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# Stress analysis of a strip under tension with a circular hole



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# Stress Analysis of a Strip Under Tension with a Circular Hole

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**Abstract.** The paper addresses stress analysis of a strip with a circular hole under uniform uniaxial tension based on circumferential stress expression $\sigma_\theta$. Stresses are analyzed in the infinite-length strips under tension with holes, the ratio of the hole radius $a$ to the strip half-width $b$ is either equal to: $\lambda = a/b = 0.1$ or $\lambda = 0.5$. Circumferential stresses are determined in selected cross-sections of the strip. The stress diagrams display local stress concentration near the hole. The analytical solutions are compared with the numerical solutions the finite elements method (FEM), the latter using the ABAQUS software due to various $\lambda$ ratios. The results show the maximum circumferential stress at hole boundary, affected by the ratio of the hole radius to the width of the strip.
The work was created as part of B.Sc. engineering thesis at Gdansk University of Technology, Faculty of Civil and Environmental Engineering.

## INTRODUCTION

Stress concentration problems are significant in structural mechanics, thus widely considered and analysed in literature. The present day technical and scientific literature often regards stress concentration problems in plates and shells. It proves validity of problem and challenge in searching for solutions. Three-dimensional solution for the stress field around a circular hole in a plate far-field tension is presented in [1]. Stress concentration factor of a plate with a circular hole under tension is analysed in the work [2] by means of the finite element method. Numerical methods to deal with stress concentration in isotropic plates with discontinuities are included in the paper [3]. The article [4] analyses an isotropic plate with a noncircular hole under uniform tension. Stress concentration in finite metallic plates with regular holes under uniaxial loading is considered in [5]. In the paper [6] elastic stress and strain fields of plates with a single central hole and two auxiliary holes subjected to uniaxial tension are investigated by means of the finite element method. The FEM-based analysis of plates with holes of different shapes: circular, square and rectangular are analysed in [7]. In [8] the authors present geometric elasticity problem of stress concentration in a plate with a circular hole under uniaxial tension, biaxial tension and shear. Theoretical solution of a rectangular plate with an arbitrarily located circular hole subjected to in-plane bending moment is presented in [9]. Interaction between two neighbouring circular holes in an orthotropic strip subjected to bending is considered in [10]. In [11] the authors analyse stress concentration in an elastic square plate with a full-strength hole under bending. The 3D FEM-based analysis of the stress and strain concentration in an elastic plate of finite thickness with a circular hole subjected to tension is included in the papers [12], [13]. The elliptical hole core is considered in [14], two interacting holes are investigated in [15]. Shape optimization of a hole in an orthotropic plate at the biaxial compression is presented in [16]. The stress concentration problem is also important in shell structures, e.g. in [17] stresses are analysed around holes in cylindrical parts of prestressed concrete protective shells. Simplified stress distributions in plate with circular hole under uniform tension can be found in the works [18], [19]. Other stress distribution functions around holes in plates and girders are included in [20], [21].

The paper concerns stress analysis of a strip under tension with a circular hole and considers the plane stress continuous theory of elasticity, addressed in [22]. In the work three well-known approaches are compared: the

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analytical ones, proposed by Girkmann [18] and Howland [20] and the FEM-based numerical solution. Sufficient accuracy of the results is observed within the scope of the stated assumptions.

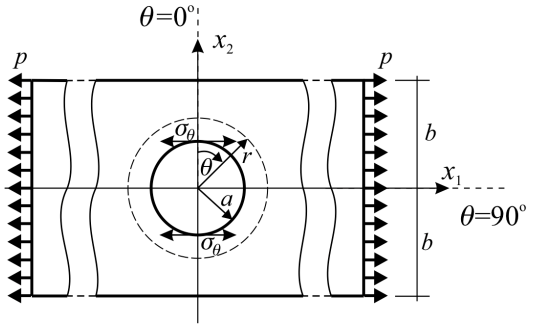


**FIGURE 1.** Static diagram.

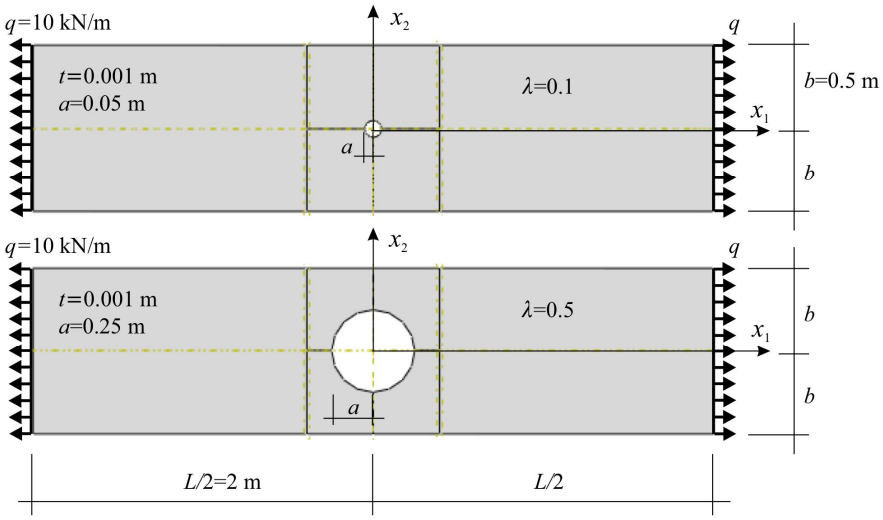


**FIGURE 2.** Numerical examples – data.

## PLANE STRESS

The paper is focused on structural analysis of plates. The structure is analysed in plane stress assuming that the in-plane loads act only, the stress components perpendicular to the plate are zero. Thus stresses in the system are presented in the form of a symmetric two-dimensional Cauchy tensor consisting of three independent components: $\sigma_{11}$, $\sigma_{22}$, $\sigma_{12}$ defined in the Cartesian coordinate system. The following basic equations of theory of elasticity hold:

- local equilibrium equations (involving Cauchy stress tensor)

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + X_1 = 0 \\ \dfrac{\partial \sigma_{12}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + X_2 = 0 \end{cases} \tag{1}$$

where $X_1$ and $X_2$ are body force components in $x_1$ and $x_2$ directions, respectively,

- the compatibility equation

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = \frac{2\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} \tag{2}$$

where $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12} = \dfrac{\gamma_{x_1 x_2}}{2}$ are strain components in the Cartesian coordinate system,

-Hook's law - linearly elastic stress-strain relations in plane stress

$$\begin{cases} \sigma_{11} = \dfrac{E}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22}) \\ \sigma_{22} = \dfrac{E}{1-\nu^2}(\varepsilon_{22} + \nu\varepsilon_{11}) \\ \sigma_{12} = 2G\varepsilon_{12} \end{cases} \tag{3}$$

where $E$ is the Young's modulus, $G$ is the shear modulus and ν is the Poisson's ratio.

A possible solution strategy for a plane stress problem is the definition of the so-called Airy stress function

$$\begin{cases} \sigma_{11} = \dfrac{\partial^2 F}{\partial x_2^2} \\ \sigma_{22} = \dfrac{\partial^2 F}{\partial x_1^2} \\ \sigma_{12} = -\dfrac{\partial^2 F}{\partial x_1 \partial x_2} - x_1 X_2 - x_2 X_1 \end{cases} \tag{4}$$

This definition is compatible with local equilibrium equations (1). The equation (2) expressed in terms of $F$ function leads to the biharmonic equation

$$\frac{\partial^4 F}{\partial x_1^4} + \frac{\partial^4 F}{\partial x_2^4} + 2\frac{\partial^4 F}{\partial x_1^2 \partial x_2^2} = 0 \tag{5}$$

The methodology, reflected equation (5) is valid only for continuous, isotropic and homogeneous material fulfilling the Hooke's law. In the case of constant body forces $X_1$, $X_2$ the equation (5) is material-invariant (constants not affected by $E$, ν), and is also valid for plane strain cases.

The biharmonic equation (5) can be rewritten in polar coordinates ($r$, $\theta$) as

$$\left(\frac{\partial^2 \bullet}{\partial r^2} + \frac{1}{r}\frac{\partial \bullet}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bullet}{\partial \theta^2}\right)\left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right) = 0 \tag{6}$$

where $\bullet$ is $\left(\dfrac{\partial^2 F}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial F}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 F}{\partial \theta^2}\right)$.

These initial considerations finally allow to formulate two main patterns describing the state of stress around the holes in thin plates. The first, simplified attempt was proposed by Girkmann [18], the second, more general, proposed by Howland [20].

## ANALYTICAL SOLUTIONS

Let us consider a strip under tension with a circular hole, the stress state is described in polar coordinates (see Fig. 1). The circular hole is situated in the middle of the strip of the width $2b$, and the radius of the hole is $a$ ($a < b$). Furthermore, a constant strip thickness $t = 1$ is assumed. The strip is subjected to tension under uniaxial load $p = const.$. The origin of the coordinate system is defined in the centre of the hole. In the analytical solutions [18], [20] the plate is infinitely thin and is made of isotropic, homogeneous and linearly elastic, Hookean material.

### Girkmann's Solution

The Girkmann's solution [18] presents a large, thin plate containing a small circular hole is subjected to simple tension. The stress distribution is expressed by equations

$$\sigma_r = \frac{p}{2}\left(1 - \frac{a^2}{r^2}\right) - \frac{p}{2}\left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right)\cos 2\theta \tag{7}$$

$$\sigma_\theta = \frac{p}{2}\left(1+\frac{a^2}{r^2}\right)+\frac{p}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta \tag{8}$$

$$\sigma_{r\theta} = -\frac{p}{2}\left(1+\frac{2a^2}{r^2}-\frac{3a^4}{r^4}\right)\sin 2\theta \tag{9}$$

where $\sigma_r$ is the radial stress, $\sigma_\theta$ is the circumferential stress and $\sigma_{r\theta}$ is the shear stress.

**TABLE 1.** Coefficients *d, e, l, m* for $\lambda = 0.1$ and 0.5.

| λ | 0.1 | 0.5 |
|---|---|---|
| $d_0$ | $5.01\cdot 10^{-3}$ | $1.32\cdot 10^{-1}$ |
| $d_2$ | $2.54\cdot 10^{-5}$ | $2.33\cdot 10^{-2}$ |
| $d_4$ | $3.15\cdot 10^{-11}$ | $3.14\cdot 10^{-4}$ |
| $d_6$ | $1.40\cdot 10^{-15}$ | $7.51\cdot 10^{-6}$ |
| $d_8$ | $5.00\cdot 10^{-20}$ | $1.45\cdot 10^{-7}$ |
| $e_2$ | $-5.08\cdot 10^{-3}$ | $-1.89\cdot 10^{-1}$ |
| $e_4$ | $-4.21\cdot 10^{-9}$ | $-1.70\cdot 10^{-3}$ |
| $e_6$ | $-1.68\cdot 10^{-13}$ | $-3.64\cdot 10^{-5}$ |
| $e_8$ | $-5.72\cdot 10^{-18}$ | $-6.70\cdot 10^{-7}$ |
| $l_2$ | $4.13\cdot 10^{-3}$ | $1.40\cdot 10^{-1}$ |
| $l_4$ | $1.06\cdot 10^{-3}$ | $3.27\cdot 10^{-2}$ |
| $l_6$ | $2.83\cdot 10^{-4}$ | $8.09\cdot 10^{-3}$ |
| $l_8$ | $7.24\cdot 10^{-5}$ | $1.95\cdot 10^{-3}$ |
| $l_{10}$ | $1.82\cdot 10^{-5}$ | $4.71\cdot 10^{-4}$ |
| $l_{12}$ | $4.65\cdot 10^{-6}$ | $1.14\cdot 10^{-4}$ |
| $l_{14}$ | $1.15\cdot 10^{-6}$ | $2.80\cdot 10^{-5}$ |
| $m_0$ | $5.69\cdot 10^{-4}$ | $1.40\cdot 10^{-2}$ |
| $m_2$ | $-1.13\cdot 10^{-3}$ | $-3.62\cdot 10^{-2}$ |
| $m_4$ | $-5.95\cdot 10^{-4}$ | $-1.78\cdot 10^{-2}$ |
| $m_6$ | $-2.28\cdot 10^{-4}$ | $-6.47\cdot 10^{-3}$ |
| $m_8$ | $-7.65\cdot 10^{-5}$ | $-2.08\cdot 10^{-3}$ |
| $m_{10}$ | $-2.39\cdot 10^{-5}$ | $-6.30\cdot 10^{-4}$ |
| $m_{12}$ | $-7.30\cdot 10^{-6}$ | $-1.84\cdot 10^{-4}$ |

## Howland's Solution

The problem of a plate with a circular hole is also addressed by R.C.J. Howland [20]. This solution assumes the strip is infinitely long.

In the analytical solution proposed by Howland, the stresses are expressed as follows

$$\sigma_r = p\Big\{\frac{1}{2}(1-\cos 2\theta)+2m_0-\frac{d_0}{\rho^2}-2\sum_{n=1}^{\infty}\Big[\frac{n(2n+1)d_{2n}}{\rho^{2n+2}}+\frac{(n+1)(2n-1)e_{2n}}{\rho^{2n}}+\ldots + n(2n-1)l_{2n}\rho^{2n-2}+(n-1)(2n+1)m_{2n}\rho^{2n}\Big]\cos 2n\theta\Big\} \tag{10}$$

$$\sigma_\theta = p\Big\{\frac{1}{2}(1+\cos 2\theta)+2m_0+\frac{d_0}{\rho^2}+2\sum_{n=1}^{\infty}\Big[\frac{n(2n+1)d_{2n}}{\rho^{2n+2}}+\frac{(n-1)(2n-1)e_{2n}}{\rho^{2n}}+\ldots + n(2n-1)l_{2n}\rho^{2n-2}+(n+1)(2n+1)m_{2n}\rho^{2n}\Big]\cos 2n\theta\Big\} \tag{11}$$

$$\sigma_{r\theta} = p\Big\{\frac{1}{2}\sin 2\theta+2\sum_{n=1}^{\infty}\Big[n(2n-1)\Big(l_{2n}\rho^{2n-2}-\frac{e_{2n}}{\rho^{2n}}\Big)+n(2n+1)\Big(m_{2n}\rho^{2n}-\frac{d_{2n}}{\rho^{2n+2}}\Big)\Big]\sin 2n\theta\Big\} \tag{12}$$

where $\rho = r / b$ is the dimensionless coordinate, $b$ is the strip half-width, $n$ is the n-th expression of the series, *e, d, l, m* are the coefficients dependent on the ratio $\lambda = a / b$ and $a$ is the hole radius.

The coefficients *e, d, l, m* are presented in the paper [20], computed for $\lambda \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$. The selected coefficient values in the case of $\lambda = 0.1$ and $\lambda = 0.5$ are listed in Tab. 1.

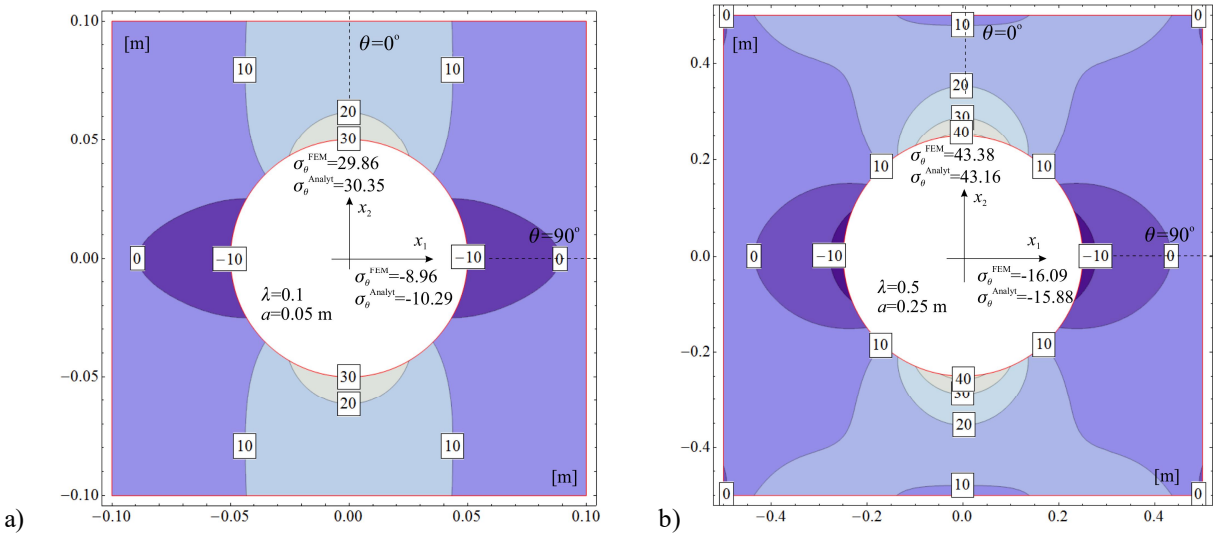


**FIGURE 3.** Howland's solution - contour lines of the circumferential stresses $\sigma_\theta$ [MPa] in the Cartesian coordinate system in the case of the hole with the radius a) a = 0.05 m and b) $a$ = 0.25 m.

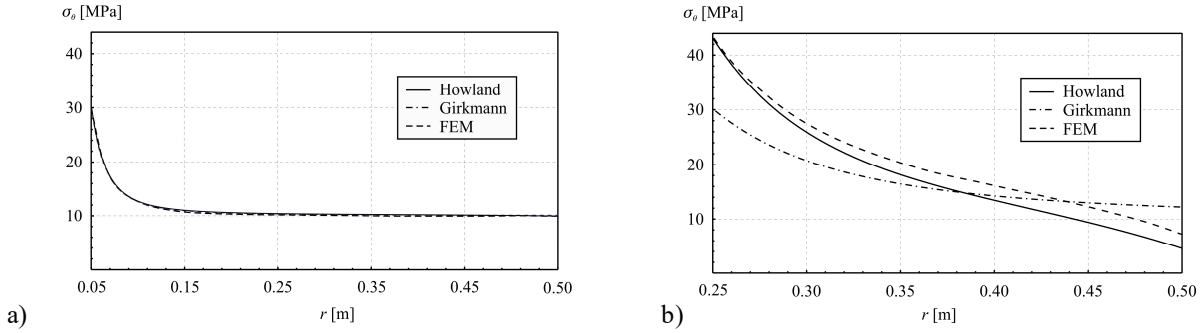


**FIGURE 4.** Circumferential stresses $\sigma_\theta$ [MPa] vs. radius $r$ - distance of a given point from the centre of the hole for $\theta = 0°$ for a) $a$ = 0.05 m and b) $a$ = 0.25 m.

## NUMERICAL EXAMPLES – RESULTS AND CONCLUSIONS

Let us consider two strips of width $2b$ = 1 m and thickness $t$ = 0.001 m with holes of different diameters. The first case assumes the hole of the radius $a$ = 0.05 m ($\lambda = a/b$ = 0.05/0.5 = 0.1), the second case shows $a$ = 0.25 m ($\lambda = a/b$ = 0.25/0.5 = 0.5). It is assumed that the strips are subjected to axial load $p=q/t$=10 MN/m$^2$ which corresponds to the edge load $q$ = 10 kN/m for $t$ = 0.001 m.

In order to confirm the analytical results, numerical analysis has been performed by the FEM ABAQUS package [23]. The stresses were determined by a general static procedure. The applied FEM models are shown in Fig. 2. The strips are modelled by 3-node triangular general-purpose shell, finite membrane-strain elements type - S3. In both numerical examples the finite element mesh is subdivided around the holes. Material behaviour is modelled by a linearly elastic procedure available in the ABAQUS ($E$ = 210 GPa, ν = 0.33).

The major numerical results are shown in Figs. 3, 4 and in Tabs. 2, 3. The stresses are compared due to two analytical approaches [18], [20] and the FEM procedures.

Tab. 2 presents the stresses due to Howland's solution, in the section $\theta = 0^o$, in both cases $a$ = 0.05 m and 0.25 m. The smaller hole case shows extreme circumferential stresses equal 30.348 MPa and for the larger hole case result is 43.158 MPa. In both cases the extreme stresses appear at the hole boundary. As the distance from the hole boundary increases, circumferential stresses decrease.

**TABLE 2.** Howland's solution - stresses [MPa] vs. radius r [m] - distance of a given point from the center of the hole for $\theta = 0^o$ for $a$ = 0.05 m and 0.25 m.

| | $a$ = 0.05 m | | | $a$ = 0.25 m | | |
|---|---|---|---|---|---|---|
| $r$[m] | $\sigma_r$ | $\sigma_\theta$ | $\sigma_{r\theta}$ | $\sigma_r$ | $\sigma_\theta$ | $\sigma_{r\theta}$ |
| 0.05 | -0.001 | 30.348 | 0.00 | | | |
| 0.10 | 2.799 | 12.298 | 0.00 | | | |
| 0.15 | 1.430 | 10.837 | 0.00 | | | |
| 0.20 | 0.805 | 10.463 | 0.00 | | | |
| 0.25 | 0.484 | 10.309 | 0.00 | 0.081 | 43.158 | 0.00 |
| 0.30 | 0.295 | 10.225 | 0.00 | 3.732 | 27.447 | 0.00 |
| 0.35 | 0.172 | 10.108 | 0.00 | 3.278 | 20.386 | 0.00 |
| 0.40 | 0.086 | 10.098 | 0.00 | 1.964 | 16.097 | 0.00 |
| 0.45 | 0.026 | 10.014 | 0.00 | 0.685 | 12.394 | 0.00 |
| 0.50 | -0.008 | 9.888 | 0.00 | -0.151 | 8.018 | 0.00 |

**TABLE 3.** Howland's solution - stresses [MPa] vs. degree of $\theta$ [°] for $r = a$ = 0.05 m and $r = a$ = 0.25 m.

| | $r = a$ = 0.05 m | | | $r = a$ = 0.25 m | | |
|---|---|---|---|---|---|---|
| $\theta$[°] | $\sigma_r$ | $\sigma_\theta$ | $\sigma_{r\theta}$ | $\sigma_r$ | $\sigma_\theta$ | $\sigma_{r\theta}$ |
| 0 | -0.001 | 30.348 | 0.00 | 0.081 | 43.158 | 0.00 |
| 15 | -0.001 | 27.623 | 0.001 | 0.066 | 37.241 | 0.008 |
| 30 | 0.00 | 20.180 | 0.002 | 0.031 | 23.125 | 0.010 |
| 45 | 0.001 | 10.016 | 0.002 | -0.007 | 7.570 | 0.008 |
| 60 | 0.003 | -0.142 | 0.002 | -0.039 | -5.084 | 0.005 |
| 75 | 0.004 | -7.575 | 0.001 | -0.059 | -13.131 | 0.002 |
| 90 | 0.004 | -10.295 | 0.00 | -0.065 | -15.880 | 0.00 |

In the considered section $\theta$=0°, the maximum radial stresses are 2.799 MPa at a distance 0.05 m from the hole boundary in the small hole case (see Tab. 2). In the case of bigger hole the maximum radial stress is 3.732 MPa at the same distance 0.05 m from the hole edge (see Tab. 2). Shear stresses in both cases are zero. Tab. 3 presents stress variation on the hole boundary in the function of θ. The circumferential stresses are extreme on the edge of the hole. Radial and shear stresses at the entire edge of the hole tend to zero. Figs. 3 and 4 present circumferential stresses. Fig. 3 shows the contours of circumferential stresses in the Cartesian coordinate system. Fig. 4 shows circumferential stresses for $\theta = 0^o$ only. Due to both hole sizes, the circumferential stresses curves indicate a local stress concentration at the edges hole boundaries. Fig. 4 compares the analytical results due to Girkmann's and Howland's solutions with the FEM its assumption. In the small hole case ($\lambda = 0.1$, $a$ = 0.05 m), all three approaches converge. However, in the larger hole area the Girkmann's solution, due its assumption significantly differs from the Howland's proposal results and the FEM solution.

Comparison of the analytical solutions with the FEM solutions leads to the conclusion that the Howland's version [20] is dedicated to stress analysis around circular holes in all the proportions of hole areas to the strips/plates areas. On the other hand the Girkmann's solution [18], is limited only to the cases of the hole area significantly smaller than the plate area (see Fig. 4).

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