vertex, so belongs to S. If $x_4y \in F$, then $x_4 \notin S$ and S is also a total dominating set of $T_k - (F \cup \{x_4z\})$. If $x_4y \notin F$, then again S is a TDS of $T_k - (F \cup \{x_4z\})$. Thus $\gamma_t(T_k) = \gamma_t(T_k - F) = \gamma_t(T_k - (F \cup \{x_4z\}))$, which is impossible. Thus $b_t(T_k^{x_4}) = k-1$ and by induction hypothesis, $T_k^{x_4}$ is in $\mathcal{R}_k$. Let $l(y') = A, l(y) = B$ and $l(x_4) = C$. Hence T_k may be obtained from R_k and $T_k^{x_4}$ by Operation $\mathcal{O}_2$.
   If $\deg_{T_k}(x_4) = 3$, then we remove x_3x_4 to obtain trees T^{x_3} and $T_k^{x_4}$ and the rest of the proof is similar to the case when $\deg_{T_k}(x_4) \geq 4$.

   (b) If x_4 is not a neighbour of a support vertex of degree 2 and $\deg_{T_k}(x_4) \geq 3$. Then x_4 is a neighbour of a support vertex z, where $z \notin \{x_3, x_5\}$ and denote by z' the leaf adjacent to z. By similar arguing as above, we conclude that $\deg_{T_k}(z) = \Delta(T_k)$ and each neighbour of z is not a support vertex and z belongs to a longest path of T_k. Denote by T^z and $T_k^{x_4}$ the two components of $T_k - x_4z$ containing z and x_4, respectively. Attach P_3 to z to obtain a tree T_k^z. Clearly $T_k^z = R_k$ and $\Delta(T_k^{x_4}) = k$. Moreover, $b_t(T_k^z) = k-1$. By similar arguments as in Case 2a we conclude that $b_t(T_k^{x_4}) = k-1$ and by induction hypothesis, $T_k^{x_4}$ is in $\mathcal{R}_k$. Let $l(x_5') = A, l(x_5) = B$ and $l(x_4) = C$. Hence T_k may be obtained from R_k and $T_k^{x_4}$ by Operation $\mathcal{O}_2$.

   (c) If x_4 is of degree 2, then denote by T^{x_4} and $T_k^{x_5}$ the two components of $T_k - x_4x_5$ containing x_4 and x_5, respectively. Attach P_2 to x_4 to obtain a tree $T_k^{x_4}$. Clearly

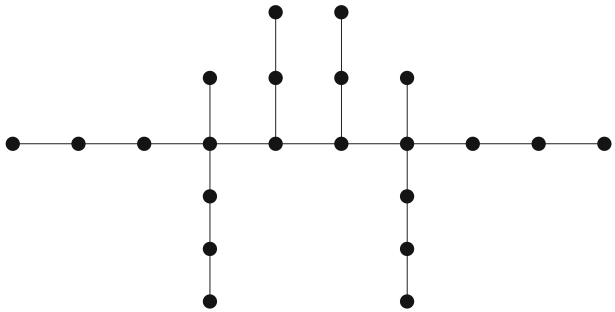

Fig. 2 Tree T

$T_k^{x_4} = R_k$ and $b_t(T_k^{x_4}) = k - 1$. If $\Delta(T_k^{x_5}) < k$, then only x_3 and x_5 are vertices of degree k in T_k and x_5 has similar properties as x_3 and $diam(T_k) = 8$. However then $b_t(T_k) < k - 1$, a contradiction. Thus $\Delta(T_k^{x_5}) = k$. By similar arguments as in Case 2a we conclude that $b_t(T_k^{x_5}) = k - 1$ and by induction hypothesis, $T_k^{x_5}$ is in $\mathcal{R}_k$. Let $l(x_5') = A$, $l(x_5) = B$ and $l(x_4) = C$. Hence T_k may be obtained from R_k and $T_k^{x_4}$ by Operation $\mathcal{O}_3$. □

As an immediate consequence of Lemmas 1 and 3, we have the following

Theorem 4 *For each $k \geq 4$, if we omit vertex labelling of trees in $\mathcal{R}_k$,*

$$\mathcal{T}_k = \mathcal{R}_k.$$

In Fig. 2 is a tree T with $b_t(T) = \Delta(T) - 1 = 3$ and containing an edge not incident with a support vertex.

We finish this section with the following open problem.

conjecture 1 T is a tree with $\Delta(T) \geq 5$ and $b_t(T) = \Delta(T) - 1$ if and only if T belongs to the family $\mathcal{R}_k$ for $k \geq 5$.

4 Upper Bounds

In this section we obtain some upper bounds for the total bondage number of a graph $G \in \mathcal{B}$ in terms of maximum and minimum degrees.

Lemma 4 *If x and y are two adjacent support vertices in a graph G, then*

$$b_t(G) \leq min\{\deg(x), \deg(y)\} - 1.$$

Proof Remove all edges incident with x with exception of pendant edges to obtain a graph H. Suppose that $\gamma_t(H) = \gamma_t(G)$. Let S be a $\gamma_t(H)$-set. Clearly, $y \in S$. Let H_1 be the component of H containing x and let $x_1 \neq x$ be a vertex of H_1 with $x_1 \in S$. Then $S - \{x_1\}$ is a TDS for G of cardinality less than $\gamma_t(G)$, a contradiction. Therefore, $\gamma_t(H) > \gamma_t(G)$ and the result follows. □

Lemma 5 *Let a graph G contain a path (x, y, z) on three vertices such that $\{x, z\} \subseteq S(G)$ and $y \notin S(G)$, then*

$$b_t(G) \leq \deg(x) + \deg(y) + \deg(z) - 5.$$

Proof Remove all edges incident with x, y, z with exception of xy and the pendant edges incident with x or z, to obtain a graph H. It is obvious that $\gamma_t(H) > \gamma_t(H+yz) \geq \gamma_t(G)$ and so the result follows. In case of $G = P_5$ the bound is sharp. □

Similarly we have the following.

Lemma 6 *If a graph G contains a path (x, y, z) on three vertices such that $\{x, y\} \cap S(G) = \emptyset$ and $z \in S(G)$, then*

$$b_t(G) \leq \deg(x) + \deg(y) + \deg(z) - 4.$$

Note that under assumptions of Lemma 5, $b_t(G) \leq 3\Delta(G) - 5$ and under assumptions of Lemma 6, $b_t(G) \leq 3\Delta(G) - 4$.

Theorem 5 *Assume that a graph G contains a path (v_1, v_2, v_3, v_4) such that $G - \{v_1, v_2, v_3, v_4\}$ has no isolated vertex. If G_1 is the subgraph induced by v_1, v_2, v_3, v_4, then*

$$b_t(G) \leq \sum_{i=1}^{4} \deg(v_i) - |E(G_1)| - 2.$$

This bound is sharp.

Proof Let G_1 be the subgraph induced by v_1, v_2, v_3, v_4. We remove all edges incident with v_1, v_2, v_3, v_4 except the edges v_1v_2 and v_3v_4 to obtain a graph H. If $\gamma_t(H) = \gamma_t(G)$, then we let S be a $\gamma_t(H)$-set. It follows that $S - \{v_1, v_4\}$ is a TDS for G of cardinality less than $\gamma_t(G)$, a contradiction. So $\gamma_t(H) > \gamma_t(G)$ and thus $b_t(G) \leq \sum_{i=1}^{4} \deg(v_i) - |E(G_1)| - 2$. The sharpness follows by Proposition 2. □

Since the graph G_1 constructed in previous theorem has at least 3 edges, $b_t(G) \leq \sum_{i=1}^{4} \deg(v_i) - 5$. Also if $\deg(v_i) = \delta(G)$ for some $1 \leq i \leq 4$, then $b_t(G) \leq 3\Delta(G) + \delta(G) - 5$.

The following upper bound is also useful.

Lemma 7 *Let (x, y, z, w) be a path in a graph G and let* $\deg(w) \leq \deg(x) = \deg(y) = \deg(z) = 2$. *Then*

$$b_t(G) \leq \deg(w) + 1.$$

Proof By Propositions 1 and 2, we assume that $G \notin \{P_4, P_5, P_6, C_4, C_5\}$. Let $a \neq y$ be a vertex adjacent to x.

Assume first that $\deg(w) = 1$. Let S_1 be a $\gamma_t(G - ax - yz)$-set. If $|S_1| = \gamma_t(G)$, then $S_1 - \{w, x\}$ is a TDS for G of cardinality smaller than $\gamma_t(G)$, a contradiction. This implies that $|S_1| > \gamma_t(G)$ and hence $b_t(G) \leq 2 = \deg(w) + 1$.

Assume next that $\deg(w) = 2$. Let $b \neq z$ be a vertex adjacent to w. If $\deg(a) = 1$ or $\deg(b) = 1$, then the situation is similar to the case when $\deg(w) = 1$. Thus we may assume that $\deg(a) \geq 2$ and $\deg(b) \geq 2$. Let S_2 be a $\gamma_t(G - ax - yz - wb)$-set. If $|S_2| = \gamma_t(G)$, then $S_2 - \{x, w\}$ is a TDS for G of cardinality smaller than $\gamma_t(G)$. This implies that $|S_2| > \gamma_t(G)$ and so $b_t(G) \leq 3 = \deg(w) + 1$. □

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