SOURCE-RELATED WAVEFIELDS IN FLUIDS AND DIELECTRICS: A NEW WAY OF THINKING ABOUT MEDIUM DYNAMICS

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Acoustic and electromagnetic wave phenomena may seem to have a proper formal representation in field theory dating from the 19th century, founded on the mathematics of complex functions. This paper shows, however, that when replacing the classical spectrum-domain approach related to the assumption of harmonic timeform of signals, with a time-domain approach imposing no restriction as to the class of time evolution of source and field signals, it is possible to perform such a description of physical effects and their local, causality-driven mechanisms, that results in new revelations as to the role of both the physical medium and the wave source in initiating and conveying phenomena, otherwise perfectly familiar, that carry energy and preserve signal waveform. It appears that causality and locality are a core of dynamic linear phenomena in physical continua, leading to the creation of doubly-dynamic states that are observed, at the approach, in "frozen" conditions.

INTRODUCTION

In the everyday practice of mathematical and applied physics, the notion of acoustic or electromagnetic field is taken for a primary one, with not much need of a strict definition. This paper presents an original approach to the field phenomenology, that reveals a fundamental role played by the local dynamic properties of fluid or dielectric medium in conveying wavefields across the space. Two-fold dynamics of a medium and the initiating impact of a given source are revealed as the leading factors of any wavefield phenomenon.

Here we show a procedure and the results of an individual particle-based synthesis of formulae, relations and equations, as well as an analysis of elementary solutions concerning, first, local mechanisms of dynamic, linear states and, second, global wave-like effects related to quasi-point sources of disturbance. Although nearly all elements of the approach are commonly known and so are in current use in theoretical and applied acoustics and electrodynamics, the approach is, to the best of the author's knowledge, totally new as to the sequence of presentation (introduction) of subsequent aspects (elements). Such a description entails a semantically unambiguous distinction between local causes and global effects. Fields appear to be related to a global view of a given dynamic state chosen as a leader of two source-related, localised, doubly-dynamic states of physical continuum carrying two kinds of energy spread in space out of the source.

1. METHODOLOGICAL FOUNDATIONS

The approach consists of the description of linear phenomena in two species of physical media, fluids and dielectrics, based on locality and causality of all dynamic actions. Acoustic and electromagnetic effects are both analysed at phenomenological level meaning things of nature as they are revealed to human intuition, susceptible to be measured where they are (locally) with "simple" means. The synthesis of fundamental relations is performed with causation principle as a rule, meaning the explicit description of the right sequences of influencing impacts and resulting events. Mere local, instantaneous medium states are analysed, without any anticipation of final results. The teleological view based on an *a priori* knowledge of final results is thus avoided, and the usual forecast of the wave-like character of field phenomena is not being applied.

It appears particularly fruitful to lay the foundation for analysis with the definition of two pairs of physical variables in each of two media, as is usually done in electromagnetics. The causation-oriented relations describe local, instantaneous effects in the two physical continua, with a noteworthy symmetry and compactness of form.

2. FLUIDS AND DIELECTRICS AS PHYSICAL CONTINUA

Fluids and dielectrics are modelled as physical continua, meaning Euclidean space endowed with specific qualities either mechanical or electromagnetic. The continuum composed of contiguous, shapeless "particles" or, otherwise, of boundless elementary areas, is the traditional model introduced by Euler for studying fluid mechanics [1] and considered by Maxwell in his search for an explanation of electric and magnetic actions at a distance [2]. Mathematically, a particle is represented by an elemental area with infinitesimal volume.

Particle qualities. Medium is assumed to be principally at rest. Every fluid particle is both inert and deformable, the respective qualities being characterized by mass density $\varrho(\mathbf{x})$ - in [kg/m^3], and elastic compliance $\varkappa(\mathbf{x})$ - in [1/Pa], as local parameters of the medium. Similarly, each dielectric particle can be polarised both electrically and magnetically, where electric permittivity $\epsilon(\mathbf{x})$ - in [F/m], and magnetic permeabililty $\mu(\mathbf{x})$ - in [H/m] play, respectively, roles of two independent medium parameters related to a given space localisation.

Two kinds of dynamic states are possible at a given point-like area when it is knocked out from static rest. We will attribute a pair of physical quantities to a given state, one of them playing the role of field "intensity" susceptible to a direct measurement, and the other one that of medium-related field "density". In this way we extend, onto fluid acoustics, the approach adopted in electromagnetics, with Maxwell's displacement current and magnetic induction as "material" aspects of electric and magnetic phenomena [3].

In Eulerian fluids, inert motion and elastic deformation will thus be described by velocity $\mathbf{v}(\mathbf{x},t)$, in [m/s], and pressure $p(\mathbf{x},t)$, in [Pa], as dynamic state intensities, on one side, and by, otherwise well-known, momentum density $\mathbf{g}(\mathbf{x},t) = \varrho(\mathbf{x})\ \mathbf{v}(\mathbf{x},t)$ in [kg · m· s^{-1}/m^3] together with dimensionless relative volume strain $s(\mathbf{x},t) = \varkappa(\mathbf{x})\, p(\mathbf{x},t)$, in [m^3/m^3], on the other side. Intuitively obvious, the notion of volume strain appears to be fundamental, as it plays the role of a quantity of elasticity, dual to that of a quantity of motion, played traditionally by momentum density.

In Maxwellian dielectrics, electrical and magnetic polarisations are described by electric field intensity $\mathbf{E}(\mathbf{x},t)$, in [V/m], and magnetic field intensity $\mathbf{H}(\mathbf{x},t)$, in [A/m], as well as by electric displacement $\mathbf{D}(\mathbf{x},t) = \epsilon(\mathbf{x})\ \mathbf{E}(\mathbf{x},t)$ in [C m /m^3] and magnetic induction $\mathbf{B}(\mathbf{x},t) = \mu(\mathbf{x})\ \mathbf{H}(\mathbf{x},t)$, in [Wb m /m^3], the latter being space densities of respective polarisation phenomena.

3. CROSS-DEPENDENCE OF "DENSITIES" ON "INTENSITIES"

The intensities and densities defined in the previous section are mutually related in a criss-cross manner, by the well-known first-order homogeneous mathematical relations of Euler and Maxwell. It is crucial for the subsequent analysis to discern their inherent causality meaning that any inhomogeneity of the space distribution of an intensity induces an instantaneous time variation of the "opposed", dual density.

Euler's relations. In fluid, any pressure gradient at an area causes a time variation of its momentum density, measured as a change of velocity:

$$\frac{\partial}{\partial t}\mathbf{g}(\mathbf{x},t) = -\ \mathbf{grad}\ p(\mathbf{x},t), \tag{1}$$

motion variation as the effect $\Leftarrow$ pressure inhomogeneity as a cause.

Similarly, the velocity divergence at the area causes a time variation of its volume strain and pressure:

$$\frac{\partial}{\partial t}s(\mathbf{x},t) = -\ \mathrm{div}\ \mathbf{v}(\mathbf{x},t), \tag{2}$$

deformation change as the effect $\Leftarrow$ flow inhomogeneity as a cause.

Both the above relations are due to Euler. The first one is Euler's equation of dynamics. The second one comes from his equation of continuity derived from the principle of mass conservation.

Maxwell's relations. In dielectrics, any curl of electric intensity at an area causes a time variation of its magnetic induction:

$$\frac{\partial}{\partial t}\mathbf{B}(\mathbf{x},t) = -\ \mathbf{curl}\ \mathbf{E}(\mathbf{x},t), \tag{3}$$

change of magnetic induction as an effect $\Leftarrow$ curly electric intensity as a cause.

In the same manner, a magnetic intensity curl at an area causes a time variation of its electric induction (i.e. displacement current):

$$\frac{\partial}{\partial t}\mathbf{D}(\mathbf{x},t) = +\ \mathbf{curl}\ \mathbf{H}(\mathbf{x},t), \tag{4}$$

change of electric induction as an effect $\Leftarrow$ curly magnetic intensity as a cause.

The change of the sign in the latter relation, compared to the first three, is worth noting, distinguishing the Maxwell displacement currents in some way.

Obvious in fluid mechanics, the above interpretation seems to be new in electromagnetics, where it is suggested that it is a time change of one field effect that creates curly space distribution of the dual effect. It appears, however, that the causality rule is to be modified. In fact, the cause and the effect are inverted compared to the classical big-scale causality interpretation, and these are space effects of given intensity that induce time variation of complementary density.

4. CONSERVATION OF POINT-ATTRIBUTED ENERGY

Instantaneous energy of local continuum area. Both in fluids and dielectric, it is possible to define the instantaneous space density of energy $w(\mathbf{x},t)$ in [J / m^3] localised at a point, namely that of inert motion and elastic strain of a fluid "particle":

$$w_{\mathrm{ac}}(\mathbf{x},t) = \frac{1}{2}\, s(\mathbf{x},t)\, p(\mathbf{x},t) + \frac{1}{2}\, \mathbf{g}(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x},t) = \frac{1}{2}\left(\varkappa\, p^2 + \varrho\, \mathrm{v}^2\right), \tag{5}$$

and that of electric and magnetic polarisations of local dielectric area (dielectric "particle"):

$$w_{\mathrm{em}}(\mathbf{x},t) = \frac{1}{2}\, \mathbf{D}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t) + \frac{1}{2}\, \mathbf{B}(\mathbf{x},t)\cdot\mathbf{H}(\mathbf{x},t) = \frac{1}{2}\left(\epsilon\, \mathrm{E}^2 + \mu\, \mathrm{H}^2\right). \tag{6}$$

Change of energy density. Now consider a time variation of local energy density. In both the media it leads to a very interesting space-related result. In a fluid it gives:

$$\frac{\partial}{\partial t} w_{\mathrm{ac}} = \varkappa\, p \frac{\partial p}{\partial t} + \varrho\, \mathbf{v}\cdot\frac{\partial \mathbf{v}}{\partial t} = p\frac{\partial s}{\partial t} + \mathbf{v}\cdot\frac{\partial \mathbf{g}}{\partial t}, \tag{7}$$

that, due to equations (1) and (2), is equivalent to:

$$\frac{\partial}{\partial t} w_{\mathrm{ac}}(\mathbf{x},t) = -\, p\, \mathrm{div}\, \mathbf{v} - \mathbf{v}\cdot\mathbf{grad}\, p = -\, \mathrm{div}\Big(p(\mathbf{x},t)\, \mathbf{v}(\mathbf{x},t)\Big). \tag{8}$$

Similarly, in a dielectric it is:

$$\frac{\partial}{\partial t} w_{\mathrm{em}} = \epsilon\, \mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t} + \mu\, \mathbf{H}\cdot\frac{\partial \mathbf{H}}{\partial t} = +\, \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\cdot\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t}, \tag{9}$$

being, due to equations (3) and (4), equivalent to:

$$\frac{\partial}{\partial t} w_{\mathrm{em}} = \mathbf{E}\cdot\mathbf{curl}\,\mathbf{H} - \mathbf{H}\cdot\mathbf{curl}\,\mathbf{E} = -\, \mathrm{div}\Big(\mathbf{E}(\mathbf{x},t)\times\mathbf{H}(\mathbf{x},t)\Big). \tag{10}$$

Local area power flux. The above effect is produced locally within mere direct neighbourhood, without any additional conditions. Any change of local energy produces instantaneously a divergent stream leaving the area, thus fulfilling the requirements of energy conservation in the absence of external sources. The terms in brackets in equations (8) and (10) have a sense of, respectively, acoustic and electromagnetic power flux density $\mathbf{S}(\mathbf{x},t)$ in [W / m^2]:

$$\mathbf{S}_{\mathrm{ac}}(\mathbf{x},t) = p(\mathbf{x},t)\, \mathbf{v}(\mathbf{x},t), \tag{11}$$

and

$$\mathbf{S}_{\mathrm{em}}(\mathbf{x},t) = \mathbf{E}(\mathbf{x},t) \times \mathbf{H}(\mathbf{x},t), \tag{12}$$

where, for the purpose of integrating the view of dynamic phenomena in the two media, the usual symbol of acoustic intensity $\mathbf{I}$, has been replaced by $\mathbf{S}_{\mathrm{ac}}$, analogous to Poynting's vector of electromagnetics $\mathbf{S}$, written here with the appropriate subscript.

Mechanical momentum in fluid and dielectric continuum. It is important to remark here that the above dot and cross products of intensities, can be applied, as well, to corresponding densities, which gives a new quantity with the dimension of momentum density [kg m^{-2} s^{-1}]:

$$\mathbf{g}_{\mathrm{ac}}(\mathbf{x},t) = s(\mathbf{x},t)\,\mathbf{g}(\mathbf{x},t) = \varkappa(\mathbf{x})\varrho(\mathbf{x})\,\mathbf{S}_{\mathrm{ac}}(\mathbf{x},t), \tag{13}$$

and

$$\mathbf{g}_{\mathrm{em}}(\mathbf{x},t) = \mathbf{D}(\mathbf{x},t) \times \mathbf{B}(\mathbf{x},t) = \epsilon(\mathbf{x})\mu(\mathbf{x})\,\mathbf{S}_{\mathrm{em}}(\mathbf{x},t). \tag{14}$$

It appears that although the notion of momentum is related to dynamics of ponderous matter, momentum density concerns both acoustic and electromagnetic wavefields. In fluids, the double-state, wave momentum is a new notion, not present in the description of acoustic phenomena, being something else than the well-known notion of particle momentum. And in dielectrics, it has no mechanics-related interpretation. However, it accompanies intrinsically the energy flow of electromagnetic wavefields.

The momentum in dielectrics, first identified by Maxwell, was next perceived by J. J. Thomson in Poynting's analysis of electromagnetic power stream. However, to the best of the author's knowledge, no such observation has ever been made in acoustics. And it is clear that the momentum density related to acoustic intensity is s times, i.e. many orders of magnitude, smaller than that of particle motion.

5. LOCAL CIRCULAR ACTIONS IN DOUBLY-DYNAMIC CONTINUA

The cross-dependence relations of Section 3 can be put in a cause-effect chain in two ways, depending on what is the primary cause of the particle out-of-rest state. The mathematical procedure concerning fluid acoustics was presented in [4] and [5], giving two second-order relations of equal value, with either the pressure or the velocity as a dynamic state variable. An analogous procedure applied to the cross relations of electromagnetics give, similarly, two second-order relations with either the electric intensity or the magnetic one the role of the variable. A closer look at all the four results thus obtained, reveals some new conclusions of fundamental importance.

First, there can be no isolated dynamic state at an area, contrary to what is suggested by a purely mathematical approach defining a field of one variable. In fact, there is always a two-fold dynamic state being a pair of coupled fields, with one of them being a leader - the one being first induced at the area, and the other one accompanying the leader, as presented below in four cases.

In a fluid, when disturbance starts from a space inhomogeneity of pressure (volume strain), pressure remains the leader at any area and time moment, being described by the second-order wave relation, while the accompanying velocity (momentum density) value is determined by Euler's first-order relation of dynamics. When a nonuniform matter flow velocity (momentum) is the primary cause of disturbance at an area, particle velocity is a leader, being accompanied by pressure determined by Euler's relation of continuity.

A self-affecting compression imbalance. The feedback loop mechanism is shown below in the example of a self-affecting compression imbalance that results in classical scalar second order relation with pressure as physical variable:

$$\text{a)}\ \ \mathbf{v}_p = -\frac{1}{\varrho}\int \mathbf{grad}\, p\, \mathrm{d}t, \qquad \text{b)}\ \ p_{\mathrm{v}p} = -\frac{1}{\varkappa}\int \mathrm{div}\, \mathbf{v}_p\, \mathrm{d}t \tag{15}$$

first effect ⇐ primary cause, final effect ⇐ secondary cause.

$$p_{\mathrm{v}p} = -\frac{1}{\varkappa}\int \mathrm{div}\left(-\frac{1}{\varrho}\int \mathbf{grad}\, p\, \mathrm{d}t\right)\mathrm{d}t = \frac{1}{\varkappa\varrho}\iint \mathrm{div}\, \mathbf{grad}\, p\, \mathrm{d}t^2, \tag{16}$$

$$p_{\mathrm{v}p} \equiv p; \qquad \frac{\partial^2 p}{\partial t^2} - \frac{1}{\varkappa\varrho}\,\mathrm{div}\,\mathbf{grad}\, p = 0; \qquad \mathbf{v}_p = -\frac{1}{\varrho}\int \mathbf{grad}\, p\, \mathrm{d}t. \tag{17}$$

A self-affecting flow imbalance. A self-affecting flow imbalance results in a dual second-order relation involving particle velocity vector as variable. Starting with:

$$\text{a)}\ \ p_{\mathrm{v}} = -\frac{1}{\varkappa}\int \mathrm{div}\, \mathbf{v}\, \mathrm{d}t, \qquad \text{b)}\ \ \mathbf{v}_{p\mathrm{v}} = -\frac{1}{\varrho}\int \mathbf{grad}\, p_{\mathrm{v}}\, \mathrm{d}t, \tag{18}$$

first effect ⇐ primary cause, final effect ⇐ secondary cause.

we get:

$$\mathbf{v}_{p\mathrm{v}} \equiv \mathbf{v}; \qquad \frac{\partial^2 \mathbf{v}}{\partial t^2} - \frac{1}{\varkappa\varrho}\,\mathbf{grad}\,\mathrm{div}\,\mathbf{v} = 0; \qquad p_{\mathrm{v}} = -\frac{1}{\varkappa}\int \mathrm{div}\, \mathbf{v}\, \mathrm{d}t. \tag{19}$$

In a dielectric, when disturbance starts from a curly inhomogeneity of electric intensity (displacement current), the electric dynamic state remains the leader at any area and time moment, being described by the second-order wave relation, while the accompanying magnetic intensity (magnetic induction) value is determined by the proper Maxwell's first-order relation. When a nonuniform curly magnetic intensity (induction) is a primary cause of disturbance at an area, the magnetic intensity is the leader, being accompanied by the electric intensity determined by the other Maxwell's relation.

A self-affecting electric polarisation imbalance. Similarly to what was performed in fluid continuum, we show below the like feedback loop mechanism of electric and magnetic local dynamic states in dielectric continuum. Self-affecting electric polarisation imbalance results in a vector second order relation with electric intensity as physical variable.

$$\text{a)}\ \ \mathbf{H}_{\mathrm{E}} = -\frac{1}{\mu}\int \mathbf{curl}\, \mathbf{E}\, \mathrm{d}t, \qquad \text{b)}\ \ \mathbf{E}_{\mathrm{HE}} = +\frac{1}{\epsilon}\int \mathbf{curl}\, \mathbf{H}_{\mathrm{E}}\, \mathrm{d}t, \tag{20}$$

first effect ⇐ primary cause, final effect ⇐ secondary cause.

$$\mathbf{E}_{\mathrm{HE}} \equiv \mathbf{E}; \qquad \frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{1}{\epsilon\mu}\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} = 0; \qquad \mathbf{H}_{\mathrm{E}} = -\frac{1}{\mu}\int \mathbf{curl}\, \mathbf{E}\, \mathrm{d}t. \tag{21}$$

A self-affecting magnetic polarisation imbalance. A self-affecting magnetic polarisation imbalance results in a dual second-order relation involving magnetic polarisation vector as variable. Starting with:

$$\text{a)}\quad \mathbf{E}_{\mathrm{H}} = +\,\frac{1}{\epsilon}\int \mathbf{curl}\,\mathbf{H}\,\mathrm{d}t, \qquad \text{b)}\quad \mathbf{H}_{\mathrm{EH}} = -\,\frac{1}{\mu}\int \mathbf{curl}\,\mathbf{E}_{\mathrm{H}}\,\mathrm{d}t \tag{22}$$

first effect ⇐ primary cause, final effect ⇐ secondary cause.

we get

$$\mathbf{H}_{\mathrm{EH}} \equiv \mathbf{H}; \quad \frac{\partial^2\mathbf{H}}{\partial t^2} + \frac{1}{\mu\epsilon}\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{H} = 0; \quad \mathbf{E}_{\mathrm{H}} = +\,\frac{1}{\epsilon}\int \mathbf{curl}\,\mathbf{H}\,\mathrm{d}t. \tag{23}$$

The cause-effect circular action chain at a given "particle" area is, in fact, a physical feedback loop effect taking place in with an unequivocal sequence of impact events depending on what is the input. Just as it happens in time invariant linear systems where space dimensions are of no concern, the output depends on input, and the result is being produced at a given time moment, without any time delay, i.e. without involving the passage of time.

The above presented "local-and-frozen circular actions" scheme seems to be the phenomenological mechanism that paradoxically rules in fluids and dielectrics all wave-like effects, though they involve space and time in a most dynamical way obvious to every observer - user of sound, radio, and light signals, as well as of all kinds of electromagnetic energy.

6. POINT-SOURCES AND THEIR LOCAL IMPACTS

The results of the previous section, otherwise well-known and mathematically obvious, carry new evidence: once leading, the dynamic state remains leading throughout all space and time travel of acoustic or electromagnetic disturbance, while the other is still adjunct to the former, accompanying it inseparably. Otherwise than is commonly presented in mathematics-only based approach, it is to be stressed that there is neither an autonomous, source independent acoustic pressure field nor an autonomous and isolated particle velocity field. And similarly, an electric intensity field or magnetic intensity field are inherently coupled one to another. The point is that a leading dynamic state characterised by a related physical quantity or field variable, is to be somehow induced in the medium in a location and time.

Two kinds of source impacts. Here we show that in each medium there are two distinct kinds of impact possible, each triggering a specific cause-effect chain of actions, proper to the primary dynamic state induced in the medium. From this location and time on, a proper sequence of local dynamic states starts, remaining internally coupled in the way related to the type of source.

In fluid acoustics, these states are described by either scalar or vector second-order equation, depending on the scalar or vector nature of the source. Hence the pressure field leads in the first case and the velocity one in the second case.

In dielectric electromagnetics, one of the second-order equations related to either electric intensity or magnetic intensity is relevant, according to the electric or magnetic character of the source.

Quasi-point source area. The formal description of a quasi-point area $\delta(\mathbf{x})$ in [1 / m^3], bases on the distribution form of definition:

$$\int_V \delta(\mathbf{x}-\mathbf{x}_0)S(\mathbf{x})dV(\mathbf{x}) = S(\mathbf{x}_0). \tag{24}$$

There are two kinds of source actions able to influence medium particles. A given point-like source is able to disrupt the state of rest, introducing a specific disequilibrium into a given area.

Matter inflow as a simple source (monopole). The source quantity $q(\mathbf{x},t) = Q(t)\ \delta(\mathbf{x})$ in [1 / s] = [(m^3 / s) / m^3], is a volume density of fluid matter flowing gently into the point area, where $Q(t)$, itself, is in [m^3 / s]. The flow is assumed to be non-inert, thus causing an elastic deformation only. It induces, around the source point, a change of strain/pressure as primary in-fluid effect that becomes, in turn, the input of the subsequent circular actions:

$$\begin{array}{ccc} \frac{\partial}{\partial t} s_{\mathrm{Q}}(\mathbf{x},t) & = & Q(t)\ \delta(\mathbf{x}) \\ \text{in-fluid primary effect} & \Leftarrow & \text{point-source action} \end{array} \qquad (25)$$

Force source as a "dipole". External force acts as an acoustic dipole source. The source quantity $\mathbf{f}(\mathbf{x},t) = \mathbf{F}(t)\delta(\mathbf{x}) = F(t)\delta(\mathbf{x})\ \hat{\boldsymbol{\imath}}_{\mathrm{F}}$, in [N / m^3], is a force density, where $F(t)$ is the force itself, in [N]. However, that a point force cannot be exerted onto a non-viscous fluid. An external force can act only by an intermediate of a contact surface, and has to be implemented to the fluid through a coupling surface. A rigid sphere is a practical intermediary that turns out to be, as well, perfectly matched to the motion of fluid particles at the interface. The resulting flow momentum/velocity around the coupling surface is then the primary in-fluid action, the input of the circular action.

$$\begin{array}{ccc} \frac{\partial}{\partial t} \mathbf{g}_{\mathrm{F}}(\mathbf{x},t) & = & \mathbf{F}(\mathbf{t})\ \delta(\mathbf{x}) \\ \text{in-fluid primary effect} & \Leftarrow & \text{point-source action} \end{array} \qquad (26)$$

Electric dipole. The quasi-point electric source $\mathbf{p}(\mathbf{x},t) = \mathbf{P}(t)\delta(\mathbf{x}) = P(t)\delta(\mathbf{x})\hat{\boldsymbol{\imath}}_{\mathrm{P}}$, in [A / m^2], is defined as an infinitesimal electric dipole of the strength $P(t)$ in [A$\cdot$ m], composed of two time-varying electric charges $q(t)$ of opposite signs, distant l one from the other. The charges time rate of change means the electric current I_{l}, flowing in a conducting filament of the length l, and $P(t) = \frac{\partial}{\partial t} q(t)\ \mathrm{l} = I_{\mathrm{l}}(t)\ \mathrm{l}$.

$$\begin{array}{ccc} \frac{\partial}{\partial t} \mathbf{D}_{\mathrm{P}}(\mathbf{x},t) & = & \mathbf{P}(t)\ \delta(\mathbf{x}) \\ \text{in-dielectric primary effect} & \Leftarrow & \text{point-source action} \end{array} \qquad (27)$$

The above equation describes the initial cause-effect sequence: the electric source action at a point and the electrical reaction of the dielectric in the direct neighbourhood.

Magnetic dipole. It appears that the magnetic effects can be produced by an electric current flowing in a conducting filament loop. In a manner analogous to the case of the electric source, a quasi-point magnetic dipole $\mathbf{m}(\mathbf{x},t) = \mathbf{M}(t)\delta(\mathbf{x}) = M(t)\delta(\mathbf{x})\ \hat{\boldsymbol{\imath}}_{\Sigma}$ in [V / m^2], is defined, where the magnetic dipole strength $M(t)$, in [V m], is related to the rate of change of the electric current I_{Σ} turning in a conducting filament loop closing a surface Σ, and involves, as well the medium permeability μ: $\mathbf{M}(t) = \mu \frac{\partial}{\partial t} I_{\Sigma}(t)\boldsymbol{\Sigma}$.

$$\begin{array}{ccc} \frac{\partial}{\partial t} \mathbf{B}_{\mathrm{M}}(\mathbf{x},t) & = & \mathbf{M}(t)\ \delta(\mathbf{x}) \\ \text{in-dielectric primary effect} & \Leftarrow & \text{point-source action} \end{array} \qquad (28)$$

Here also the initial cause-effect sequence is present: the magnetic source action at a point and the magnetic reaction of the dielectric in the direct neighbourhood. The magnetic dipole acts on a purely magnetic principle, influencing merely the magnetic dynamics, with no electric displacement dynamic state involved on a primary foot.

7. LOCAL CIRCULAR ACTIONS INITIATED BY SOURCES

Compression-initiated imbalance. Compression imbalance induced by external matter inertia-less inflow:

$$p_{\mathrm{Q}}(\mathbf{x},t) = \frac{1}{\varkappa}\int Q(t)\delta(\mathbf{x})\mathrm{d}t, \tag{29}$$

in-fluid primary effect $\Leftarrow$ external cause (source operation).

$$p_{\mathrm{Q}}(\mathbf{x},t) = \frac{1}{\varkappa\varrho}\iint \mathrm{div}\,\mathbf{grad}\, p_{\mathrm{Q}}(\mathbf{x},t)\mathrm{d}t^2 + \frac{1}{\varkappa}\int Q(t)\delta(\mathbf{x})\mathrm{d}t, \tag{30}$$

$$\frac{\partial^2}{\partial t^2}p_{\mathrm{Q}}(\mathbf{x},t) - \frac{1}{\varkappa\varrho}\,\mathrm{div}\,\mathbf{grad}\, p_{\mathrm{Q}}(\mathbf{x},t) = \frac{1}{\varkappa}\frac{\partial}{\partial t}Q(t)\delta(\mathbf{x}), \tag{31}$$

$$\mathbf{v}_{\mathrm{Q}}(\mathbf{x},t) = -\frac{1}{\varrho}\int \mathbf{grad}\, p_{\mathrm{Q}}(\mathbf{x},t)\mathrm{d}t. \tag{32}$$

No motion effect at the source point is observed in the beginning, only an elastic reaction of fluid continuum, that means the $\varrho \equiv 0$ inertia-less start of the disturbance.

Motion-initiated imbalance. Flow imbalance induced by external deformation-less force impact:

$$\mathbf{v}_{\mathrm{F}}(\mathbf{x},t) = \frac{1}{\varrho}\int \mathbf{F}(t)\delta(\mathbf{x})\mathrm{d}t, \tag{33}$$

in-fluid primary effect $\Leftarrow$ external cause (source operation).

$$\mathbf{v}_{\mathrm{F}}(\mathbf{x},t) = \frac{1}{\varrho\varkappa}\iint \mathbf{grad}\,\mathrm{div}\,\mathbf{v}_{\mathrm{F}}(\mathbf{x},t)\mathrm{d}t^2 + \frac{1}{\varrho}\int \mathbf{F}(t)\delta(\mathbf{x})\mathrm{d}t, \tag{34}$$

$$\frac{\partial^2}{\partial t^2}\mathbf{v}_{\mathrm{F}}(\mathbf{x},t) - \frac{1}{\varrho\varkappa}\,\mathbf{grad}\,\mathrm{div}\,\mathbf{v}_{\mathrm{F}}(\mathbf{x},t) = \frac{1}{\varrho}\frac{\partial}{\partial t}F(t)\delta(\mathbf{x})\;\hat{\boldsymbol{\imath}}_{\mathrm{F}}, \tag{35}$$

$$p_{\mathrm{F}}(\mathbf{x},t) = -\frac{1}{\varkappa}\int \mathrm{div}\,\mathbf{v}_{\mathrm{F}}(\mathbf{x},t)\mathrm{d}t. \tag{36}$$

No deformation effect at the source point is observed, only a motion reaction of fluid continuum, that means a $\varkappa \equiv 0$ start "scenario".

Electric polarisation. Electric polarisation imbalance induced by magnetism-less electric dipole:

$$\mathbf{E}_{\mathrm{P}}(\mathbf{x},t) \;=\; \frac{1}{\epsilon}\int \mathbf{P}(t)\delta(\mathbf{x})\;\mathrm{d}t \tag{37}$$

in dielectric effect $\Leftarrow$ external cause (source operation) (38)

$$\mathbf{E}_{\mathrm{P}}(\mathbf{x},t) = -\frac{1}{\epsilon\mu}\iint \mathbf{curl\,curl}\,\mathbf{E}_{\mathrm{P}}(\mathbf{x},t)\,\mathrm{d}t^2 + \frac{1}{\epsilon}\int \mathbf{P}(t)\delta(\mathbf{x})\mathrm{d}t, \tag{39}$$

$$\frac{\partial^2}{\partial t^2}\mathbf{E}_{\mathrm{P}}(\mathbf{x},t) + \frac{1}{\epsilon\mu}\,\mathbf{curl}\,\mathrm{curl}\,\mathbf{E}_{\mathrm{P}}(\mathbf{x},t) = \frac{1}{\epsilon}\frac{\partial}{\partial t}P(t)\delta(\mathbf{x})\,\hat{\boldsymbol{\imath}}_{\mathrm{P}}, \tag{40}$$

$$\mathbf{H}_{\mathrm{P}}(\mathbf{x},t) = -\frac{1}{\mu}\int \mathbf{curl}\,\mathbf{E}_{\mathrm{P}}(\mathbf{x},t)\,\mathrm{d}t. \tag{41}$$

No magnetic effect at the electric dipole point is observed, only an electric reaction of dielectric continuum, that means the $\mu \equiv 0$ "scenario" at the start.

The above equation describes the cause-effect sequence: source action at a point - vacuum reaction in the surrounding neighbourhood.

Magnetic polarisation. Magnetic polarisation imbalance induced by the magnetic dipole:

$$\mathbf{H}_{\mathrm{M}}(\mathbf{x},t) \;=\; \frac{1}{\mu}\int \mathbf{M}(t)\delta(\mathbf{x})\,\mathrm{d}t \tag{42}$$

$$\text{in-dielectric initial effect} \;\Leftarrow\; \text{external cause (source operation)} \tag{43}$$

$$\mathbf{H}_{\mathrm{M}}(\mathbf{x},t) = -\frac{1}{\mu\epsilon}\iint \mathbf{curl\,curl}\,\mathbf{H}_{\mathrm{M}}(\mathbf{x},t)\,\mathrm{d}t^2 + \frac{1}{\mu}\int \mathbf{M}(t)\delta(\mathbf{x})\,\mathrm{d}t, \tag{44}$$

$$\frac{\partial^2}{\partial t^2}\mathbf{H}_{\mathrm{M}}(\mathbf{x},t) + \frac{1}{\mu\epsilon}\,\mathbf{curl\,curl}\,\mathbf{H}_{\mathrm{M}}(\mathbf{x},t) = \frac{1}{\mu}\frac{\partial}{\partial t}M(t)\delta(\mathbf{x})\,\hat{\boldsymbol{\imath}}_{\mathrm{M}}, \tag{45}$$

$$\mathbf{E}_{\mathrm{M}}(\mathbf{x},t) = +\frac{1}{\epsilon}\int \mathbf{curl}\,\mathbf{H}_{\mathrm{M}}(\mathbf{x},t)\,\mathrm{d}t. \tag{46}$$

No electric effect at the source point is observed, only magnetic reaction of dielectric continuum, meaning an $\epsilon \equiv 0$ start "scenario".

Although direct means to apply magnetic-only effects in physical space, we will follow the derivation as if it were possible, in analogous way as was made with point force applied to fluid.

8. "AT-A-DISTANCE" IMPLICATIONS OF POINT SOURCES ACTION

In fundamental wavefields initiated by point sources, up to three components can be distinguished, each following the source quantity signal form, though in different way. The time evolution of the wave component is that of time source quantity time differential. The "induction" component follows directly the source timeform, and the quasi-static one that of the source quantity integral.

Scalar fundamental solutions. In the case of an extra matter inflow/outflow $Q(t)$, i.e. a scalar source, the cause-effect chain leads to a double time integral relation that can be reduced to the equivalent differential form [4]:

$$\frac{\partial^2}{\partial t^2}p_{\mathrm{Q}}(\mathbf{x},t) - \frac{1}{\varkappa\varrho}\,\mathrm{div}\,\mathbf{grad}\,p_{\mathrm{Q}}(\mathbf{x},t) = \frac{1}{\varkappa}\frac{\partial}{\partial t}Q(t)\delta(\mathbf{x}). \tag{47}$$

For calculating the field described by the above equation, use is made of the pure mathematical time-space differential equation related to an impulsive scalar point source, known as the inhomogeneous wave equation, with the so-called Green's function as its solution, with the time evolution variable in a delayed form, namely $(t - r/c)$, related to the distance r from the source and a medium dynamic parameter c being the celerity of the disturbance transport spread. The celerity appears to be equal to the inverted geometric mean of the relative medium parameters:

$$c_{\mathrm{ac}}(\mathbf{x}) = \Big(\varkappa(\mathbf{x})\varrho(\mathbf{x})\Big)^{-1/2} \quad \text{and} \quad c_{\mathrm{em}}(\mathbf{x}) = \Big(\epsilon(\mathbf{x})\mu(\mathbf{x})\Big)^{-1/2} \tag{48}$$

Pressure and velocity in the inflow-source wavefield. By virtue of the superposition principle, the solution of the equation (47) becomes:

$$p_{\mathrm{Q}}(\mathbf{x}, t) = \frac{\varrho}{4\pi r}\frac{\partial}{\partial t}Q(t - r/c), \tag{49}$$

with the accompanying velocity field calculated from rel. 15 a), as follows:

$$\begin{aligned} \mathbf{v}_Q(r,t) &= \hat{\boldsymbol{r}}\,\sqrt{\varkappa\varrho}\,\frac{1}{4\pi r}\left[\frac{\partial}{\partial t}Q(t-r/c) + \frac{c}{r}Q(t-r/c)\right] \\ &= \mathbf{v}_{Q\mathrm{a}}(r,t) + \mathbf{v}_{Q\mathrm{d}}(r,t) \end{aligned} \tag{50}$$

Irrotational velocity vector equations. The action-reaction causal chain initiated by an external force $F(t)$, leads to the following equation dual to eq. 47:

$$\frac{\partial^2}{\partial t^2}\mathbf{v}_{\mathrm{F}}(\mathbf{x},t) - \frac{1}{\varrho\varkappa}\,\mathbf{grad}\,\mathrm{div}\,\mathbf{v}_{\mathrm{F}}(\mathbf{x},t) = \frac{1}{\varrho}\frac{\partial}{\partial t}F(t)\hat{\boldsymbol{\imath}}_{\mathrm{F}}\delta(\mathbf{x}). \tag{51}$$

Velocity and pressure in the force-source wavefield. As shown in [4], the solution of the equation (51) is:

$$\begin{aligned} \mathbf{v}_{\mathrm{F}}(r,\vartheta,t) &= \hat{\boldsymbol{r}}\cos\vartheta\,\varkappa\,\frac{1}{4\pi r}\,\frac{\partial}{\partial t}F(t-r/c) \\ &+ (2\hat{\boldsymbol{r}}\cos\vartheta + \hat{\boldsymbol{\vartheta}}\,\sin\vartheta)\sqrt{\frac{\varkappa}{\varrho}}\,\frac{1}{4\pi r^2}\,F(t-r/c) \\ &+ (2\hat{\boldsymbol{r}}\cos\vartheta + \hat{\boldsymbol{\vartheta}}\sin\vartheta)\,\frac{1}{\varrho}\,\frac{1}{4\pi r^3}\int F(t-r/c)\mathrm{d}t \\ &= \mathbf{v}_{\mathrm{Fac}}(r,\vartheta,t) + \mathbf{v}_{\mathrm{Fhd}}(r,\vartheta,t) + \mathbf{v}_{\mathrm{Fqs}}(r,\vartheta,t) \end{aligned} \tag{52}$$

with the accompanying velocity field calculated from rel. 18 a), as follows:

$$\begin{aligned} p_{\mathrm{F}}(\mathbf{x},t) &= \cos\vartheta\,\sqrt{\varrho\varkappa}\,\frac{1}{4\pi r}\,\left[\frac{\partial}{\partial t}\,F(t-r/c) + \frac{c}{r}F(t-r/c)\right] \\ &= p_{\mathrm{Fac}}(r,\vartheta,t) + p_{\mathrm{Fhd}}(r,\vartheta,t) \end{aligned} \tag{53}$$

Nondivergent electric intensity vector equation. The cause-effect chain initiated by an electric dipole $P(t)$, leads to the following equation analogous to eq. 51:

$$\frac{\partial^2}{\partial t^2}\mathbf{E}_{\mathrm{P}}(\mathbf{x},t) + \frac{1}{\epsilon\mu}\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}_{\mathrm{P}}(\mathbf{x},t) = \frac{1}{\epsilon}\frac{\partial}{\partial t}P(t)\hat{\boldsymbol{\imath}}_{\mathrm{P}}\delta(\mathbf{x}). \tag{54}$$

This case is much similar to the former one with, however, significant difference in the radiation terms.

Electric and magnetic intensities in the electric-dipole wavefield. The fundamental solution concerning the electric intensity are well known:

$$
\begin{aligned}
\mathbf{E}_{\mathrm{P}}(r,\vartheta,t) &= \hat{\boldsymbol{\vartheta}} \sin\vartheta \, \mu \, \frac{1}{4\pi r} \, \frac{\partial}{\partial t} P(t-r/c) \\
&+ (2\hat{\boldsymbol{r}}\cos\vartheta + \hat{\boldsymbol{\vartheta}}\sin\vartheta) \sqrt{\frac{\mu}{\epsilon}} \, \frac{1}{4\pi r^2} \, P(t-r/c) \\
&+ (2\hat{\boldsymbol{r}}\cos\vartheta + \hat{\boldsymbol{\vartheta}}\sin\vartheta) \, \frac{1}{\epsilon} \, \frac{1}{4\pi r^3} \int P(t-r/c)\mathrm{d}t \\
&= \mathbf{E}_{\mathrm{Prad}}(r,\vartheta,t) + \mathbf{E}_{\mathrm{Pind}}(r,\vartheta,t) + \mathbf{E}_{\mathrm{Pqes}}(r,\vartheta,t)
\end{aligned}
\tag{55}
$$

with the accompanying magnetic intensity field calculated from rel. 18 a), as follows:

$$
\begin{aligned}
\mathbf{H}_{\mathrm{P}}(r,\vartheta,t) &= +\hat{\boldsymbol{\varphi}} \sin\vartheta \, \sqrt{\epsilon\mu} \, \frac{1}{4\pi r} \left[\frac{\partial}{\partial t} P(t-r/c) + \frac{c}{r} P(t-r/c) \right] \\
&= \mathbf{H}_{\mathrm{Prad}}(r,\vartheta,t) + \mathbf{H}_{\mathrm{Pind}}(r,\vartheta,t)
\end{aligned}
\tag{56}
$$

Nondivergent magnetic intensity vector equations. The cause-effect chain initiated by a magnetic dipole $M(t)$, leads to the following equation dual to eq. 54:

$$
\frac{\partial^2}{\partial t^2} \mathbf{H}_{\mathrm{M}}(\mathbf{x},t) + \frac{1}{\mu\epsilon} \, \mathbf{curl}\,\mathbf{curl}\, \mathbf{H}_{\mathrm{M}}(\mathbf{x},t) = \frac{1}{\mu} \frac{\partial}{\partial t} P(t) \hat{\boldsymbol{\imath}}_{\mathrm{M}} \delta(\mathbf{x}). \tag{57}
$$

The solutions turns out to be perfectly symmetric to the former ones.

Magnetic intensity in the magnetic-dipole wavefield. Here the fundamental solution concerning the magnetic intensity is:

$$
\begin{aligned}
\mathbf{H}_{\mathrm{M}}(r,\vartheta,t) &= \hat{\boldsymbol{\vartheta}} \sin\vartheta \, \epsilon \, \frac{1}{4\pi r} \, \frac{\partial}{\partial t} M(t-r/c) \\
&+ (2\hat{\boldsymbol{r}}\cos\vartheta + \hat{\boldsymbol{\vartheta}}\sin\vartheta) \sqrt{\frac{\epsilon}{\mu}} \, \frac{1}{4\pi r^2} \, M(t-r/c) \\
&+ (2\hat{\boldsymbol{r}}\cos\vartheta + \hat{\boldsymbol{\vartheta}}\sin\vartheta) \, \frac{1}{\mu} \, \frac{1}{4\pi r^3} \int M(t-r/c)\mathrm{d}t \\
&= \mathbf{H}_{\mathrm{Mrad}}(r,\vartheta,t) + \mathbf{H}_{\mathrm{Mind}}(r,\vartheta,t) + \mathbf{H}_{\mathrm{Mqms}}(r,\vartheta,t)
\end{aligned}
\tag{58}
$$

with the accompanying electric intensity field calculated from rel. 18 a), as follows:

$$
\begin{aligned}
\mathbf{E}_{\mathrm{M}}(r,\vartheta,t) &= -\,\hat{\boldsymbol{\varphi}} \, \sin\vartheta \, \sqrt{\mu\epsilon} \, \frac{1}{4\pi r} \left[\frac{\partial}{\partial t} M(t-r/c) + \frac{c}{r} M(t-r/c) \right] \\
&= \mathbf{E}_{\mathrm{Mrad}}(r,\vartheta,t) + \mathbf{E}_{\mathrm{Mind}}(r,\vartheta,t)
\end{aligned}
\tag{59}
$$

A symmetry between the two fundamental electromagnetic fields is much closer than between the fundamental acoustic fields.

Components of inhomogeneous equations fundamental solutions. In fundamental wavefields initiated by point sources, up to three components can be distinguished, each following the source quantity timeform but in different ways. The time evolution of the wave component is that of time source quantity time differential. The "induction" component follows directly the source timeform, and the quasi-static one that of the source quantity time integral.

In the radiation terms of the source-point wavefields, indicated with rad subscripts, dual dynamic quantities are proportional to each other, enabling the use of the notions of medium impedance and admittance, otherwise related to mere plane waves.

The approach covers all kinds of signals with no restriction other than the limitation of finite integral of source quantity waveform, meaning the finite source energy. And it helps to recognise and identify the fundamental aspects of the phenomena.

The fundamental, core effects can hardly be picked out with the time-frequency approach, and even more so with the space-frequency notion, the first being acausal, spanned onto all the past and all the future, the second - spread onto all the space.

9. LOCAL PROVENIENCE OF DYNAMIC PHENOMENA CELERITY

As it was shown in Section 4, the physical quantities of practical importance related to local energy, power stream, and momentum have appeared to be principally the same, though the two media are fundamentally different, and so are their local dynamic states. It is worth stressing here that the analysis there performed concerned merely a given elemental area of medium without leaving it whatsoever, and the notion of celerity appears for the first time only in the solutions of the second-order dynamic state relations known as homogeneous wave equations.

The mutual proportionality of dual wavefield intensities and densities leads directly to a significant observation. The principle of energy and momentum transport appears to be the same in both media, the disturbance they are related to, being conveyed by the medium with its proper celerity:

$$\mathrm{S}_{\mathrm{ac}} = w_{\mathrm{ac}}\, c_{\mathrm{ac}} \quad \text{and} \quad \mathrm{S}_{\mathrm{em}} = w_{\mathrm{em}}\, c_{\mathrm{em}} \tag{60}$$

The above means, as well, the following formulae related energy and momentum densities with corresponding celerities :

$$\mathrm{g}_{\mathrm{ac}} = w_{\mathrm{ac}}/c_{\mathrm{ac}} \quad \text{and} \quad \mathrm{g}_{\mathrm{em}} = w_{\mathrm{em}}/c_{\mathrm{em}} \tag{61}$$

The latter relations, however, have no direct interpretation if not that momenta of plane wave disturbances are extremely small, especially in the case of electromagnetics. The results of the Section give strong arguments for treating the dynamic phenomena of acoustics and electromagnetics on the same foot.

10. CONCLUSIONS

The consequent application of phenomena-founded description of local effects reveals new, weighty aspects of wavefield behaviour. It becomes clear that the true reason for huge wave celerity resides directly in every elementary region of the medium.

When dealing with dynamic phenomena in physical spaces, namely acoustic and electromagnetic wavefields, a mathematics-founded view prevails, preferring an approach focused on waves and their global features related to largely known class of possible solutions of wave equation.

It turns out to be more beneficial to clearly distinguish between the phenomenon-focused description of two-fold dynamics acting at each point of physical space, and the classical formal description based on field theory mathematics, and admit unambiguous definitions of current notions, otherwise use in an ambiguous, context-defined way. Hence a distinction is introduced and consistently followed between notions of formula, relation and equation. A formula defines a quantity in terms of other quantities and is mostly a description of several aspects of the same phenomenon. It is a somehow internal relation. A relation describes equilibrium conditions at force in any moment in each medium area. It results in a locally defined dynamic action between autonomous quantities being somehow external to each other. An equation describes a particular situation related to an action of a source. The idea to strive for semantic unambiguity was is inspired by Whithead's view of space, time, and physical phenomena discussed in [6].

In the sense of the above definitions, a formula serves to calculate, at a point, the value of one quantity from the known values of other quantities. In terms of mathematics, relation means an homogeneous equation that allows to calculate the time-space distribution of a quantity in continuum, in absence of sources, thus having myriads of possible solutions. And there is one and only one possible solution of what we call equation, being the mathematically inhomogeneous one.

In common practice, equation is a term equally used in all three of the above mentioned cases. Each of them being semantically so different, such a habit promotes a blurred thinking and mitigates against a precise understanding of the true meaning of a particular formal description of physical phenomena. In the author's opinion, this can be the reason for the existing oversight of the dynamic essence of wavefield phenomena.

It appears that the classical methodology, although giving results that are both mathematically correct and technologically useful, overlooks, to a large extent, local causality mechanisms, thus ignoring effects of significant phenomenological consequences.

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